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Quasi-k-normal ring

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Abstract

In [4] Wei and Libin defined Quasi normal ring. In this paper we attempt to define Quasi-k-normal ring by using the action of k-potent element .A ring is called Quasi-k-normal ring if $ae = 0 \Rightarrow eaRe = 0$ *for* $a \in N(R)$*and* $e \in K(R)$*, where* $K(R) = \{e \in R | e^k = e\}$*. Several analogous results give in[4] is defined here. we find here that a ring is quasi-k-normal if and only if* $eR(1 - e^{k-1})Re = 0$ *for each* $e \in K(R)$*. Also we get a ring is quasi-k-normal ring if and only if* $T_n(R, R)$ *is quasi-k-normal ring.*

Key Words: *Abelian rings; Quasi-k-normal rings;* Π*-Regular rings*

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1. Introduction

In [3] Parmenter and Stewart defined the notion of normal ring as for $x, y \in R$ and $xRy = 0$ then $ann(x) + ann(y) = R$ where $ann(x) = \{z \in R : zRx = 0\}$. In [4] Wei and Libin generalized normal rings and defined quasi normal ring on the basis of normal ring. Here we define quasi-k-normal ring in terms of k-potent elements as defined in [2]. Here we study several properties of quasi-k-normal rings.

2. Preliminaries

All rings considered here are associative with identity. We use the symbols $K(R), E(R), N(R)$ for set of all k-potents, idempotents and nilpotents respectively. Again for any subset X of a ring R, we denote $r(X) = r_R(X)$ and $l(X) = l_R(X)$ for right annihilator and left annihilator of X respectively. Again $E_{ij}(1i, jn)$ denote $n \times n$ matrix units over R, and write $V = \sum_{i=1}^{n-1} E_{i,i+1}$ for $n2$. Also we use the symbol $T_n(R, R)$ to denote the ring of $n \times n$ upper triangular matrices whose principal diagonal elements are equals and others belong to R and $V_n(R, R) = RE_n + RV + RV^2 + ... + RV^{n-1}$ for $n2$, where E_n is $n \times n$ identity matrix over E.

A ring is called quasi-k-normal if $ae = o \Rightarrow eaRe = 0$ for $e \in K(R)$ and $a \in N(R)$.Again a ring is semiabelian if every idempotents of R is either left semicentral or right semicentral. A ring is called abelian [1] if every idempotent of R is central. Semiabelian rings are quasi-k-normal. An example of quasi-k-normal ring is reversible ring R ($ab = 0 \Rightarrow ba = 0$ for $a, b \in R$).

3. Results and Discussion

We start with the foloowing results.

Theorem 2.1. The following conditions are equivalent for a ring R :

1. R is quasi-k-normal.
2. $eR(1 - e^{k-1})Re = 0; \forall e \in K(R)$.
3. $eR(1 - e^{k-1})$ is right ideal of R; $\forall e \in K(R)$.
4. $(1 - e^{k-1})Re$ is left ideal of R $\forall e \in K(R)$.

5. $[(1-e^{k-1})R \cap N(R)]e \subseteq r(eR) \cap N(R)e \ \forall e \in K(R)$.
6. $ea = 0 \Rightarrow eRae = 0 \ \forall a \in N(R)$ and $e \in K(R)$.

Proof:

$(1) \Rightarrow (2)$: For any $a \in R, e \in K(R)$, let $h = e^{k-1}a - e^{k-1}ae^{k-1} = e^{k-1}a(1-e^{k-1})$. So $h^2 = e^{k-1}a(e^{k-1} - e^{2k-2})a(1-e^{k-1}) = 0 \Rightarrow h \in N(R)$ and $he = e^{k-1}a(e - e^k) = 0$. Now $h \in N(R), he = 0$. So by definition of quasi-k-normal ring, $ehRe = 0 \Rightarrow e.e^{k-1}a(1-e^{k-1})Re = 0 \Rightarrow ea(1-e^{k-1})Re = 0 \Rightarrow eR(1-e^{k-1})Re = 0$ as $a \in R$ is arbitrary.

$(6) \Rightarrow (2)$: Let $h = ae^{k-1} - e^{k-1}ae^{k-1} = (1-e^{k-1})ae^{k-1}$;$e \in K(R)$.Then $eh = 0, h^2 = 0$. So $eh = 0, h \in N(R) \Rightarrow eRhe = 0 \Rightarrow eR(1-e^{k-1})ae^k = 0 \Rightarrow eR(1-e^{k-1})Re = 0$ as $a \in R$ is arbitrary.

