Quasi-k-normal ring

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Abstract

In [4] Wei and Libin defined Quasi normal ring. In this paper we attempt to define Quasi-k-normal ring by using the action of k-potent element. A ring is called Quasi-k-normal ring if \( ae = 0 \Rightarrow eaRe = 0 \) for \( a \in N(R) \) and \( e \in K(R) \), where \( K(R) = \{ e \in R | e^k = e \} \). Several analogous results given in [4] is defined here. We find here that a ring is quasi-k-normal if and only if \( eR(1 - e^{k-1})Re = 0 \) for each \( e \in K(R) \). Also we get a ring is quasi-k-normal ring if and only if \( T_{\alpha}(R, R) \) is quasi-k-normal ring.

Key Words: Abelian rings; Quasi-k-normal rings; \( \Pi \)-Regular rings

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1. Introduction

In [3] Parmenter and Stewart defined the notion of normal ring as for \(x, y \in R\) and \(xRy = 0\) then \(ann(x) + ann(y) = R\) where \(ann(x) = \{z \in R : zRx = 0\}\). In [4] Wei and Libin generalized normal rings and defined quasi-normal ring on the basis of normal ring. Here we define quasi-k-normal ring in terms of k-potent elements as defined in [2]. Here we study several properties of quasi-k-normal rings.

2. Preliminaries

All rings considered here are associative with identity. We use the symbols \(K(R), E(R), N(R)\) for set of all k-potents, idempotents and nilpotents respectively. Again for any subset \(X\) of a ring \(R\), we denote \(r(X) = r_R(X)\) and \(l(X) = l_R(X)\) for right annihilator and left annihilator of \(X\) respectively. Again \(E_{ij}(1 \leq i, j \leq n)\) denote \(n \times n\) matrix units over \(R\), and write \(V = \sum_{i=1}^{n-1} E_{i,i+1}\) for \(n\). Also we use the symbol \(T_n(R, R)\) to denote the ring of \(n \times n\) upper triangular matrices whose principal diagonal elements are equals and others belong to \(R\) and \(V_n(R, R) = RE_n + RV + RV^2 + \ldots + RV^{n-1}\) for \(n > 2\), where \(E_n\) is \(n \times n\) identity matrix over \(E\).

A ring is called quasi-k-normal if \(ae = 0 \Rightarrow eaRe = 0\) for \(e \in K(R)\) and \(a \in N(R)\). Again a ring is semiabelian if every idempotent of \(R\) is either left semicentral or right semicentral. A ring is called abelian [1] if every idempotent of \(R\) is central. Semiabelian rings are quasi-k-normal. An example of quasi-k-normal ring is reversible ring \(R\) (\(ab = 0 \Rightarrow ba = 0\) for \(a, b \in R\)).

3. Results and Discussion

We start with the following results.

**Theorem 2.1.** The following conditions are equivalent for a ring \(R\):

1. \(R\) is quasi-k-normal.
2. \(eR(1 - e^{k-1})Re = 0; \forall e \in K(R)\).
3. \(eR(1 - e^{k-1})\) is right ideal of \(R; \forall e \in K(R)\).
4. \((1 - e^{k-1})Re\) is left ideal of \(R \forall e \in K(R)\).
5. \([(1 - e^{k-1})R \cap N(R)]e \subseteq r(eR) \cap N(R)e \ \forall e \in K(R)\).

6. \(ea = 0 \Rightarrow eRae = 0 \ \forall a \in N(R) \text{ and } e \in K(R)\).

**Proof:**

(1) \(\Rightarrow\) (2) : For any \(a \in R, e \in K(R)\), let \(h = e^{k-1}a - e^{k-1}ae^{k-1} = e^{k-1}a(1 - e^{k-1})\). So \(h^2 = e^{k-1}a(e^{k-1} - e^{2k-2})a(1 - e^{k-1}) = 0 \Rightarrow h \in N(R)\) and \(he = e^{k-1}a(e - e^k) = 0\). Now \(h \in N(R), he = 0\). So by definition of quasi-k-normal ring, \(ehRe = 0 \Rightarrow e.e^{k-1}a(1 - e^{k-1})Re = 0 \Rightarrow e(a - e^{k-1})Re = 0 \Rightarrow eR(1 - e^{k-1})Re = 0\) as \(a \in R\) is arbitrary.

