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Regularity of the extensions of a double fuzzy topological space

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Abstract

This paper studies the regularity of the extensions of a double fuzzy topological space. The interrelations among certain families of closed sets in a double fuzzy topological space and its extensions are also investigated.

Keywords: *Double fuzzy topology; Closed set; Regular space*

2010 Mathematics Subject Classification: *54A40.*

1. Introduction

The study on the extensions of a double fuzzy topological space was initiated in [8] and the authors examined whether the connectedness and compactness of a given double fuzzy topological space are carried over to its extensions. Further, in [11] it is proved that the extensions need not preserve the normality of a double fuzzy topological space and obtained some conditions which ensure the normality of the extended space. Recently, some interrelations among the families of (r, s)-regular fuzzy closed sets in a double fuzzy topological space and its extensions were obtained in [9, 12]. Also, some studies on mixed fuzzy topological spaces can be seen in [7, 13, 14, 15, 16].

In this paper, we inquire the regularity of the extension of a regular double fuzzy topological space. Though the extensions do not preserve regularity in general, some conditions which ensure the regularity of the extended space are obtained. Moreover, certain families of closed sets in a double fuzzy topological space and its extensions are investigated and some types of extensions under which these families remain unchanged are identified.

2. Preliminaries

Throughout this paper X stands for a non-empty set and some particular sets are identified as, $I = [0, 1]$, $I_0 = (0, 1]$, $I_1 = [0, 1)$ and $I_0 \oplus I_1 = \{(r, s) \in I_0 \times I_1 : r + s \leq 1\}$.

Definition 2.1. *[5] Let (τ, τ^*) be a pair of functions from I^X to I satisfying*

(i) $\tau(f) + \tau^*(f) \leq 1, \forall f \in I^X$,

(ii) $\tau(\underline{0}) = \tau(\underline{1}) = 1, \tau^*(\underline{0}) = \tau^*(\underline{1}) = 0$,

(iii) $\tau(f_1 \wedge f_2) \geq \tau(f_1) \wedge \tau(f_2)$ *and* $\tau^*(f_1 \wedge f_2) \leq \tau^*(f_1) \vee \tau^*(f_2), f_i \in I^X, i = 1, 2$,

(iv) $\tau(\bigvee_{i\in\Delta} f_i) \geq \bigwedge_{i\in\Delta} \tau(f_i)$ *and* $\tau^*(\bigvee_{i\in\Delta} f_i) \leq \bigvee_{i\in\Delta} \tau^*(f_i), f_i \in I^X, i \in \Delta$

Then the pair (τ, τ^*) is called a double fuzzy topology on X and the triplet (X, τ, τ^*) is called a double fuzzy topological space or dfts in short.

Definition 2.2. *[3] Let (X, τ, τ^*) be a dfts. For $(r, s) \in I_0 \oplus I_1$, a fuzzy set f is called*

(i) (r, s)-fuzzy open if $\tau(f) \geq r$ and $\tau^*(f) \leq s$ and

(ii) (r, s)-fuzzy closed if f^c is (r, s)-fuzzy open.

The collections of all (r, s)-fuzzy open sets and (r, s)-fuzzy closed sets are respectively denoted by $\mathcal{O}_{\tau,r,s}$ and $\mathcal{C}_{\tau,r,s}$.

Definition 2.3. *[3] Let (X, τ, τ^*) be a dfts. For each $(r, s) \in I_0 \oplus I_1$, $f \in I^X$ the operator $C_{\tau,\tau^*} : I^X \times I \times I \to I^X$ defined by*

$$C_{\tau,\tau^*}(f, r, s) = \bigwedge\{g \in I^X | f \leq g, \tau(g^c) \geq r, \tau^*(g^c) \leq s\}$$

is called the double fuzzy closure operator on (X, τ, τ^*).

Definition 2.4. *[3] Let (X, τ, τ^*) be a dfts. For each $(r, s) \in I_0 \oplus I_1$, $f \in I^X$ the operator $I_{\tau,\tau^*} : I^X \times I \times I \to I^X$ defined by*

$$I_{\tau,\tau^*}(f, r, s) = \bigvee\{g \in I^X | f \geq g, \tau(g) \geq r, \tau^*(g) \leq s\}$$

is called the double fuzzy interior operator on (X, τ, τ^*).

Definition 2.5. *[6] Let (X, τ, τ^*) be a dfts, $f \in I^X$, $(r, s) \in I \oplus I$. Then*

1. *f is called (r, s)-regular fuzzy open (or (r, s)-rfo) if $f = I_{\tau,\tau^*}\Big(C_{\tau,\tau^*}(f, r, s), r, s\Big)$.*
2. *f is called (r, s)-regular fuzzy closed (or (r, s)-rfc) if $f = C_{\tau,\tau^*}\Big(I_{\tau,\tau^*}(f, r, s), r, s\Big)$.*

We denote the collection of all (r, s)-rfc sets by $R\mathcal{C}_{\tau,r,s}$ and the collection of all (r, s)-rfo sets by $R\mathcal{O}_{\tau,r,s}$.

