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I-convergent triple fuzzy normed spaces

Tanweer Jalal
National Institute of Technology Srinagar, India
and
Ishfaq Ahmad Malik
National Institute of Technology Srinagar, India
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Abstract

In this paper we introduce the concept of lacunary ideal convergence of triple sequences in fuzzy normed spaces and the relation between lacunary convergence and lacunary ideal convergence is investigated for triple sequences in fuzzy normed spaces. Concept of limit point and cluster point for triple sequences in fuzzy normed spaces and theorems related to these concepts are also given.

Keywords and phrases: *Triple sequence, Statistical Convergence, Fuzzy Normed Space, Lacunary Convergence, I-convergence.*

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1. Introduction

The theory of fuzzy sets was introduced by Zadeh [34] in 1965. Subsequently several authors discussed various aspects of the theory and applications of fuzzy sets such as fuzzy topological spaces, similarity relations and fuzzy orderings, fuzzy measures of fuzzy events, fuzzy mathematical programming. In 2005 Saadati and Vaezpour [19] introduced Fuzzy normed spaces by means of definition that was closely modeled on the theory of (classical) normed spaces. Matloka [16] introduced the notion of convergence of sequence of fuzzy numbers. Nanda [18] studied the sequences of fuzzy numbers and Sencimen and Pehlivan [23] introduced the notions of a statistically convergent sequence and a statistically Cauchy sequence in a fuzzy normed linear space. The concepts of I-convergence, I*-convergence, and I Cauchy sequence were studied by Hazarika [9] in a fuzzy normed linear space.

A triple sequence (real or complex) is a function $x : \mathbf{N}\times\mathbf{N}\times\mathbf{N} \to \mathbf{R}(\mathbf{C})$, where $\mathbf{N}, \mathbf{R} \text{and} \mathbf{C}$ are the set of natural numbers, real numbers, and complex numbers respectively. We denote by ${}_3\omega$ the class of all complex triple sequence (x_{pqr}), where $p, q, r \in \mathbf{N}$. Then under the coordinate wise addition and scalar multiplication ${}_3\omega$ is a linear space. A triple sequence can be represented by a matrix in the form of a box in three dimensions, like in case of double sequences matrix is in the form of a square.

The different types of notions of triple sequences and their statistical convergence were introduced and investigated initially by Sahiner et. al [21]. Later Debnath et.al [1, 2], Esi et.al [3, 4, 5], and many authors have studied it further and obtained various results [25, 28, 31].

Statistical convergence was introduced by Fast [6] and later on it was studied by Fridy [7, 8] from the sequence space point of view and linked it with summability theory. The notion of statistical convergent double sequence was introduced by Mursaleen and Edely [17].

I-convergence is a generalization of the statistical convergence. Kostyrko et. al. [20] introduced the notion of I-convergence of real sequence and studied its several properties. Later Jalal [10, 11], Salat et. al. [20] and many other researchers contributed in its study. Sahiner and Tripathy [22] studied I-related properties in triple sequence spaces and showed some interesting results. Tripathy [24] extended the concept of I-convergent to double sequence and later Kumar [15] obtained some results on I-

convergent double sequence. Recently Jalal and Malik [12, 13] extended the concept of n-norms to triple sequence spaces and proved several algebraic and topological properties.

A sequence $x = (x_k)$ of fuzzy numbers is said to be convergent to a fuzzy number x_0 if there exists a positive integer k_0 such that $D(x_k, x_0) < \varepsilon$, for every $\varepsilon > 0$ and $k > k_0$. A sequence $x = (x_k)$ of fuzzy numbers convergent to level wise to x_0 if and only if $\lim_{k\to\infty} [x_k]_\alpha^- = [x_0]_\alpha^-$ and $\lim_{k\to\infty} [x_k]_\alpha^+ = [x_0]_\alpha^+$, where $[x_k]_\alpha = \left[(x_k)_\alpha^+, (x_k)_\alpha^+\right]$ and $[x_0]_\alpha = \left[(x_0)_\alpha^+, (x_0)_\alpha^+\right]$, for every $\alpha \in (0,1)$.