$(2) \Rightarrow (3)$: Let $e \in K(R)$. By (2), $eR(1-e^{k-1})Re = 0 \Rightarrow eR(1-e^{k-1})Re^{k-1} = 0$
Now $eR(1-e^{k-1})R = eR(1-e^{k-1})R(1-e^{k-1}) \subseteq eR(1-e^{k-1})$ $[R(1-e^{k-1})R \subseteq R]$
Again $eR(1-e^{k-1}) = eR(1-e^{k-1}).1 \subseteq eR(1-e^{k-1})R$
Therefore $eR(1-e^{k-1}) = eR(1-e^{k-1})R$. So $eR(1-e^{k-1})$ is left ideal of R.

$(3) \Rightarrow (4)$: Following (3),we have $eR(1-e^{k-1}) = eR(1-e^{k-1})R \Rightarrow eR(e-e^k) = eR(1-e^{k-1})Re \Rightarrow eR(1-e^{k-1})Re = 0 \Rightarrow e^{k-1}R(1-e^{k-1})Re = 0$.
Now, $R(1-e^{k-1})Re = (1-e^{k-1})R(1-e^{k-1})Re \subseteq (1-e^{k-1})Re$ $[R(1-e^{k-1})R \subseteq R]$. Therefore $eR(1-e^{k-1})Re \subseteq (1-e^{k-1})Re$.
Again, $(1-e^{k-1})Re = 1.(1-e^{k-1})Re \subseteq R(1-e^{k-1})Re$.
Therefore $(1-e^{k-1})Re = R(1-e^{k-1})Re$ which implies $(1-e^{k-1})Re$, is right left ideal of R.

$(4) \Rightarrow (5)$: Let $x \in (1-e^{k-1})R \cap N(R)$
Therefore $x = (1-e^{k-1})b$ for some $b \in R$ and $xe \in N(R)e$. So $ex = (e-e^k)b = 0 \Rightarrow e^{k-1}x = 0$. So $x = x - e^{k-1} = (1-e^{k-1})x$. Again $(1-e^{k-1})Re$ is left ideal of R. So $eR(1-e^{k-1})Re \subseteq (1-e^{k-1})Re$. Now $eRxe = eR(1-e^{k-1})xe \subseteq eR(1-e^{k-1})Re \subseteq (1-e^{k-1})Re$.Therefore $e^kRxe \subseteq (e^{k-1} - e^{2k-2})Re \Rightarrow eRxe \subseteq (e^{k-1} - e^{k-1})Re = 0 \Rightarrow eRxe = 0 \Rightarrow xe \in r(eR)$. Hence $xe \in r(eR) \cap N(R)e$ and finally we get $[(1-e^{k-1})R \cap N(R)]e \subseteq r(eR) \cap N(R)e$.

$(5) \Rightarrow (6)$: Let $ea = 0 \Rightarrow e^{k-1}a = 0$ for some $a \in N(R), e \in K(R)$. Now $ae = ae - e^{k-1}ae = (1-e^{k-1})ae \in [(1-e^{k-1})R \cap N(R)]e \subseteq r(eR) \cap N(R)e \Rightarrow ae \in r(eR) \cap N(R)e \Rightarrow ae \in r(eR) \Rightarrow eRae = 0$.

(2) $\Rightarrow$ (1) : Let $ae = 0$ for some $a \in N(R), e \in K(R)$. So by hypothesis $eR(1 - e^{k-1})Re = 0$. Now $ae^{k-1} = 0$, which implies $a = a - ae^{k-1} = a(1 - e^{k-1})$. Therefore $eaRe = ea(1 - e^{k-1})Re \subseteq eR(1 - e^{k-1})Re$. Hence $eaRe = 0$.

Corollary 2.2

1. The following conditions are equivalent for a ring R
   (i) R is quasi-k-normal.
   (ii) For any $e \in K(R); x, y \in R$,$exye = exe^{k-1}ye$.
2. Semiabelian rings are quasi-k-normal.
3. Let R be quasi-k-normal ring. If $e \in E(R)$ with $ReR = R$ then $e^{k-1} = 1$.

Proof.