Abbas [1] introduced the concept of (r, s)-generalized fuzzy closed sets in a dfts as the following:

Definition 2.6. *[1] Let (X, τ, τ^*) be a dfts, $f, h \in I^X$, $(r, s) \in I_0 \oplus I_1$, then f is called*

1. *(r, s)-generalized fuzzy closed (for short, (r, s)-gfc) set if $C_{\tau,\tau^*}(f, r, s) \leq h$ whenever $f \leq h$ and $h \in \mathcal{O}_{\tau,r,s}$.*

2. (r,s)-generalized fuzzy open (for short, (r,s)-gfo) set if f^c is a (r,s)-gfc set.

We denote the collection of all (r,s)-gfc sets by $G\mathcal{C}_{\tau,r,s}$ and that of (r,s)-gfo sets by $G\mathcal{O}_{\tau,r,s}$.

The (r,s)-regular generalized fuzzy open sets and (r,s)-regular generalized fuzzy closed sets were introduced by Ghareeb in [2] as follows:

Definition 2.7. *[2] Let (X,τ,τ^*) be a dfts, $f,h \in I^X, (r,s) \in I_0 \oplus I_1$, then f is called*

1. *(r,s)-regular generalized fuzzy closed (for short, (r,s)-rgfc) set if $C_{\tau,\tau^*}(f,r,s) \leq h$ whenever $f \leq h$ and $h \in R\mathcal{O}_{\tau,r,s}$.*

2. *(r,s)-regular generalized fuzzy open (for short, (r,s)-rgfo) set if f^c is a (r,s)-gfc set.*

We denote the collection of all (r,s)-rgfc sets by $RG\mathcal{C}_{\tau,r,s}$ and the collection of all (r,s)-rgfo sets by $RG\mathcal{O}_{\tau,r,s}$.

In [4], the authors introduced the concepts of (r,s)-fuzzy b-closed sets and (r,s)-fuzzy b-open sets in a dfts.

Definition 2.8. *[4] Let (X,τ,τ^*) be a dfts. A fuzzy set f is called*

1. *(r,s)-fuzzy b-closed (briefly, (r,s)-fbc) if*

$$(I_{\tau,\tau^*}(C_{\tau,\tau^*}(f,r,s),r,s)) \wedge (C_{\tau,\tau^*}(I_{\tau,\tau^*}(f,r,s),r,s)) \leq f$$

2. *(r,s)-fuzzy b-open (briefly, (r,s)-fbo) iff f^c is (r,s)-fbc set.*

The collection of all (r,s)-fbc sets is denoted by $b\mathcal{C}_{\tau,r,s}$ and that of (r,s)-fbo sets is denoted by $b\mathcal{O}_{\tau,r,s}$.

Mohammed et. al.[4] introduced the concepts (r,s)-generalized fuzzy b-closed sets and (r,s)-generalized fuzzy b-open sets in terms of the double fuzzy b-closure and double fuzzy b-interior operators as defined below:

Definition 2.9. *[4] Let (X,τ,τ^*) be a dfts. Then double fuzzy b-closure operator and double fuzzy b-interior operator are defined by*
$bC_{\tau,\tau^*}(f,r,s) = \wedge\{h \in I^X : f \leq h \text{ and } h \in b\mathcal{C}_{\tau,r,s}\}$,
$bI_{\tau,\tau^*}(f,r,s) = \vee\{h \in I^X : h \leq f \text{ and } h \in b\mathcal{O}_{\tau,r,s}\}$

where (r,s) $\in I_0 \oplus I_1$.

Definition 2.10. *[4] Let (X, τ, τ^*) be a dfts, $f \in I^X$, $(r, s) \in I_0 \oplus I_1$, then f is called*

1. *(r, s)-generalized fuzzy b-closed (or, (r, s)-gfbc) set if $bC_{\tau,\tau^*}(f, r, s) \leq h$ whenever $f \leq h$ and $h \in \mathcal{O}_{\tau,r,s}$.*
2. *(r, s)-generalized fuzzy b-open (or, (r, s)-gfbo) set if f^c is a (r, s)-gfbc set.*

We denote the collection of all (r, s)-gfbc sets by $Gb\mathcal{C}_{\tau,r,s}$ and the collection of all (r, s)-gfbo sets by $Gb\mathcal{O}_{\tau,r,s}$.