A sequence $x = (x_k)$ of fuzzy numbers is said to be statistically convergent to a fuzzy number x_0 if every $\varepsilon > 0$,

$$\lim_n \frac{1}{n} \left|\{k \le n : \bar{d}(x_k, x_0) \ge \varepsilon\}\right| = 0.$$

Let X be a vector space over $\mathbf{R}$, $\|\cdot\| : X \to L^*(\mathbf{R})$, and let the mappings $L, R : [0,1]\times[0,1] \to [0,1]$ be symmetric, non-decreasing in both arguments, and satisfy $L(0,0) = 0$ and $R(1,1) = 1$. The quadruple $(X, \|\cdot\|, L, R)$ is called a fuzzy normed linear space (briefly, $(X, \|\cdot\|)$ FNS) and $\|\cdot\|$ is a fuzzy norm if the following axioms are satisfied:

(1) $\|x\| = \tilde{0}$ if and only if $x = 0$,

(2) $\|rx\| = |r| \odot \|x\|$ for $x \in X, r \in \mathbf{R}$,

(3) For all $x, y \in X$,

a) $\|x+y\|(s+t) \ge L(\|x\|(s), \|y\|(t)),]$ whenever $s \le \|x\|_1^-, t \le \|y\|_1^-$ and $s + t \le \|x+y\|_1^-$.

b) $\|x+y\|(s+t) \le R(\|x\|(s), \|y\|(t)),]$ whenever $s \ge \|x\|_1^-, t \ge \|y\|_1^-$ and $s + t \ge \|x+y\|_1^-$.

Let $(X, \|\cdot\|)$ be a fuzzy normed linear space. A sequence $(x_n)_{n=1}^\infty$ in X is convergent to $L \in X$ with respect to the fuzzy norm on X and it is denoted by $x_n FN{\to}L$, provided $(D) - \lim_{n\to\infty} \|x_n - L\| = \tilde{0}$, i.e., for every $\varepsilon > 0$ there exists an $N(\varepsilon) \in \mathbf{N}$ such that $D(\|x_n L\|, \tilde{0}) < \varepsilon$, for all $n \ge N(\varepsilon)$. This means that for every $\varepsilon > 0$ there exists an integer $N(\varepsilon) \in \mathbf{N}$ such that

$$\sup_{\alpha\in[0,1]} \|x_n - L\|_\alpha^+ = \|x_n - L\|_0^+ < \varepsilon,$$

for all $n \geq N(\varepsilon)$.

From now on wards we write TFNS for "Triple Fuzzy Normed Space". Let $(X, \|\cdot\|)$ be an TFNS, if for every $\varepsilon > 0$ there exist a number $N = N(\varepsilon)$ such that

$$D\left(\|x_{pqr} - L, \tilde{0}\|\right) < \varepsilon$$

for all, $p, q, r \geq N$, then the triple sequence $x = (x_{pqr})$ is said to be convergent to $L \in X$ with respect to the fuzzy norm on X, and we denote it by $x_{pqr}[]TFNL$. Which means that for every $\varepsilon > 0$ there exists a number $N = N(\varepsilon)$ such that $\sup_{\alpha \in [0,1]} \|x_{pqr} - L\|_{\alpha}^{+} = \|x_{pqr} - L\|_{0}^{+} < \varepsilon$, for all $p, q, r \geq N$.

2. Definitions and Preliminaries

Let $K \subset \mathbf{N} \times \mathbf{N} \times \mathbf{N}$ and C_{ijk} be the number of $(p, q, r) \in K$ such that $p \leq i, q \leq j, r \leq k$. If the sequence $\left\{\dfrac{C_{ijk}}{i \cdot j \cdot k}\right\}$ has a limit in Pringsheim's sense then we say that the sub-sequence K has triple natural density denoted by

$$\delta_3(K) = \lim_{i,j,k \to \infty} \frac{C_{ijk}}{i \cdot j \cdot k}.$$

If for every $\varepsilon > 0$,

$$\delta_3\left(\left\{(i, j, k) \in \mathbf{N} \times \mathbf{N} \times \mathbf{N} : D\left(\|x_{ijk} - L, \tilde{0}\|\right) \geq \varepsilon\right\}\right) = 0,$$

then the triple sequence $x = (x_{ijk})$ is said to be statistically convergent to $L \in X$ with respect to the fuzzy norm on X.

This implies that $\|x_{ijk} - L\|_{0}^{+} < \varepsilon$ for almost all i, j, k and each $\varepsilon > 0$, and we denote it as $\mathrm{F}S_3 - \lim \|x_{ijk} - L\| = \tilde{0}$ or $x_{ijk}[]FS_3L$.

A class I of subsets of X is said to be ideal if the following conditions hold good

(i) $A, B \in I \Rightarrow A \cup B \in I$

(ii) $\in I, B \subset A \Rightarrow B \in I$

If $X \notin I$, I is a nontrivial ideal. If $\{x\} \in I$ for each $x \in X$, the nontrivial ideal I in X is called admissible.

If $X \neq \phi$, a filter in X is a class of subsets of X, having the following properties

(i) $A, B \in \Rightarrow A \cap B \in$
(ii) $A \in, B \subseteq A \Rightarrow B \in$

Let I be a nontrivial ideal in $X \neq \phi$. The class $(I) = \{M \subset X : (\exists A \in I)(M = X \setminus A)\}$ is a filter associated with I on X.