1. R is quasi-k-normal $\Leftrightarrow eR(1-e^{k-1})Re = 0 \Rightarrow exye = exe^{k-1}ye; x, y \in R$. For $k = 2$ we get the following beautiful result by Junchao [4] $(ex)^n e = ex^n e = e(xe)^n$.
2. As R is semiabelian so for $e_1 \in E(R)$, we have $e_1 r = e_1 r e_1$ or $re_1 = e_1 r e_1 \ \forall r \in R$. Let $e \in K(R) \Rightarrow e^k = e \Rightarrow (e^{k-1})^2 = e^{k-1} \Rightarrow e^{k-1} \in E(R)$. Now $eR(1-e^{k-1})Re = e.e^{k-1}R(1-e^{k-1})Re \subseteq e.e^{k-1}Re^{k-1}(1-e^{k-1})Re = 0$ or $eR(1-e^{k-1})Re = eR(1-e^{k-1})Re^{k-1}.e \subseteq eR(1 - e^{k-1})e^{k-1}Re^{k-1} = 0$. So $eR(1 - e^{k-1})Re = 0$ in both cases implies R is quasi-k-normal.
3. As R is quasi-k-normal ring so by **Theorem 2.1**,$eR(1-e^{k-1})Re = 0$. As $ReR = R$ so $R(1-e^{k-1})R = ReR(1-e^{k-1})ReR = 0$, which implies $1(1 - e^{k-1})1 = 0 \Rightarrow e^{k-1} = 0$.

A ring is called directly finite if $xy = 1 \Rightarrow yx = 1 \forall x, y \in R$. A ring is called left min abel if for every $e \in ME_1(R) = \{e \in E(R)$:$Re$ is minimal left ideal of R $\}$, e is left semicentral in R.

Theorem 2.4 Quasi-k-normal rings are directly finite.

Proof. Let $ab = 1$ where $a, b \in R$. Let $e = ba$. Now $e^2 = baba = ba = e \Rightarrow e^k = e$ so $e \in K(R)$ as idempotents are always k-potent; $ae = aba = a, eb = bab = b$. As R is quasi-k-normal so $eR(1 - e^{k-1})Re = 0 \Rightarrow eb(1 - e)ae = 0 \Rightarrow b(1 - e)a = 0 \Rightarrow ba = bea = b^2a^2 \Rightarrow bab = b^2a^2b \Rightarrow b = b^2a \Rightarrow ab = ab^2a \Rightarrow ab = ba \Rightarrow ba = 1$. Therefore R is directly finite.

Theorem 2.5 Quasi-k-normal ring are left min-abel.

Proof. Let $e \in ME_1(R)$ and $a \in R$. As e is idempotent it is clearly k-potent and in that case quasi-k-normal ring takes the form of quasi normal ring which is left min abel due to Junchao[4]. Hence the result.

A ring R is called left idempotent reflexive if $aRe = 0$ implies $eRa = 0$ for all $a \in R$ and $e \in K(R)$.

Theorem 2.6: The following conditions are equivalent for a ring R

1. R is abelian.
2. R is semiabelian and left idempotent reflexive.
3. R is quasi-k-normal and left idempotent reflexive.

Proof: By **Corollary 2.2**, $(1) \Rightarrow (2) \Rightarrow (3)$
$(3) \Rightarrow (1)$: As for idempotents, quasi-k-normal ring takes the form of quasi normal ring. So we have the result directly from Junchao[4].

Theorem 2.7: A ring R is quasi-k-normal ring if and only if $T_n(R, R)$ is quasi-k-normal ring.

Proof. Suppose R is quasi-k-normal. We show that $S = T_n(R, R)$ is quasi-k-normal by inducting on n. For $n = 1$ it is trivial. Suppose $S_1 = T_{n-1}(R, R)$ is quasi-k-normal for any $n2$, that is for any

$$E_1 = \begin{pmatrix} (e & e_{12} & e_{13} & \cdots & e_{1n-1} \\ 0 & e & e_{23} & \cdots & e_{2n-1} \\ 0 & 0 & e & \cdots & e_{3n-1} \\ .. & .. & .. & .. & .. \\ 0 & 0 & 0 & \cdots & e \end{pmatrix} \in K(S_1),$$