Definition 2.11. *[8] Let (X, τ, τ^*) be a dfts and $g \in I^X$. For $\alpha \in I_0$ and $\beta \in I_1$ with $\alpha \geq \tau(g)$, $\beta \leq \tau^*(g)$ and $\alpha + \beta \leq 1$, define $\tau_0, \tau_0^* : I^X \to I$ by*

(i) $\tau_0(g) = \alpha$, $\tau_0^*(g) = \beta$ *and*

(ii) *for all* $f \in I^X \setminus \{g\}$,

$$\begin{aligned} \tau_0(f) &= \max\{\tau(f), \bigvee\{\tau(f_1) \wedge \tau(f_2) \wedge \alpha : (f_1, f_2) \in R_g f\}\} \\ \tau_0^*(f) &= \min\{\tau^*(f), \bigwedge\{\tau^*(f_1) \vee \tau^*(f_2) \vee \beta : (f_1, f_2) \in R_g f\}\} \end{aligned}$$

where $R_g f = \{(f_1, f_2) : f = f_1 \vee (f_2 \wedge g); f_1, f_2 \in I^X\}$

Then the triplet (X, τ_0, τ_0^*) is a dfts called the (g, α, β)-extension of (X, τ, τ^*).

3. Regularity and extensions

This section answers the question, "whether the extensions of a regular double fuzzy topological space remain regular or not?".

The regularity axiom in a double fuzzy topological space is defined as follows:

Definition 3.1. *A dfts (X, τ, τ^*) is said to be regular if for any fuzzy point x_λ in X and any $f \in \mathcal{O}_{\tau,r,s}$ with $x_\lambda \in f$, there exists $h \in \mathcal{O}_{\tau,r,s}$ such that $x_\lambda \in h \leq C_{\tau,\tau^*}(h, r, s) \leq f$.*

In general, the regularity of a dfts is not carried over to its extensions as shown below:

Example 3.2. *Let $X = \{x, y\}$ and τ, τ^* be as follows:*

$$\tau(f) = \begin{cases} 1, & \text{if } f \in \{\underline{0}, \underline{1}\} \\ \frac{1}{4}, & \text{if } f \in \left\{x_{\frac{1}{2}}, x_1, y_1, x_{\frac{1}{2}} \vee y_1\right\} \\ 0, & \text{otherwise.} \end{cases}$$

and

$$\tau^*(f) = \begin{cases} 0, & \text{if } f \in \{\underline{0}, \underline{1}\} \\ \frac{1}{3}, & \text{if } f \in \left\{x_{\frac{1}{2}}, x_1, y_1, x_{\frac{1}{2}} \vee y_1\right\} \\ 1, & \text{otherwise.} \end{cases}$$

Then, (X, τ, τ^*) is a regular dfts.

Let (X, τ_0, τ_0^*) be the $(g, \frac{1}{4}, \frac{1}{3})$-extension of (X, τ, τ^*) with $g = y_{\frac{1}{4}}$. Then, the resulting double fuzzy topology (τ_0, τ_0^*) on X is given by,

$$\tau_0(f) = \begin{cases} 1, & \text{if } f \in \{\underline{0}, \underline{1}\} \\ \frac{1}{4}, & \text{if } f \in \left\{x_{\frac{1}{2}}, x_1, y_1, , x_{\frac{1}{2}} \vee y_1, y_{\frac{1}{4}}, x_1 \vee y_{\frac{1}{4}}, x_{\frac{1}{2}} \vee y_{\frac{1}{4}}\right\} \\ 0, & \text{elsewhere} \end{cases}$$

and

$$\tau_0^*(f) = \begin{cases} 0, & \text{if } f \in \{\underline{0}, \underline{1}\} \\ \frac{1}{3}, & \text{if } f \in \left\{x_{\frac{1}{2}}, x_1, y_1, , x_{\frac{1}{2}} \vee y_1, y_{\frac{1}{4}}, x_1 \vee y_{\frac{1}{4}}, x_{\frac{1}{2}} \vee y_{\frac{1}{4}}\right\} \\ 1, & \text{elsewhere.} \end{cases}$$

But (X, τ_0, τ_0^*) is not regular since for $(r, s) = (\frac{1}{4}, \frac{1}{3})$, $x_\lambda = y_{\frac{1}{4}}$ and $f = x_{\frac{1}{2}} \vee y_{\frac{1}{4}} \in \mathcal{O}_{\tau,r,s}$ with $x_\lambda \in f$ there does not exist $h \in \mathcal{O}_{\tau,r,s}$ such that $x_\lambda \in h \leq C_{\tau,\tau^*}(h, r, s) \leq f$.

Lemma 3.3. *Let (X, τ_0, τ_0^*) be the (g, α, β)-extension of a dfts (X, τ, τ^*) with $\tau(f) + \tau^*(f) = 1$, $\forall f \in I^X$ and $R(\tau)$, the range of τ, is finite. Let $(r, s) \in I_0 \oplus I_1$ and $f \in \mathcal{O}_{\tau_0,r,s}$. Then there exist $f_1, f_2 \in \mathcal{O}_{\tau,r,s}$ such that $f = f_1 \vee (f_2 \wedge g)$.*

Since $\tau(f) + \tau^*(f) = 1$ and $R(\tau)$ is finite, $R(\tau^*)$ is finite.