Definition 2.1. *[21] A triple sequence (x_{pqr}) is said to be convergent to L in Pringsheim's sense if for every $\varepsilon > 0$, there exists $\mathbf{N} \in \mathbf{N}$ such that*

$$|x_{pqr} - L| < \varepsilon \quad \textit{whenever} \quad p \geq \mathbf{N}, q \geq \mathbf{N}, r \geq \mathbf{N}$$

and write as $\lim_{p,q,r \to \infty} x_{pqr} = L$.

Note: A triple sequence is convergent in Pringsheim's sense may not be bounded [21].

Example Consider the sequence (x_{pqr}) defined by

$$x_{pqr} = \begin{cases} p + q & \text{for all } p = q \text{ and} r = 1 \\ \frac{1}{p^2 qr} & \text{otherwise.} \end{cases}$$

Then $x_{pqr} \to 0$ in Pringsheim's sense but is unbounded.

Definition 2.2. *A triple sequence (x_{pqr}) is said to be I-convergent to a number L if for every $\varepsilon > 0$,*

$$\{(p, q, r) \in \mathbf{N} \times \mathbf{N} \times \mathbf{N} : |x_{pqr} - L|) \geq \varepsilon\} \in I.$$

In this case we write $I - \lim x_{pqr} = L$.

Definition 2.3. *A triple sequence (x_{pqr}) is said to be I-null if $L = 0$. In this case we write* $I - \lim x_{pqr} = 0$.

Definition 2.4. *[21] A triple sequence (x_{pqr}) is said to be Cauchy sequence if for every $\varepsilon > 0$, there exists $\mathbf{N} \in \mathbf{N}$ such that*

$$|x_{pqr} - x_{lmn}| < \varepsilon \qquad \textit{whenever}\ p \geq l \geq \mathbf{N}, q \geq m \geq \mathbf{N}, r \geq n \geq \mathbf{N}.$$

Definition 2.5. *A triple sequence (x_{pqr}) is said to be $I-$Cauchy sequence if for every $\varepsilon > 0$, there exists $\mathbf{N} \in \mathbf{N}$ such that*

$$\{(p, q, r) \in \mathbf{N} \times \mathbf{N} \times \mathbf{N} : |x_{pqr} - a_{lmn}| \geq \varepsilon\} \in I$$

whenever $p \geq l \geq \mathbf{N}, q \geq m \geq \mathbf{N}, r \geq n \geq \mathbf{N}$.

Definition 2.6. *[21] A triple sequence* (x_{pqr}) *is said to be bounded if there exists* $M > 0$*, such that* $|x_{pqr}| < M$ *for all* $p, q, r \in \mathbf{N}$.

Definition 2.7. *A triple sequence* (x_{pqr}) *is said to be* $I-$*bounded if there exists* $M > 0$*, such that* $\{(p, q, r) \in \mathbf{N} \times \mathbf{N} \times \mathbf{N} : |x_{pqr}| \geq M\} \in I$ *for all* $p, q, r \in \mathbf{N}$.

Let $(X, \|\cdot\|)$ be a fuzzy normed space. A sequence $x = (x_n)_{n\in\mathbf{N}}$ in X is said to be I-convergent to $L \in X$ with respect to fuzzy norm on X if for each $\varepsilon > 0$, the set $A(\varepsilon) = \{n \in N : \|x_n L\|_0^+ \geq \varepsilon\}$ belongs to I. In this case,we write $x_n FI{\rightarrow}L$. The element L is called the I-limit of x in X.

A sequence (x_n) in X is said to be I^*-convergent to L in X with respect to the fuzzy norm on X if there exists a set $M \in F(I)$, $M = \{t_1 < t_2 < \cdots\} \subset \mathbf{N}$ such that $\lim_{k\to\infty} \|x_{t_k} L\| = 0$.

Let $(X, \|\cdot\|)$ be a fuzzy normed space. A sequence $x = (x_{ijk})$ in X is said to be I_3-convergent to $L \in X$ with respect to fuzzy norm on X, if for each $\varepsilon > 0$, the set $A(\varepsilon) = \{(i, j, k) \in \mathbf{N} \times \mathbf{N} \times \mathbf{N} : \|x_{ijk} L\|_0^+ \geq \varepsilon\}$ belongs to I_3. In this case, it is denoted by $x_{ijk} FI_3{\rightarrow}L$. The element L is called the I_3-limit of x in X.

The triple sequence $\theta_3 = \{(r_k, s_l, t_m)\}$ is called a triple lacunary sequence if there exist three increasing sequences of integers such that

$$r_0 = 0\ ,\ h_k = r_k r_{k1} \to \infty, s_0 = 0\ ,\ h_l = s_l s_{l1} \to \infty,$$

and $\quad t_0 = 0, h_m = t_m t_{m-1} \to \infty,$ as $k, l, m \to \infty$.