(6) \(\Rightarrow\) (2) : Let \(h = ae^{k-1} - e^{k-1}ae^{k-1} = (1 - e^{k-1})ae^{k-1}; e \in K(R)\). Then \(eh = 0, h^2 = 0\). So \(eh = 0, h \in N(R) \Rightarrow ehRe = 0 \Rightarrow eR(1 - e^{k-1})ae^k = 0 \Rightarrow eR(1 - e^{k-1})Re = 0\) as \(a \in R\) is arbitrary.

(2) \(\Rightarrow\) (3) : Let \(e \in K(R)\). By (2), \(eR(1 - e^{k-1})Re = 0 \Rightarrow eR(1 - e^{k-1})Re^{k-1} = 0\). Now \(eR(1 - e^{k-1})R = eR(1 - e^{k-1})R(1 - e^{k-1}) \subseteq eR(1 - e^{k-1})\).

\[R(1 - e^{k-1})R \subseteq R\]

Again \(eR(1 - e^{k-1}) = eR(1 - e^{k-1})R \subseteq eR(1 - e^{k-1})R\). Therefore \(eR(1 - e^{k-1}) = eR(1 - e^{k-1})R\). So \(eR(1 - e^{k-1})\) is left ideal of \(R\).

(3) \(\Rightarrow\) (4) : Following (3), we have \(eR(1 - e^{k-1}) = eR(1 - e^{k-1})R \Rightarrow eR(e - e^k) = eR(1 - e^{k-1})R \Rightarrow eR(1 - e^{k-1})Re = 0 \Rightarrow e^{k-1}R(1 - e^{k-1})Re = 0\). Now, \(R(1 - e^{k-1})Re = (1 - e^{k-1})R(1 - e^{k-1})Re \subseteq (1 - e^{k-1})R \subseteq R\). Therefore \(eR(1 - e^{k-1})Re \subseteq (1 - e^{k-1})Re\).

Again, \(1 - e^{k-1})Re = 1(1 - e^{k-1})Re \subseteq R(1 - e^{k-1})Re\). Therefore \((1 - e^{k-1})Re = R(1 - e^{k-1})Re\) which implies \((1 - e^{k-1})Re\) is left ideal of \(R\).

(4) \(\Rightarrow\) (5) : Let \(x \in (1 - e^{k-1})R \cap N(R)\).

Therefore \(x = (1 - e^{k-1})b\) for some \(b \in R\) and \(xe \in N(R)e\). So \(ex = (e - e^k)b = 0 \Rightarrow e^{k-1}x = 0\). So \(x = x - e^{k-1} = (1 - e^{k-1})x\). Again \((1 - e^{k-1})Re\) is left ideal of \(R\). So \(eR(1 - e^{k-1})Re \subseteq (1 - e^{k-1})Re\). Now \(eRxe = eR(1 - e^{k-1})xe \subseteq eR(1 - e^{k-1})Re \subseteq (1 - e^{k-1})Re\). Therefore \(e^{k-1}xe \subseteq (1 - e^{k-1})xe \Rightarrow eRxe \subseteq (1 - e^{k-1})Re = 0 \Rightarrow eRxe = 0 \Rightarrow xe \in r(eR)\). Hence \(xe \in r(eR) \cap N(R)e\) and finally we get \([1 - e^{k-1})R \cap N(R)e \subseteq r(eR) \cap N(R)e\).

(5) \(\Rightarrow\) (6) : Let \(ea = 0 \Rightarrow e^{k-1}a = 0\) for some \(a \in N(R), e \in K(R)\). Now \(ae = ae - e^{k-1}ae = (1 - e^{k-1})ae \in [(1 - e^{k-1})R \cap N(R)e \subseteq r(eR) \cap N(R)e \Rightarrow ae \in r(eR) \cap N(R)e \Rightarrow ae \in r(eR) \Rightarrow eRae = 0\).
Let \( ae = 0 \) for some \( a \in N(R), e \in K(R) \). So by hypothesis \( eR(1 - e^{k-1})Re = 0 \). Now \( ae^{k-1} = 0 \), which implies \( a = a - ae^{k-1} = a(1 - e^{k-1}) \). Therefore \( eaRe = ea(1 - e^{k-1})Re \subseteq eR(1 - e^{k-1})Re \). Hence \( eaRe = 0 \).

**Corollary 2.2**

1. The following conditions are equivalent for a ring \( R \)
   
   (i) \( R \) is quasi-k-normal.
   
   (ii) For any \( e \in K(R); x, y \in R, exe = exe^{k-1}ye \).