Let $f \in \mathcal{O}_{\tau_0,r,s}$. Then, from the definition of (g, α, β)-extension, we have either $f \in \mathcal{O}_{\tau,r,s}$ or $\tau(f) \geq r$ and $\bigwedge\{\tau^*(f_1) \vee \tau^*(f_2) \vee \beta : (f_1, f_2) \in R_g f\} \leq s$ or $\bigvee\{\tau(f_1) \wedge \tau(f_2) \wedge \alpha : (f_1, f_2) \in R_g f\} \geq r$ and $\tau^*(f) \leq s$ or $\bigvee\{\tau(f_1) \wedge$

$\tau(f_2)\wedge\alpha : (f_1, f_2) \in R_g f\} \geq r$ and $\bigwedge\{\tau^*(f_1)\vee\tau^*(f_2)\vee\beta : (f_1, f_2) \in R_g f\} \leq s$

Case 1: $f \in \mathcal{O}_{\tau,r,s}$
Then, there exist $f, \underline{0} \in \mathcal{O}_{\tau,r,s}$ such that $(f, \underline{0}) \in R_g f$.

Case 2: $\tau(f) \geq r$ and $\bigwedge\{\tau^*(f_1) \vee \tau^*(f_2) \vee \beta : (f_1, f_2) \in R_g f\} \leq s$
$\bigwedge\{\tau^*(f_1) \vee \tau^*(f_2) \vee \beta : (f_1, f_2) \in R_g f\} \leq s$
$\Longrightarrow \exists (f_1, f_2) \in R_g f$ such that $\tau^*(f_1) \vee \tau^*(f_2) \vee \beta \leq s$, since $R(\tau^*)$ is finite
$\Longrightarrow \exists (f_1, f_2) \in R_g f$ such that $\tau^*(f_1) \vee \tau^*(f_2) \leq s$
$\Longrightarrow \exists (f_1, f_2) \in R_g f$ such that

$$\tau(f_1) \wedge \tau(f_2) = 1 - (\tau^*(f_1) \vee \tau^*(f_2)) \geq 1 - s \geq r.$$

Thus, there exist $f_1, f_2 \in \mathcal{O}_{\tau,r,s}$ such that $f = f_1 \vee (f_2 \wedge g)$.

Case 3: $\bigvee\{\tau(f_1) \wedge \tau(f_2) \wedge \alpha : (f_1, f_2) \in R_g f\} \geq r$ and $\tau^*(f) \leq s$
$\bigvee\{\tau f_1) \wedge \tau(f_2) \vee \alpha : (f_1, f_2) \in R_g f\} \geq r$
$\Longrightarrow \exists (f_1, f_2) \in R_g f$ such that $\tau(f_1) \wedge \tau(f_2) \wedge \alpha \geq r$, since $R(\tau)$ is finite
$\Longrightarrow \exists (f_1, f_2) \in R_g f$ such that $\tau(f_1) \wedge \tau(f_2) \geq r$
$\Longrightarrow \exists (f_1, f_2) \in R_g f$ such that

$$\tau^*(f_1) \vee \tau^*(f_2) = 1 - (\tau(f_1) \wedge \tau(f_2)) \leq 1 - r \leq s.$$

i.e., there exist $f_1, f_2 \in \mathcal{O}_{\tau,r,s}$ such that $f = f_1 \vee (f_2 \wedge g)$.

Case 4: $\bigvee\{\tau(f_1) \wedge \tau(f_2) \wedge \alpha : (f_1, f_2) \in R_g f\} \geq r$ and
$\bigwedge\{\tau^*(f_1) \vee \tau^*(f_2) \vee \beta : (f_1, f_2) \in R_g f\} \leq s$
Proceeding as in Case 3 we get $f_1, f_2 \in \mathcal{O}_{\tau,r,s}$ such that
$f = f_1 \vee (f_2 \wedge g)$. ∎

Now, the following theorem gives a sufficient condition for the extension to be regular.

Theorem 3.4. *Let (X, τ, τ^*) be a regular dfts such that $\tau(f) + \tau^*(f) = 1$, $\forall f \in I^X$ and $R(\tau)$ is finite. Then the $(g, 1, 0)$-extension of (X, τ, τ^*) is regular, provided $g \in \mathcal{C}_{\tau,1,0}$.*

Let (X, τ_0, τ_0^*) be the $(g, 1, 0)$-extension of (X, τ, τ^*) where $g \in \mathcal{C}_{\tau,1,0}$ and let $f \in \mathcal{O}_{\tau_0,r,s}$. Then by lemma 3.3, there exist $f_1, f_2 \in \mathcal{O}_{\tau,r,s}$ such that $f = f_1 \vee (f_2 \wedge g)$.