We use following notations in the sequel:

$$r_{jkl} = r_j s_k t_l \quad , \quad h_{jkl} = h_j h_k h_l,$$

$$C_{ijk} = \{(r, s, t) : r_{i-1} < r \leq r_i, s_{j-1} < s \leq s_j \text{ and } t_{k-1} < r \leq t_k\}.$$

Different properties of lacunary sequence spaces can be found in [26, 27, 29, 30, 32].

In this paper we introduce the concept of $F\theta_3-$convergence and $FI_{\theta_3}-$convergence. We gave the definition of $FI_{\theta_3}-$limit point and $FI_{\theta_3}-$cluster point.

Definition 2.8. *A sequence* $x = (x_{rst})_{(r,s,t)\in\mathbf{N}\times\mathbf{N}\times\mathbf{N}}$ *in* X *is said to be* $F\theta_3$*-convergent to* $L_1 \in X$ *with respect to fuzzy norm on* X *if for each*

$\varepsilon > 0$, there exists $n_0 \in \mathbf{N}$ such that

$$\frac{1}{h_{klm}} \sum_{(r,s,t)\in C_{klm}} D(\|x_{rst} - L_1\|, \tilde{0}) < \varepsilon$$

for all $k, l, m \geq n_0$. In this case, we write $x_{rst}F\theta_3{\to}L_1$ or $x_{rst} \to L1(F\theta_3)$, or $F\theta_3 - \lim_{r,s,t\to\infty} x_{rst} = L_1$. The element L_1 is called the $F\theta_3$-limit of x in X.

Definition 2.9. *A sequence $x = (x_{rst})$ in X is said to be lacunary FI_3 $-$convergent to $l_1 \in X$ with respect to fuzzy norm on X if for each $\varepsilon > 0$, the set*

$$\left\{(j,k,l) \in \mathbf{N}\times\mathbf{N}\times\mathbf{N} : \frac{1}{h_{jkl}} \sum_{(r,s,t)\in C_{jkl}} D\left(\|x_{rst} - l_1\|, \tilde{0}\right) \geq \varepsilon\right\}$$

belongs to I_3. In this case, we write $x_{rst}[]FI_{\theta_3}l_1$ or $x_{rst} \to l_1(FI_{\theta_3})$, or $FI_{\theta_3} - \lim_{r,s,t\to\infty} x_{rst} = l_1$. The number l_1 is called the $FI_{\theta_3}-$limit of (x_{rst}) in X.

3. Lacunary I_3-Convergence

Theorem 3.1. *If $x = (x_{rst})$ in X is $F\theta_3-$convergent, then $F\theta_3 - \lim x$ is unique.*

Proof. Let, $F\theta_3 - \lim x = l_1$ and $F\theta_3 - \lim x = l_2$. Then, for any $\varepsilon > 0$, there exists $n_1, n_2 \in \mathbf{N}$ such that

$$\frac{1}{h_{klm}} \sum_{(r,s,t)\in C_{klm}} \|x_{rst} - l_1\|_0^+ < \frac{\varepsilon}{2} \quad \forall \;\; k,l,m \geq n_1$$

and

$$\frac{1}{h_{klm}} \sum_{(r,s,t)\in C_{klm}} \|x_{rst} - l_2\|_0^+ < \frac{\varepsilon}{2} \quad \forall \;\; k,l,m \geq n_2.$$

Let $n_0 = \max\{n_1, n_2\}$, then for $k, l, m \geq n_0$, we have for $(a,b,c) \in \mathbf{N}\times\mathbf{N}\times\mathbf{N}$,

$$\|x_{abc} - l_1\|_0^+ < \frac{1}{h_{klm}} \sum_{(r,s,t)\in C_{klm}} \|x_{rst} - l_1\|_0^+ < \frac{\varepsilon}{2}$$

and

$$\|x_{abc} - l_2\|_0^+ < \frac{1}{h_{klm}} \sum_{(r,s,t)\in C_{klm}} \|x_{rst} - l_1\|_0^+ < \frac{\varepsilon}{2}.$$

Therefore,

$$\|l_1 - l_2\|_0^+ \leq \|x_{abc} - l_1\|_0^+ + \|x_{abc} - l_2\|_0^+ < \varepsilon.$$

Since, $\varepsilon > 0$ is arbitrary, we have $l_1 = l_2$. □

Lemma 3.2. *For every $\varepsilon > 0$, the following statements are equivalent*

(a) $FI_{\theta_3} - \lim_{r,s,t\to\infty} x_{rst} = l_1$,

(b) $\left\{(j,k,l) \in \mathbf{N}\times\mathbf{N}\times\mathbf{N} : \frac{1}{h_{jkl}} \sum_{(r,s,t)\in C_{jkl}} \|x_{rst} - l_1\|_0^+ \geq \varepsilon\right\} \in I_3$

(c) $\left\{(j,k,l) \in \mathbf{N}\times\mathbf{N}\times\mathbf{N} : \frac{1}{h_{jkl}} \sum_{(r,s,t)\in C_{jkl}} \|x_{rst} - l_1\|_0^+ < \varepsilon\right\} \in (I_3)$

(d) $FI_{\theta_3} - \lim_{r,s,t\to\infty} \|x_{rst} - l_1\|_0^+ = 0$.