2. Semiabelian rings are quasi-k-normal.

3. Let \( R \) be quasi-k-normal ring. If \( e \in E(R) \) with \( ReR = R \) then \( e^{k-1} = 1 \).

**Proof.**

1. R is quasi-k-normal \( \iff eR(1 - e^{k-1})Re = 0 \Rightarrow exe = exe^{k-1}ye; x, y \in R \). For \( k = 2 \) we get the following beautiful result by Junchao [4] \((ex)^ne = exne = e(xe)^n\).

2. As \( R \) is semiabelian so for \( e_1 \in E(R) \), we have \( e_1r = e_1re_1 \) or \( re_1 = e_1re_1 \) \( \forall r \in R \). Let \( e \in K(R) \Rightarrow e = e \Rightarrow (e^{k-1})^2 = e^{k-1} \in E(R) \). Now \( eR(1 - e^{k-1})Re = e.e^{k-1}R(1 - e^{k-1})Re \subseteq e.e^{k-1}Re^{k-1}(1 - e^{k-1})Re = 0 \) or \( eR(1 - e^{k-1})Re = eR(1 - e^{k-1})Re \) \( e \subseteq eR(1 - e^{k-1})e^{k-1}e^{k-1}Re = 0 \). So \( eR(1 - e^{k-1})Re = 0 \) in both cases implies \( R \) is quasi-k-normal.

3. As \( R \) is quasi-k-normal ring so by **Theorem 2.1**, \( eR(1 - e^{k-1})Re = 0 \). As \( ReR = R \) so \( R(1 - e^{k-1})R = ReR(1 - e^{k-1})ReR = 0 \), which implies \( 1(1 - e^{k-1})1 = 0 \Rightarrow e^{k-1} = 0 \).

A ring is called directly finite if \( xy = 1 \Rightarrow yx = 1 \forall x, y \in R \). A ring is called left min abel if for every \( e \in ME_1(R) = \{ e \in E(R): Re \) is minimal left ideal of \( R \} \), \( e \) is left semicentral in \( R \).

**Theorem 2.4** Quasi-k-normal rings are directly finite.
**Proof.** Let \( ab = 1 \) where \( a, b \in R \). Let \( e = ba \). Now \( e^2 = baba = ba = e \Rightarrow e^k = e \) so \( e \in K(R) \) as idempotents are always k-potent; \( ae = aba = a, eb = bab = b \). As \( R \) is quasi-k-normal so \( eR(1 - e^{k-1})Re = 0 \Rightarrow eb(1 - e)ae = 0 \Rightarrow (1 - e)a = 0 \Rightarrow ba = bea = b^2a^2 \Rightarrow bab = b^2a^2b \Rightarrow b = b^2a \Rightarrow ab = ab^2a \Rightarrow ab = ba \Rightarrow ba = 1 \). Therefore \( R \) is directly finite.

**Theorem 2.5** Quasi-k-normal ring are left min-abel.

**Proof.** Let \( e \in ME_1(R) \) and \( a \in R \). As \( e \) is idempotent it is clearly k-potent and in that case quasi-k-normal ring takes the form of quasi normal ring which is left min abel due to Junchao[4]. Hence the result.

A ring \( R \) is called left idempotent reflexive if \( aRe = 0 \) implies \( eRa = 0 \) for all \( a \in R \) and \( e \in K(R) \).

**Theorem 2.6:** The following conditions are equivalent for a ring \( R \)

1. \( R \) is abelian.
2. \( R \) is semiabelian and left idempotent reflexive.
3. \( R \) is quasi-k-normal and left idempotent reflexive.

**Proof:** By Corollary 2.2, (1) \( \Rightarrow \) (2) \( \Rightarrow \) (3)
(3) \( \Rightarrow \) (1) : As for idempotents, quasi-k-normal ring takes the form of quasi normal ring. So we have the result directly from Junchao[4].

**Theorem 2.7:** A ring \( R \) is quasi-k-normal ring if and only if \( T_n(R, R) \) is quasi-k-normal ring.

**Proof.** Suppose \( R \) is quasi-k-normal. We show that \( S = T_n(R, R) \) is quasi-k-normal by inducting on \( n \). For \( n = 1 \) it is trivial. Suppose \( S_1 = T_{n-1}(R, R) \) is quasi-k-normal for any \( n \), that is for any
Thus we have

\[ E_1 S_1 (1 - E_1^{k-1}) S_1 E_1 = 0. \]  
\[ E_1 S_1 (1 - E_1^{k-1}) R^{n-1} e = 0. \]  
\[ E_1 R^{n-1} (1 - e^{k-1}) R e = 0. \]  
\[ e R (1 - e^{k-1}) R e = 0. \]