Now for a fuzzy point x_λ,

$$\begin{aligned} x_\lambda \in f &\Longrightarrow x_\lambda \in f_1 \vee (f_2 \wedge g) \\ &\Longrightarrow \text{either } x_\lambda \in f_1 \text{ or } x_\lambda \in f_2 \wedge g \end{aligned}$$

If $x_\lambda \in f_1$, then $\exists h \in \mathcal{O}_{\tau,r,s}$ such that $x_\lambda \in h \leq C_{\tau,\tau^*}(h,r,s) \leq f_1$. Clearly, $\exists h \in \mathcal{O}_{\tau_0,r,s}$ such that $x_\lambda \in h \leq C_{\tau_0,\tau_0^*}(h,r,s) \leq f_1 \leq f$.
Further, $x_\lambda \in f_2 \wedge g \Longrightarrow x_\lambda \in f_2$ and $x_\lambda \in g$. Then, $\exists h^* \in \mathcal{O}_{\tau,r,s}$ such that $x_\lambda \in h^* \leq C_{\tau,\tau^*}(h^*,r,s) \leq f_2$ so that $x_\lambda \in h^* \leq C_{\tau_0,\tau_0^*}(h^*,r,s) \leq f_2$.
Hence, $x_\lambda \in h^* \wedge g \leq C_{\tau_0,\tau_0^*}(h^* \wedge g, r, s) \leq f_2 \wedge g$ and $\tau_0\Big(h^* \wedge g\Big) \geq \tau_0(h^*) \wedge \tau_0(g) \geq r$. Similarly, $\tau_0^*\Big(h^* \wedge g\Big) \leq s$. Thus, $h^* \wedge g \in \mathcal{O}_{\tau_0,r,s}$. ∎

Remark 3.5. *The converse of the above theorem is not true in general. For example, let $X = \{y, z\}$ and $\tau, \tau^* : I^X \to I$ be as follows:*

$$\tau(f) = \begin{cases} 1, & \text{if } f \in \{\underline{0}, \underline{1}, y_1\} \\ \frac{9}{20}, & \text{if } f = y_{\frac{1}{2}} \\ 0, & \text{otherwise.} \end{cases}$$

and

$$\tau^*(f) = \begin{cases} 0, & \text{if } f \in \{\underline{0}, \underline{1}, y_1\} \\ \frac{11}{20}, & \text{if } f = y_{\frac{1}{2}} \\ 1, & \text{otherwise.} \end{cases}$$

Then, (X, τ, τ^*) is not a regular dfts since for any $(r,s) \leq (\frac{9}{20}, \frac{11}{20})$, $y_1 \in \mathcal{O}_{\tau,r,s}$ and the fuzzy point $y_{\frac{1}{2}} \in y_1$, but not exist $h \in \mathcal{O}_{\tau,r,s}$ such that $y_{\frac{1}{2}} \in h \leq C_{\tau,\tau^*}(h,r,s) \leq y_1$. Also, note that $\tau(f) + \tau^*(f) = 1$, $\forall f \in I^X$.

Now, consider the $(g,1,0)$ extension (X, τ_0, τ_0^*) of (X, τ, τ^*), where $g = z_1 \in \mathcal{C}_{\tau,1,0}$. Then τ_0 and τ_0^* are given by

$$\tau_0(f) = \begin{cases} 1, & \text{if } f \in \{\underline{0}, \underline{1}, z_1, y_1\} \\ \frac{9}{20}, & \text{if } f \in \{y_{\frac{1}{2}}, y_{\frac{1}{2}} \vee z_1\} \\ 0, & \text{otherwise.} \end{cases}$$

and

$$\tau_0^*(f) = \begin{cases} 0, & \text{if } f \in \{\underline{0}, \underline{1}, z_1, y_1\} \\ \frac{11}{20}, & \text{if } f \in \{y_{\frac{1}{2}}, y_{\frac{1}{2}} \vee z_1\} \\ 1, & \text{otherwise.} \end{cases}$$

Clearly, (X, τ_0, τ_0^*) is a regular dfts since for any fuzzy point x_λ in X and any $f \in \mathcal{O}_{\tau,r,s}$, $(r,s) \in I_0 \oplus I_1$ with $x_\lambda \in f$, by taking $h = f$, the condition for regularity is satisfied.