Theorem 3.3. *If $x = (x_{rst})$ in X is lacunary I_3-convergent with respect to fuzzy norm on X, then $FI_{\theta_3} - \lim x$ is unique.*

Proof. The proof of this theorem is similar to proof of Theorem 3.1. □

Theorem 3.4. *Let $x = (x_{rst})$ and $y = (y_{rst})$ be the two triple triple sequences in X. Then*

(i) *if $FI_{\theta_3} - \lim x_{rst} = l_1$, and $FI_{\theta_3} - \lim y_{rst} = l_2$, then $FI_{\theta_3} - \lim(x_{rst} + y_{rst}) = l_1 + l_2$;*

(ii) *if $FI_{\theta_3} - \lim x_{rst} = l_1$, then $FI_{\theta_3} - \lim cx_{rst} = l_1$, for $c \in \mathbf{R} - \{0\}$.*

Proof. For any $\varepsilon > 0$, we define the following sets

$$A_1 = \left\{(j,k,l) \in \mathbf{N}\times\mathbf{N}\times\mathbf{N} : \frac{1}{h_{jkl}} \sum_{(r,s,t)\in C_{jkl}} \|x_{rst} - l_1\|_0^+ \geq \frac{\varepsilon}{2}\right\},$$

and

$$A_2 = \left\{(j,k,l) \in \mathbf{N}\times\mathbf{N}\times\mathbf{N} : \frac{1}{h_{jkl}} \sum_{(r,s,t)\in C_{jkl}} \|y_{rst} - l_2\|_0^+ \geq \frac{\varepsilon}{2}\right\}.$$

Since, $FI_{\theta_3} - \lim x_{rst} = l_1$, and $FI_{\theta_3} - \lim y_{rst} = l_2$, using Lemma 3.2, we have $A_1, A_2 \in I_3$ for all $\varepsilon > 0$.

Now, let $A_3 = A_1 \cup A_2$. Then, $A_3 \in I_3$, so the complement $(A_3)^c$ is a nonempty set in (I_3). We claim that

$$(A_3)^c \subset \left\{(j,k,l) \in \mathbf{N} \times \mathbf{N} \times \mathbf{N} : \frac{1}{h_{jkl}} \sum_{(r,s,t) \in C_{jkl}} \|(x_{rst} + y_{rst}) - (l_1 + l_2)\|_0^+ < \varepsilon\right\}.$$

Let $(j,k,l) \in (A_3)^c$, then we have

$$\frac{1}{h_{jkl}} \sum_{(r,s,t) \in C_{jkl}} \|x_{rst} - l_1\|_0^+ < \frac{\varepsilon}{2} \quad \text{and} \quad \frac{1}{h_{jkl}} \sum_{(r,s,t) \in C_{jkl}} \|y_{rst} - l_2\|_0^+ < \frac{\varepsilon}{2}.$$

Now, for $(a,b,c) \in \mathbf{N} \times \mathbf{N} \times \mathbf{N}$, we have

$$\|x_{abc} - l_1\|_0^+ < \frac{1}{h_{jkl}} \sum_{(r,s,t) \in C_{jkl}} \|x_{rst} - l_1\|_0^+ < \frac{\varepsilon}{2},$$

and

$$\|y_{abc} - l_2\|_0^+ < \frac{1}{h_{jkl}} \sum_{(r,s,t) \in C_{jkl}} \|y_{rst} - l_2\|_0^+ < \frac{\varepsilon}{2}.$$

Then, we have

$$\|(x_{abc} + y_{abc}) - (l_1 + l_2)\|_0^+ \leq \|x_{abc} - l_1\|_0^+ + \|y_{abc} - l_2\|_0^+ < \varepsilon.$$

Hence,

$$(A_3)^c \subset \left\{(j,k,l) \in \mathbf{N} \times \mathbf{N} \times \mathbf{N} : \frac{1}{h_{jkl}} \sum_{(r,s,t) \in C_{jkl}} \|(x_{rst} + y_{rst}) - (l_1 + l_2)\|_0^+ < \varepsilon\right\}.$$

Since, $(A_3)^c \in (I_3)$, we have

$$\left\{(j,k,l) \in \mathbf{N} \times \mathbf{N} \times \mathbf{N} : \frac{1}{h_{jkl}} \sum_{(r,s,t) \in C_{jkl}} \|(x_{rst} + y_{rst}) - (l_1 + l_2)\|_0^+ \geq \varepsilon\right\} \in I_3.$$

Therefore, $FI_{\theta_3} - \lim(x_{rst} + y_{rst}) = l_1 + l_2$. This proves part (i) of the theorem.