Thus we have

$$E_1 S_1 (1 - {E_1}^{k-1}) S_1 E_1 = 0. \tag{3.1}$$

$$E_1 S_1 (1 - {E_1}^{k-1}) R^{n-1} e = 0. \tag{3.2}$$

$$E_1 R^{n-1} (1 - e^{k-1}) Re = 0. \tag{3.3}$$

$$eR(1 - e^{k-1}) Re = 0. \tag{3.4}$$

Now for n, let $E = \begin{matrix} E_1 & \alpha \\ 0 & e \end{matrix} \in K(S)$ where $\alpha \in \mathbf{R}^{n-1}$ and $E_1 \in S_1$, $e \in K(R)$ and

$$E^k = E.$$

$$\Rightarrow \begin{pmatrix} {E_1}^k & {E_1}^{k-1}\alpha + {E_1}^{k-2}\alpha e + ... + E_1 \alpha e^{k-2} + \alpha e^{k-1} \\ 0 & e^k \end{pmatrix} = \begin{pmatrix} E_1 & \alpha \\ 0 & e \end{pmatrix}$$

$$\Rightarrow {E_1}^{k-1}\alpha + {E_1}^{k-2}\alpha e + ... + E_1 \alpha e^{k-2} + \alpha e^{k-1} = \alpha \tag{3.5}$$

Now suppose that $A = \begin{pmatrix} A_1 & \xi_1 \\ 0 & a \end{pmatrix}$, $B = \begin{pmatrix} B_1 & \xi_2 \\ 0 & b \end{pmatrix} \in S$, where $A_1, B_1 \in S_1; \xi_1, \xi_2 \in \mathbf{R}^{n-1}; a, b \in R$.

Now

$$EA(1 - E^{k-1})BE$$

$$= \begin{pmatrix} A_1 & \xi_1 \\ 0 & a \end{pmatrix} \begin{pmatrix} 1 - {E_1}^{k-1} & -y \\ 0 & 1 - e^{k-1} \end{pmatrix} \begin{pmatrix} B_1 & \xi_2 \\ 0 & b \end{pmatrix} \begin{pmatrix} E_1 & \alpha \\ 0 & e \end{pmatrix}$$

$$= \begin{pmatrix} E_1A_1(1-{E_1}^{k-1})B_1E_1 & x \\ 0 & ea(1-e^{k-1})be \end{pmatrix}$$

Where

$$\begin{aligned} x &= E_1A_1(1-{E_1}^{k-1})B_1\alpha + E_1A_1(1-{E_1}^{k-1})\xi_2 e - E_1A_1ybe \\ &\quad + E_1\xi_1(1-e^{k-1})be + \alpha a(1-e^{k-1})be \end{aligned} \tag{3.6}$$

and

$$y = {E_1}^{k-2}\alpha + {E_1}^{k-3}\alpha e + ... + E_1\alpha e^{k-3} + \alpha e^{k-2} \tag{3.7}$$

.

Now,$E_1A_1(1-{E_1}^{k-1})B_1E_1 \in E_1S_1(1-{E_1}^{k-1})S_1E_1$. So by(1),we have,

$$E_1A_1(1-{E_1}^{k-1})B_1E_1 = 0 \tag{3.8}$$

.

Again $ea(1-e^{k-1})be \in eR(1-e^{k-1})Re$, So by (4) we have

$$ea(1-e^{k-1})be = 0. \tag{3.9}$$

.

Now for x, by using(5)

$$E_1A_1(1-{E_1}^{k-1})B_1\alpha = E_1A_1(1-{E_1}^{k-1})B_1({E_1}^{k-1}\alpha + {E_1}^{k-2}\alpha e + ... + E_1\alpha e^{k-2} + \alpha e^{k-1})$$

$$= E_1A_1(1-{E_1}^{k-1})B_1{E_1}^{k-1}\alpha + E_1A_1(1-{E_1}^{k-1})B_1{E_1}^{k-2}\alpha e + ... + E_1A_1(1-{E_1}^{k-1})B_1\alpha e^{k-1}$$

$$= 0 \tag{3.10}$$

[Using (6) and $B_1\alpha \in \mathbf{R}^{n-1}$ so $E_1A_1(1-{E_1}^{k-1})B_1\alpha e \in E_1S_1(1-{E_1}^{k-1})R^{n-1}e = 0$ using(2)]

Again $E_1A_1(1-{E_1}^{k-1})\xi_2 e \in E_1S_1(1-{E_1}^{k-1})R^{n-1}e$. So by using(2) we have

$$E_1A_1(1-{E_1}^{k-1})\xi_2 e = 0. \tag{3.11}$$

.