The conditions $\tau(f)+\tau^*(f)=1$ and $g \in \mathcal{C}_{\tau,1,0}$ in Theorem 3.4 is not necessary as verified below:

Example 3.6. *Let $X = I$ and define a double fuzzy topology (τ,τ^*) on X as follows:*

$$\tau(f)=\begin{cases} 1, & \text{if } f \in \{\underline{0},\underline{1}\} \\ \frac{9}{20}, & \text{if } f \in \{(\frac{1}{4})_{\frac{1}{3}},(\frac{7}{25})_{\frac{1}{3}},(\frac{1}{4})^c_{\frac{1}{3}},(\frac{7}{25})^c_{\frac{1}{3}}\} \\ 0, & \text{otherwise.} \end{cases}$$

and

$$\tau^*(f)=\begin{cases} 0, & \text{if } f \in \{\underline{0},\underline{1}\} \\ \frac{2}{5}, & \text{if } f \in \{(\frac{1}{4})_{\frac{1}{3}},(\frac{7}{25})_{\frac{1}{3}},(\frac{1}{4})^c_{\frac{1}{3}},(\frac{7}{25})^c_{\frac{1}{3}}\} \\ 1, & \text{otherwise.} \end{cases}$$

Clearly, (X,τ,τ^*) is a regular dfts.

Let (X,τ_0,τ_0^*) be the $(g,1,0)$-extension of (X,τ,τ^*) where $g=(\underline{\frac{1}{2}}) \notin \mathcal{C}_{\tau,1,0}$. Then, τ_0 and τ_0^* are given by

$$\tau_0(f)=\begin{cases} 1, & \text{if } f \in \left\{\underline{0},\underline{1},(\underline{\frac{1}{2}})\right\} \\ \frac{9}{20}, & \text{if } f \in \{(\frac{1}{4})_{\frac{1}{3}},(\frac{7}{25})_{\frac{1}{3}},(\frac{1}{4})^c_{\frac{1}{3}},(\frac{7}{25})^c_{\frac{1}{3}}\} \\ 0, & \text{otherwise.} \end{cases}$$

and

$$\tau_0^*(f)=\begin{cases} 0, & \text{if } f \in \left\{\underline{0},\underline{1},(\underline{\frac{1}{2}})\right\} \\ \frac{2}{5}, & \text{if } f \in \{(\frac{1}{4})_{\frac{1}{3}},(\frac{7}{25})_{\frac{1}{3}},(\frac{1}{4})^c_{\frac{1}{3}},(\frac{7}{25})^c_{\frac{1}{3}}\} \\ 1, & \text{otherwise.} \end{cases}$$

Now, for any fuzzy point x_λ and $f \in \mathcal{O}_{\tau,r,s}$, $(r,s) \in I_0 \oplus I_1$ with $x_\lambda \in f$, there exists $h = f$ such that $x_\lambda \in h \leq C_{\tau_0,\tau_0^*}(h,r,s) \leq f$. Hence, (X,τ_0,τ_0^*) is regular.

Remark 3.7. *Consider the double fuzzy topological space (X,τ,τ^*) and its extension (X,τ_0,τ_0^*) as defined in Remark 3.5. Then it follows that the regularity of the extension does not guarantee the regularity of the original space.*

4. Closed sets and extensions

In [10], it is shown that the partial ordering in $I_0 \oplus I_1$ induces an ordering in the collection of all $\mathcal{C}_{\tau,r,s}$. However, the same partial ordering does not induce an order for $G\mathcal{C}_{\tau,r,s}$, $RG\mathcal{C}_{\tau,r,s}$, $b\mathcal{C}_{\tau,r,s}$ and $Gb\mathcal{C}_{\tau,r,s}$ as shown below:

Example 4.1. *Let $X = \{a, b, c\}$, and define a double fuzzy topology on X as follows:*

$$\tau(f) = \begin{cases} 1, & \text{if } f \in \{\underline{0}, \underline{1}\} \\ 1-\alpha, & \text{if } f = \underline{\alpha}, \alpha \in I \setminus \{0,1\} \\ 0, & \text{otherwise.} \end{cases}$$

and

$$\tau^*(f) = \begin{cases} 0, & \text{if } f \in \{\underline{0}, \underline{1}\} \\ \alpha, & \text{if } f = \underline{\alpha}, \alpha \in I \setminus \{0,1\} \\ 1, & \text{otherwise.} \end{cases}$$

Now, for $(r_1, s_1) = (\frac{2}{3}, \frac{1}{3})$, $C_{\tau,\tau^*}\Big((\underline{\tfrac{2}{5}}), r_1, s_1\Big) = (\underline{\tfrac{2}{3}})$ and hence $(\underline{\tfrac{2}{5}}) \in G\mathcal{C}_{\tau,r_1,s_1}$ since the only $f \in \mathcal{O}_{\tau,r_1,s_1}$ such that $(\underline{\tfrac{2}{5}}) \leq f$ is $f = \underline{1}$.