Again, let $FI_{\theta_3} - \lim x_{rst} = l_1$. Then, for $\varepsilon > 0$ and $c \in \mathbf{R} - \{0\}$, we define the following sets:

$$D = \left\{ (j,k,l) \in \mathbf{N} \times \mathbf{N} \times \mathbf{N} : \frac{1}{h_{jkl}} \sum_{(r,s,t) \in C_{jkl}} \|x_{rst} - l_1\|_0^+ < \frac{\varepsilon}{|c|} \right\}.$$

So, $D \in (I_3)$. Let $(j,k,l) \in D$, then we have

$\frac{1}{h_{jkl}} \sum_{(r,s,t) \in C_{jkl}} \|x_{rst} - l_1\|_0^+ < \frac{\varepsilon}{|c|}$

$\Rightarrow \frac{|c|}{h_{jkl}} \sum_{(r,s,t) \in C_{jkl}} \|x_{rst} - l_1\|_0^+ < |c| \times \frac{\varepsilon}{|c|}$

$\Rightarrow \frac{1}{h_{jkl}} \sum_{(r,s,t) \in C_{jkl}} |c| \|x_{rst} - l_1\|_0^+ < \varepsilon$

$\Rightarrow \frac{1}{h_{jkl}} \sum_{(r,s,t) \in C_{jkl}} \|cx_{rst} - cl_1\|_0^+ < \varepsilon.$

Therefore,

$$D \subset \left\{ (j,k,l) \in \mathbf{N} \times \mathbf{N} \times \mathbf{N} : \frac{1}{h_{jkl}} \sum_{(r,s,t) \in C_{jkl}} \|cx_{rst} - cl_1\|_0^+ < \varepsilon \right\}$$

and

$$\left\{ (j,k,l) \in \mathbf{N} \times \mathbf{N} \times \mathbf{N} : \frac{1}{h_{jkl}} \sum_{(r,s,t) \in C_{jkl}} \|cx_{rst} - cl_1\|_0^+ < \varepsilon \right\} \in (I_3).$$

Hence, $FI_{\theta_3} - \lim c\, x_{rst} = cl_1$. □

Theorem 3.5. *Let $x = (x_{rst})$ be a triple sequence in X. If $F\theta_3 - \lim x = l_1$, then $FI_{\theta_3} - \lim x = l_1$.*

Proof. Assuming, $F\theta_3 - \lim x = l_1$, we have for every $\varepsilon > 0$, there exists $n_0 \in \mathbf{N}$ such that

$$\frac{1}{h_{jkl}} \sum_{(r,s,t) \in C_{jkl}} \|x_{rst} - l_1\|_0^+ < \varepsilon$$

for all $j,k,l \geq n_0$. Therefore, the set

$$=$$

$$\left\{ (j,k,l) \in \mathbf{N} \times \mathbf{N} \times \mathbf{N} : \tfrac{1}{h_{jkl}} \textstyle\sum_{(r,s,t) \in C_{jkl}} \|x_{rst} - l_1\|_0^+ \geq \varepsilon \right\}$$

$$\subset (\mathbf{N} \times \mathbf{N} \times \{1,2,\ldots,(n_0 - 1)\}) \cup (\{1,2,\ldots,(n_0 - 1)\} \times \mathbf{N} \times \mathbf{N}).$$

So, we have $E \in I_3$. Hence, $FI_{\theta_3} - \lim x = l_1$. □

Theorem 3.6. *Let $x = (x_{rst})$ be a triple sequence in X. If $F\theta_3 - \lim x = l_1$, then there exists a subsequence $(x_{r_i s_j t_k})$ such that $x_{r_i s_j t_k} [] TFN l_1$*

Proof. Let $F\theta_3 - \lim x = l_1$. Then, for every $\varepsilon > 0$, there exists $n_0 \in \mathbf{N}$ such that

$$\frac{1}{h_{abc}} \sum_{(r,s,t)\in C_{abc}} \|x_{rst} - l_1\|_0^+ < \varepsilon$$

for all $a, b, c \geq n_0$. Clearly, for each $a, b, c \geq n_0$, we select a $(r_i, s_j, t_k) \in C_{abc}$ such that

$$\|x_{r_i s_j t_k} - l_1\|_0^+ < \frac{1}{h_{abc}} \sum_{(r,s,t)\in C_{abc}} \|x_{rst} - l_1\|_0^+ < \varepsilon$$

It follows that $x_{r_i s_j t_k} [] TFN l_1$. □

4. Limit point and cluster point

In this section, we introduce the notions of $FI_{\theta_3}-$limit point and $FI_{\theta_3}-$cluster point for triple sequences in a fuzzy normed space. Also, we examine the relations between $FI_{\theta_3}-$limit point and $FI_{\theta_3}-$cluster point of triple sequences in fuzzy normed space.