Now for \( n \), let \( E = \begin{pmatrix} E_1 & \alpha \\ 0 & e \end{pmatrix} \in K(S) \) where \( \alpha \in \mathbb{R}^{n-1} \) and \( E_1 \in S_1 \), \( e \in K(R) \) and

\[ E^k = E. \]
\[ \Rightarrow \begin{pmatrix} E_1^k & E_1^{k-1} \alpha + E_1^{k-2} \alpha e + \ldots + E_1 \alpha e^{k-2} + \alpha e^{k-1} \\ 0 & e^k \end{pmatrix} = \begin{pmatrix} E_1 & \alpha \\ 0 & e \end{pmatrix} \]

\[ \Rightarrow E_1^{k-1} \alpha + E_1^{k-2} \alpha e + \ldots + E_1 \alpha e^{k-2} + \alpha e^{k-1} = \alpha \]

Now suppose that \( A = \begin{pmatrix} A_1 & \xi_1 \\ 0 & a \end{pmatrix} , B = \begin{pmatrix} B_1 & \xi_2 \\ 0 & b \end{pmatrix} \in S \), where \( A_1, B_1 \in S_1 ; \xi_1, \xi_2 \in \mathbb{R}^{n-1} ; a, b \in R \).

Now

\[ EA (1 - E_1^{k-1}) B E \]

\[ = \begin{pmatrix} A_1 & \xi_1 \\ 0 & a \end{pmatrix} \begin{pmatrix} 1 - E_1^{k-1} & -y \\ 0 & 1 - e^{k-1} \end{pmatrix} \begin{pmatrix} B_1 & \xi_2 \\ 0 & b \end{pmatrix} \begin{pmatrix} E_1 & \alpha \\ 0 & e \end{pmatrix} \]
\[
\begin{pmatrix}
E_1 A_1 (1 - E_1^{k-1}) B_1 E_1 & x \\
0 & ea(1 - e^{k-1})be
\end{pmatrix}
\]

Where
\[
x = E_1 A_1 (1 - E_1^{k-1}) B_1 \alpha + E_1 A_1 (1 - E_1^{k-1}) \xi_2 e - E_1 A_1 y be
\]

and
\[
y = E_1^{k-2} \alpha + E_1^{k-3} \alpha e + \cdots + E_1 \alpha e^{k-3} + \alpha e^{k-2}
\]

Now, \(E_1 A_1 (1 - E_1^{k-1}) B_1 E_1 \in E_1 S_1 (1 - E_1^{k-1}) S_1 E_1\). So by (1), we have,
\[
E_1 A_1 (1 - E_1^{k-1}) B_1 E_1 = 0
\]

Again \(ea(1 - e^{k-1})be \in eR(1 - e^{k-1}) Re\), So by (4) we have
\[
ea(1 - e^{k-1})be = 0.
\]

Now for \(x\), by using (5)
\[
E_1 A_1 (1 - E_1^{k-1}) B_1 \alpha = E_1 A_1 (1 - E_1^{k-1}) B_1 (E_1^{k-1} \alpha + E_1^{k-2} \alpha e + \cdots + E_1 \alpha e^{k-2} + \alpha e^{k-1})
\]
\[
= E_1 A_1 (1 - E_1^{k-1}) B_1 E_1^{k-1} \alpha + E_1 A_1 (1 - E_1^{k-1}) B_1 E_1^{k-2} \alpha e + \cdots + E_1 A_1 (1 - E_1^{k-1}) B_1 \alpha e^{k-1}
\]

(3.10)

[Using (6) and \(B_1 \alpha \in R^{n-1}\) so \(E_1 A_1 (1 - E_1^{k-1}) B_1 \alpha e \in E_1 S_1 (1 - E_1^{k-1}) R^{n-1} e = 0\) using (2)]

Again \(E_1 A_1 (1 - E_1^{k-1}) \xi_2 e \in E_1 S_1 (1 - E_1^{k-1}) R^{n-1} e\). So by using (2) we have
\[
E_1 A_1 (1 - E_1^{k-1}) \xi_2 e = 0.
\]

Again \(E_1 \xi_1 (1 - e^{k-1})be \in E_1 R^{n-1}(1 - e^{k-1}) Re\). So by using (3) we have
\[
E_1 \xi_1 (1 - e^{k-1})be = 0.
\]

Again \(aa(1 - e^{k-1})be \in eR(1 - e^{k-1}) Re\). So by using (4) we have
\[
aa(1 - e^{k-1})be = 0.
\]
Using (10), (11), (12), (13) we get

\[ x = -E_1 A_1 y b e \]

So we have to show \( E_1 A_1 y b e = 0 \).