Again, for $(r_2, s_2) = (\frac{3}{5}, \frac{1}{3})$, $C_{\tau,\tau^*}\Big((\underline{\tfrac{2}{5}}), r_2, s_2\Big) = (\underline{\tfrac{3}{5}})$ and hence $(\underline{\tfrac{2}{5}}) \notin G\mathcal{C}_{\tau,r_2,s_2}$ since $C_{\tau,\tau^*}\Big((\underline{\tfrac{2}{5}}), r_2, s_2\Big) \not\leq (\underline{\tfrac{2}{5}})$ where $(\underline{\tfrac{2}{5}}) \in \mathcal{O}_{\tau,r_2,s_2}$.

i.e., $G\mathcal{C}_{\tau,r_1,s_1} \not\subseteq G\mathcal{C}_{\tau,r_2,s_2}$ for $(r_1, s_1), (r_2, s_2) \in I_0 \oplus I_1$ with $(r_2, s_2) \leq (r_1, s_1)$.

Again, $R\mathcal{O}_{\tau,r_1,s_1} = \{\underline{0}, \underline{1}, (\underline{\tfrac{1}{3}})\}$ and $(\underline{\tfrac{2}{5}}) \in RG\mathcal{C}_{\tau,r_1,s_1}$. But, $(\underline{\tfrac{2}{5}}) \notin RG\mathcal{C}_{\tau,r_2,s_2}$ since $(\underline{\tfrac{2}{5}}) \in R\mathcal{O}_{\tau,r_2,s_2}$ and $C_{\tau,\tau^*}\Big((\underline{\tfrac{2}{5}}), r_2, s_2\Big) = (\underline{\tfrac{3}{5}}) \not\leq (\underline{\tfrac{2}{5}})$.

Further, $(\underline{\tfrac{1}{3}}) \in b\mathcal{C}_{\tau,r_1,s_1}$. But, $(\underline{\tfrac{1}{3}}) \notin b\mathcal{C}_{\tau,r_2,s_2}$ since $C_{\tau,\tau^*}\Big((\underline{\tfrac{1}{3}}), r_2, s_2\Big) = (\underline{\tfrac{3}{5}})$ and $I_{\tau,\tau^*}\Big(C_{\tau,\tau^*}\Big((\underline{\tfrac{1}{3}}), r_2, s_2\Big), r_2, s_2\Big) = (\underline{\tfrac{2}{5}})$.

However, we have the following theorem establishing inclusions among certain families of closed in a double fuzzy topological space.

Theorem 4.2. *Let (X, τ, τ^*) be a dfts. Then, for any $(r_1, s_1), (r_2, s_2) \in I_0 \oplus I_1$ with $(r_2, s_2) \leq (r_1, s_1)$,*

1. $\mathcal{C}_{\tau,r_1,s_1} \subseteq G\mathcal{C}_{\tau,r_2,s_2} \subseteq RG\mathcal{C}_{\tau,r_2,s_2}$
2. $\mathcal{C}_{\tau,r_1,s_1} \subseteq b\mathcal{C}_{\tau,r_2,s_2} \subseteq Gb\mathcal{C}_{\tau,r_2,s_2}$

Note that in general, the families $G\mathcal{C}_{\tau,r,s}$, $RG\mathcal{C}_{\tau,r,s}$, $b\mathcal{C}_{\tau,r,s}$ and $Gb\mathcal{C}_{\tau,r,s}$ do not remain intact while taking extensions.

Example 4.3. *Consider the dfts (X, τ, τ^*) defined in Example 4.1 and the fuzzy set g defined by*

$$g(x) = \begin{cases} \frac{1}{2}, & \text{if } x = a \text{or } x = b \\ \frac{9}{20}, & \text{if } x = c \end{cases}$$

Let (X, τ_0, τ_0^*) be the $(g, \frac{2}{3}, \frac{1}{3})$-extension of (X, τ, τ^*). Then, for $(r_1, s_1) = (\frac{2}{3}, \frac{1}{3})$ we have $(\underline{\frac{2}{5}}) \in G\mathcal{C}_{\tau, r_1, s_1}$. But, $(\underline{\frac{2}{5}}) \notin G\mathcal{C}_{\tau_0, r_1, s_1}$ since $(\underline{\frac{2}{5}}) \leq g \in \mathcal{O}_{\tau_0, r_1, s_1}$ but $C_{\tau_0, \tau_0^*}\Big((\underline{\frac{2}{5}}), r_1, s_1) = g^c \not\leq g$. Hence, $G\mathcal{C}_{\tau, r_1, s_1} \not\subseteq G\mathcal{C}_{\tau_0, r_1, s_1}$.