Definition 4.1. *An element $L \in X$ is said to be an $FI_{\theta_3}-$limit point of $x = (x_{rst})$ if there is a set $M_1 = \{r_1 < r_2 < \cdots < r_a < \cdots\} \subset \mathbf{N}$, $M_2 = \{s_1 < s_2 < \cdots < s_b < \cdots\} \subset \mathbf{N}$ and $M_3 = \{t_1 < t_2 < \cdots < t_c < \cdots\} \subset \mathbf{N}$ such that the set $M' = \{(j,k,l) \in \mathbf{N}\times\mathbf{N}\times\mathbf{N} : (r_a, s_b, t_c) \in C_{jkl}\} \notin I_3$, and $F\theta_3 - \lim x_{r_a s_b t_c} = l$.*
We denote the set of all $FI_{\theta_3}-$limit points of x as $\Lambda_F^{I_{\theta_3}}(x)$.

Definition 4.2. *An element $L \in X$ is said to be an $FI_{\theta_3}-$cluster point of $x = (x_{rst})$ if for every $\varepsilon > 0$, we have*

$$\left\{(j,k,l) \in \mathbf{N}\times\mathbf{N}\times\mathbf{N} : \frac{1}{h_{jkl}} \sum_{(r,s,t)\in C_{jkl}} \|x_{rst} - l\|_0^+ < \varepsilon\right\} \notin I_3.$$

We denote the set of all $FI_{\theta_3}-$cluster point of x as $\Gamma_F^{I_{\theta_3}}(x)$.

Theorem 4.3. *For each $x = (x_{rst})$ in X, we have $\Lambda_F^{I_{\theta_3}}(x) \subset \Gamma_F^{I_{\theta_3}}(x)$.*

Proof. Let $l \in \Lambda_F^{I_{\theta_3}}(x)$, then there exists three sets $M_1, M_2, M_3 \subset \mathbf{N}$ such that $M' \notin I_3$, where M_1, M_2, M_3 and M' are as in Definition 4.1, and also $F\theta_3 - \lim x_{r_a s_b t_c} = l$. Thus, for every $\varepsilon > 0$ there exists $n_0 \in \mathbf{N}$ such that

$$\frac{1}{h_{jkl}} \sum_{(r_a, s_b, t_c) \in C_{jkl}} \|x_{r_a s_b t_c} - l\|_0^+ < \varepsilon,$$

for all $j, k, l \geq n_0$. Then we get

$$\left\{(j,k,l) \in \mathbf{N} \times \mathbf{N} \times \mathbf{N} : \tfrac{1}{h_{jkl}} \textstyle\sum_{(r,s,t) \in C_{jkl}} \|x_{rst} - l\|_0^+ < \varepsilon\right\}$$

$$\supseteq M' - \{\{r_1, r_2, \ldots, r_{n_0}\} \times \{s_1, s_2, \ldots, s_{n_0}\} \times \{t_1, t_2, \ldots, t_{n_0}\}\}.$$

Therefore, we have
$M' - \{\{r_1, r_2, \ldots, r_{n_0}\} \times \{s_1, s_2, \ldots, s_{n_0}\} \times \{t_1, t_2, \ldots, t_{n_0}\}\} \notin I_3$
and as such $D \notin I_3$. Consequently, $l \in \Gamma_F^{I_{\theta_3}}(x)$. □

Theorem 4.4. *For every triple sequence $x = (x_{rst})$, the following statements are equivalent:*

(i) l is an FI_{θ_3}−limit point of x,

(ii) there exists two triple sequences $y = (y_{rst})$ and $z = (z_{rst})$ in X such that $x = y + z$, $F\theta_3 - \lim y = l$ and $\{(j,k,l) \in \mathbf{N} \times \mathbf{N} \times \mathbf{N} : (r,s,t) \in C_{jkl}, z_{rst} \neq 0\} \in I_3$.