Using (5) \( E_1^{k-1} \alpha + E_1^{k-2} \alpha e + ... + E_1 \alpha e^{k-2} + \alpha e^{k-1} = \alpha \)

\[ \Rightarrow E_1 (E_1^{k-2} \alpha + E_1^{k-3} \alpha e + E_1^{k-4} \alpha e^2 + ... + E_1 \alpha e^{k-2}) = \alpha (1 - e^{k-1}) \]

(3.14) \[ \Rightarrow E_1 y = \alpha (1 - e^{k-1}) \]

Similarly, \( (E_1^{k-2} \alpha + E_1^{k-3} \alpha e + E_1^{k-4} \alpha e^2 + ... + E_1 \alpha e^{k-2}) e = (1 - E_1^{k-1}) \alpha \)

(3.15) \[ \Rightarrow ye = (1 - E_1^{k-1}) \alpha \]

Now \( b \in R; y \in \mathbb{R}^{n-1}; A_1 \in M_{(n-1) \times (n-1)}, \) so \( A_1 y \in \mathbb{R}^{n-1}. \) So by using (3) we get

\[ E_1 A_1 y (1 - e^{k-1}) b e = 0 \]

\[ \Rightarrow E_1 A_1 y b e \]

\[ = E_1 A_1 y e^{k-1} b e \]

\[ = E_1 A_1 y e^{k-2} b e \ [k \geq 2] \]

\[ = E_1 A_1 (1 - E_1^{k-1}) \alpha e^{k-2} b e \ [\text{Using (15)}] \]

\[ \subset E_1 S_1 (1 - E_1^{k-1}) R^{n-1} e \ [A_1 \in S_1; \alpha e^{k-2} b \in \mathbb{R}^{n-1}] \]

Using (3) we get \( E_1 A_1 y b e = 0 \)

Therefore \( x = 0 \)

Hence \( E A (1 - E^{k-1}) B E = 0. \)

So \( T_n (R, R) \) is quasi-k-normal. Converse part is quite obvious.

By using the above theorem we can the get the following corollaries.
Corollary 2.8: A ring $R$ is quasi-k-normal if and only if $V_n(R, R)$ is quasi-k-normal for $n \geq 2$.

Since there is a ring isomorphism $\theta : V_n(R, R) = RE_n + RV + RV^2 + \ldots + RV^{n-1} \to R[x](x^n)$ defined by $\theta(r_0E_n + r_1V + r_2V^2 + \ldots + r_{n-1}V^{n-1}) = r_0 + r_1x + r_2x^2 + \ldots + r_{n-1}x^{n-1} + (x^n)$, So using Corollary 2.8 we get following.

Corollary 2.9: A ring $R$ is quasi-k-normal if and only if $R[x](x^n)$ is quasi-k-normal for $n \geq 2$.

Theorem 2.10 If $R$ is subdirect product of family of a quasi-k-normal rings $\{R_i : i \in I\}$ then $R$ is quasi-k-normal.

Proof. Let $R_i = RA_i$ where $A_i$ are ideals such that $\cap_{i \in I} A_i = 0$. Let $e \in K(R)$. Then $e_i = e + A_i \in K(R_i)$ for $i \in I$. As each $R_i$ is quasi-k-normal, $e_i R_i(1-e_i^{k-1})e_i = 0$ for $i \in I$ which implies $e R(1-e^{k-1})Re \subseteq A_i \forall i \in I$ implies $e R(1-e^{k-1})Re \subseteq \cap_{i \in I} A_i$. Therefore $e R(1-e^{k-1})Re = 0$.

Theorem 2.11: Let $I$ be an ideal of a ring $R$ and k-potents can be lifted modulo $I$. If $R$ is quasi-k-normal, then $RI$ is quasi-k-normal.

Proof. Let $g + I$ is k-potent of $RI$ so there exist $e \in K(R)$ such that $g + I = e + I$. Now $(g + I)(RI)((1 + I) - (g + I)^{k-1})(RI)(g + I) = (e + I)(RI)((1 + I) - (e + I)^{k-1})(RI)(e + I) = e R(1-e^{k-1})Re + I = I$ as $R$ is quasi-k-normal. So $RI$ is quasi-k-normal.

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