Again $E_1\xi_1(1-e^{k-1})be \in E_1R^{n-1}(1-e^{k-1})Re$. So by using (3)we have

$$E_1\xi_1(1-e^{k-1})be = 0. \tag{3.12}$$

Again $\alpha a(1-e^{k-1})be \in eR(1-e^{k-1})Re$. So by using(4)we have

$$\alpha a(1-e^{k-1})be = 0. \tag{3.13}$$

Using(10),(11),(12,(13)we get

$$x = -E_1A_1ybe$$

So we have to show $E_1A_1ybe = 0$.

Using(5) ${E_1}^{k-1}\alpha + {E_1}^{k-2}\alpha e + ... + E_1\alpha e^{k-2} + \alpha e^{k-1} = \alpha$

$\Rightarrow E_1({E_1}^{k-2}\alpha + {E_1}^{k-3}\alpha e + {E_1}^{k-4}\alpha e^2 + ... + E_1\alpha e^{k-2}) = \alpha(1 - e^{k-1})$

(3.14) $$\Rightarrow E_1y = \alpha(1 - e^{k-1})$$

Similarly,$({E_1}^{k-2}\alpha+{E_1}^{k-3}\alpha e+{E_1}^{k-4}\alpha e^2+...+E_1\alpha e^{k-2})e = (1-{E_1}^{k-1})\alpha$

(3.15) $$\Rightarrow ye = (1 - {E_1}^{k-1})\alpha$$

Now $b \in R; y \in \mathbf{R}^{n-1}; A_1 \in M_{(n-1)\times(n-1)}$, so $A_1y \in \mathbf{R}^{n-1}$. So by using(3)we get

$E_1A_1y(1 - e^{k-1})be = 0$

$\Rightarrow E_1A_1ybe$

$= E_1A_1ye^{k-1}be$

$= E_1A_1ye.e^{k-2}be$ $[k \geq 2]$

$= E_1A_1(1 - {E_1}^{k-1})\alpha e^{k-2}be$ [Using(15)]

$\subseteq E_1S_1(1 - {E_1}^{k-1})R^{n-1}e$ $[A_1 \in S_1; \alpha e^{k-2}b \in \mathbf{R}^{n-1}]$

Using(3) we get $E_1A_1ybe = 0$

Therefore $x = 0$

Hence $EA(1 - E^{k-1})BE = 0$.

So $T_n(R, R)$ is quasi-k-normal. Converse part is quite obvious.

By using the above theorem we can the get the following corollaries.

Corollary 2.8: A ring R is quasi-k-normal if and only if $V_n(R, R)$ is quasi-k-normal for $n \geq 2$.

Since there is a ring isomorphism $\theta : V_n(R, R) = RE_n + RV + RV^2 + \ldots. + RV^{n-1} \to R[x](x^n)$ defined by $\theta(r_0E_n + r_1V + r_2V^2 + \ldots. + r_{n-1}V^{n-1}) = r_0 + r_1x + r_2x^2 + \ldots. + r_{n-1}x^{n-1} + (x^n)$, So using **Corollary 2.8** we get following.

Corollary 2.9: A ring R is quasi-k-normal if and only if $R[x](x^n)$ is quasi-k-normal for $n \geq 2$.

Theorem 2.10 If R is subdirect product of family of a quasi-k-normal rings $\{R_i : i \in I\}$ then R is quasi-k-normal.

Proof. Let $R_i = RA_i$ where A_i are ideals such that $\cap_{i \in I} A_i = 0$.Let $e \in K(R)$. Then $e_i = e + A_i \in K(R_i)$ for $i \in I$. As each R_i is quasi-k-normal, $e_i R_i (1 - {e_i}^{k-1}) R_i e_i = 0$ for $i \in I$ which implies $eR(1 - e^{k-1})Re \subseteq A_i$ $\forall i \in I$ implies $eR(1 - e^{k-1})Re \subseteq \cap_{i \in I} A_i$. Therefore $eR(1 - e^{k-1})Re = 0$.

Theorem 2.11: Let I be an ideal of a ring R and k-potents can be lifted modulo I. If R is quasi-k-normal, then RI is quasi-k-normal.

Proof. let $g + I$ is k-potent of RI so there exist $e \in K(R)$ such that $g + I = e + I$. Now $(g + I)(RI)((1 + I) - (g + I)^{k-1})(RI)(g + I) = (e + I)(RI)((1 + I) - (e + I)^{k-1})(RI)(e + I) = eR(1 - e^{k-1})Re + I = I$ as R is quasi-k-normal. So RI is quasi-k-normal.

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