Proof. Suppose (i) hold, then there exists sets M_1, M_2, M_3 and M' as in Definition 4.1 such that $M' \notin I_3$ and $F\theta_3 - \lim x_{r_i s_j t_k} = l$. Define the sequences y and z as follows:

$$y_{rst} = \begin{cases} x_{rst}, & \text{if } (r,s,t) \in C_{jkl}, (j,k,l) \in M', \\ l, & \text{otherwise} \end{cases}$$

and

$$z_{rst} = \begin{cases} 0, & \text{if } (r,s,t) \in C_{jkl}, (j,k,l) \in M', \\ x_{rst} - l, & \text{otherwise.} \end{cases}$$

It suffices to consider the case $(r,s,t) \in C_{jkl}$, such that $(j,k,l) \in \mathbf{N} \times \mathbf{N} \times \mathbf{N} \setminus M'$. For each $\varepsilon > 0$, we have $\|y_{rst} - l\|_0^+ = 0 < \varepsilon$. Thus,

$$\frac{1}{h_{jkl}} \sum_{(r,s,t) \in C_{jkl}} \|y_{rst} - l\|_0^+ = 0 < \varepsilon.$$

Therefore, $F\theta_3 - \lim y = l$. Now

$$\{(j,k,l) \in \mathbf{N}\times\mathbf{N}\times\mathbf{N} : (r,s,t) \in C_{jkl}, z_{rst} \neq 0\} \subset \mathbf{N}\times\mathbf{N}\times\mathbf{N} \setminus M'.$$

But, $\mathbf{N}\times\mathbf{N}\times\mathbf{N} \setminus M' \in I_3$, and so

$$\{(j,k,l) \in \mathbf{N}\times\mathbf{N}\times\mathbf{N} : (r,s,t) \in C_{jkl}, z_{rst} \neq 0\} \in I_3.$$

Now, suppose that (ii) holds. Let $M' = \{(j,k,l) \in \mathbf{N}\times\mathbf{N}\times\mathbf{N} : (r,s,t) \in C_{jkl}, z_{rst} = 0\}$. Then, clearly $M' \in F(I_3)$ and so it is an infinite set. Construct the sets $M_1 = \{r_1 < r_2 < \cdots < r_a < \cdots\} \subset \mathbf{N}$, $M_2 = \{s_1 < s_2 < \cdots < s_b < \cdots\} \subset \mathbf{N}$ and $M_3 = \{t_1 < t_2 < \cdots < t_c < \cdots\} \subset \mathbf{N}$ such that $(r_i, s_j, t_k) \in C_{jkl}$ and $z_{r_i s_j t_k} = 0$. Since, $x_{r_i s_j t_k} = y_{r_i s_j t_k}$ and $F\theta_3 - \lim y = l$,
We obtain $F\theta_3 - \lim x_{r_i s_j t_k} = l$.
This completes the proof. □

Theorem 4.5. *If there is an $FI_{\theta_3}-$convergent sequence $y = (y_{rst}) \in X$ such that $\{(r,s,t) \in \mathbf{N}\times\mathbf{N}\times\mathbf{N} : y_{rst} \neq x_{rst}\} \in I_3$, then $x = (x_{rst})$ is also $FI_{\theta_3}-$convergent.*

Proof. Suppose that $\{(r,s,t) \in \mathbf{N}\times\mathbf{N}\times\mathbf{N} : y_{rst} \neq x_{rst}\} \in I_3$ and $FI_{\theta_3}-\lim y = l$. Then, for every $\varepsilon > 0$, the set

$$\left\{(j,k,l) \in \mathbf{N}\times\mathbf{N}\times\mathbf{N} : \frac{1}{h_{jkl}} \sum_{(r,s,t)\in C_{jkl}} \|y_{rst} - l\|_0^+ \geq \varepsilon\right\} \in I_3.$$

For every $\varepsilon > 0$, we get

$$\left\{(j,k,l) \in \mathbf{N}\times\mathbf{N}\times\mathbf{N} : \tfrac{1}{h_{jkl}} \textstyle\sum_{(r,s,t)\in C_{jkl}} \|x_{rst} - l\|_0^+ \geq \varepsilon\right\}$$
$$\subseteq$$
$$\{(j,k,l) \in \mathbf{N}\times\mathbf{N}\times\mathbf{N} : y_{rst} \neq x_{rst}\} \ \cup$$

$$\left\{(j,k,l) \in \mathbf{N}\times\mathbf{N}\times\mathbf{N} : \tfrac{1}{h_{jkl}} \textstyle\sum_{(r,s,t)\in C_{jkl}} \|y_{rst} - l\|_0^+ \geq \varepsilon\right\}.$$

Therefore, we have

$$\left\{(j,k,l) \in \mathbf{N}\times\mathbf{N}\times\mathbf{N} : \frac{1}{h_{jkl}} \sum_{(r,s,t)\in C_{jkl}} \|x_{rst} - l\|_0^+ \geq \varepsilon\right\} \in I_3.$$

This completes the proof of the theorem. □

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Tanweer Jalal
Department of Mathematics
National Institute of Technology
Hazratbal, Srinagar-190006
India
e-mail: tjalal@nitsri.net
Corresponding author

and

Ishfaq Ahmad Malik
Department of Mathematics
National Institute of Technology
Hazratbal, Srinagar-190006
India
e-mail: ishfaq_2phd15@nitsri.net