Now, $g \in R\mathcal{O}_{\tau_0, r_1, s_1}$ since $I_{\tau_0, \tau_0^*}\Big(C_{\tau_0, \tau_0^*}(g, r_1, s_1), r_1, s_1\Big) = I_{\tau_0, \tau_0^*}(g^c, r_1, s_1) = g$. Then, $(\underline{\frac{2}{5}}) \leq g$ but $C_{\tau_0, \tau_0^*}\Big((\underline{\frac{2}{5}}), r_1, s_1) \not\leq g$ which shows that $RG\mathcal{C}_{\tau, r_1, s_1} \not\subseteq RG\mathcal{C}_{\tau_0, r_1, s_1}$ since $(\underline{\frac{2}{5}}) \in RG\mathcal{C}_{\tau, r_1, s_1}$.

Again, $(\underline{\frac{1}{3}}) \in b\mathcal{C}_{\tau, r_1, s_1}$. Also, $I_{\tau_0, \tau_0^*}\Big(C_{\tau_0, \tau_0^*}\Big((\underline{\frac{1}{3}}), r_1, s_1\Big), r_1, s_1\Big) = I_{\tau_0, \tau_0^*}(g^c, r_1, s_1) = g$ and $C_{\tau_0, \tau_0^*}\Big(I_{\tau_0, \tau_0^*}\Big((\underline{\frac{1}{3}}), r_1, s_1\Big), r_1, s_1\Big) = C_{\tau_0, \tau_0^*}\Big((\underline{\frac{1}{3}}), r_1, s_1\Big) = g^c$.
i.e, $(\underline{\frac{1}{3}}) \notin b\mathcal{C}_{\tau_0, r_1, s_1}$.

Further, $(\underline{\frac{1}{3}}) \in Gb\mathcal{C}_{\tau, r_1, s_1}$ and $b\mathcal{C}_{\tau_0, r_1, s_1} = \Big\{f \in I^X : g \leq f \leq \underline{1}\Big\} \vee \{\underline{0}\}$. Then, $bC_{\tau_0, \tau_0^*}\Big((\underline{\frac{1}{3}}), r_1, s_1\Big) = g$ and therefore $(\underline{\frac{1}{3}}) \notin Gb\mathcal{C}_{\tau_0, r_1, s_1}$ since $(\underline{\frac{1}{3}}) \in \mathcal{O}_{\tau_0, r_1, s_1}$ but $bC_{\tau_0, \tau_0^*}\Big((\underline{\frac{1}{3}}), r_1, s_1\Big) \not\leq (\underline{\frac{1}{3}})$.
i.e., $Gb\mathcal{C}_{\tau, r_1, s_1} \not\subseteq Gb\mathcal{C}_{\tau_0, r_1, s_1}$.

However, the following theorem identifies some families that remain intact under extensions.

Theorem 4.4. *Let (X, τ, τ^*) be a dfts and (X, τ_0, τ_0^*) be its (g, α, β)-extension. Then for any $(r, s) \in I_0 \oplus I_1$ such that $(\alpha, \beta) \leq (r, s)$, $G\mathcal{C}_{\tau, r, s} = G\mathcal{C}_{\tau_0, r, s}$, $RG\mathcal{C}_{\tau, r, s} = RG\mathcal{C}_{\tau_0, r, s}$, $b\mathcal{C}_{\tau, r, s} = b\mathcal{C}_{\tau_0, r, s}$ and $Gb\mathcal{C}_{\tau, r, s} = Gb\mathcal{C}_{\tau_0, r, s}$.*

Since $\tau(f_1) \wedge \tau(f_2) \wedge \alpha \leq \alpha < r$, $\tau^*(f_1) \vee \tau^*(f_2) \vee \beta \geq \beta > s$ for all $f_1, f_2 \in I^X$, we have $\mathcal{O}_{\tau, r, s} = \mathcal{O}_{\tau_0, r, s}$. Therefore, $\mathcal{C}_{\tau, r, s} = \mathcal{C}_{\tau_0, r, s}$.

Hence, $I_{\tau_0, \tau_0^*}(f, r, s) = I_{\tau, \tau^*}(f, r, s)$ and $C_{\tau_0, \tau_0^*}(f, r, s) = C_{\tau, \tau^*}(f, r, s)$, for all $f \in I^X$.

Again, $C_{\tau_0, \tau_0^*}\Big(I_{\tau_0, \tau_0^*}(f, r, s), r, s\Big) = C_{\tau, \tau^*}\Big(I_{\tau, \tau^*}(f, r, s), r, s\Big)$ and $I_{\tau_0, \tau_0^*}\Big(C_{\tau_0, \tau_0^*}(f, r, s), r, s\Big) = I_{\tau, \tau^*}\Big(C_{\tau, \tau^*}(f, r, s), r, s\Big)$. Hence the proof. ∎

5. Conclusion

Looking into the regularity of a given double fuzzy topological space and its extensions, it is found that the regularity of the given space does not guarantee regularity of the extended space. However, certain situations under which the regularity of a double fuzzy topological space is carried over to its extensions are obtained. Further, some types of extensions which keep the families of certain types of closed sets intact are identified.

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