Proyecciones Journal of Mathematics
Vol. 42, N^o 3, pp. 571-597, June 2023.
Universidad Católica del Norte
Antofagasta - Chile

10.22199/issn.0717-6279-4818

Path-connectedness and topological closure of some sets related to the non-compact Stiefel manifold

Nizar El Idrissi
Universitè Ibn Tofail, Maroc
Samir Kabbaj
Universitè Ibn Tofail, Maroc
Pr. Brahim Moalige
Universitè Ibn Tofail, Maroc
Received : March 2021. Accepted : October 2022

Abstract

If H is a Hilbert space, the non-compact Stiefel manifold $St(n, H)$ consists of independent n-tuples in H. In this article, we contribute to the topological study of non-compact Stiefel manifolds, mainly by proving two results on the path-connectedness and topological closure of some sets related to the non-compact Stiefel manifold. In the first part, after introducing and proving an essential lemma, we prove that $\bigcap_{j\in J}(U(j) + St(n, H))$ is path-connected by polygonal paths under a condition on the codimension of the span of the components of the translating J-family. Then, in the second part, we show that the topological closure of $St(n, H) \cap S$ contains all polynomial paths contained in S and passing through a point in $St(n, H)$. As a consequence, we prove that $St(n, H)$ is relatively dense in a certain class of subsets which we illustrate with many examples from frame theory coming from the study of the solutions of some linear and quadratic equations which are finite-dimensional continuous frames. Since $St(n, L^2(X, \mu; \mathbf{F}))$ is isometric to $\mathcal{F}^{\mathbf{F}}_{(X,\mu),n}$, this article is also a contribution to the theory of finite-dimensional continuous Hilbert space frames.

Mathematics Subject Classification. *57N20; 42C15; 54D05; 54D99.*

Keywords: *Stiefel manifold, continuous frame, path-connected space, topological closure, dense subset.*

1. Introduction

Duffin and Shaeffer introduced in 1952 [13] the notion of a Hilbert space frame to study some deep problems in nonharmonic Fourier series. However, the general idea of signal decomposition in terms of elementary signals was known to Gabor [16] in 1946. The landmark paper of Daubechies, Grossmann, and Meyer [12] (1986) accelerated the development of the theory of frames which then became more widely known to the mathematical community. Nowadays, frames have a wide range of applications in both engineering science and mathematics: they have found applications in signal processing, image processing, data compression, and sampling theory. They are also used in Banach space theory. Intuitively, a frame in a Hilbert space K is an overcomplete basis allowing non-unique linear expansions, though technically, it must satisfy a double inequality called the frame inequality. There are many generalizations of frames in the literature, for instance frames in Banach space [10] or Hilbert C*-modules [15]. A general introduction to frame theory can be found in ([8],[10]).

The space $\mathcal{F}^{\mathbf{F}}_{(X,\mu),n}$ of continuous frames indexed by (X,μ) and with values in $\mathbf{F}^n$ is isometric to the Stiefel manifold $St(n, L^2(X,\mu;\mathbf{F}))$. If H is a Hilbert space, the non-compact Stiefel manifold $St(n,H)$ is the set of independent n-frames in H, where an independent n-frame simply denotes an independent n-tuple. Stiefel manifolds are studied in differential topology and are one of the fundamental examples in this area. Even though the theory of finite dimensional Stiefel manifolds is generally well-known ([24],[20],[23]), there are still some aspects under study ([22],[25]). The theory of infinite dimensional Stiefel manifolds is less studied and some recent results can be found in ([5],[19]).

There have been also many studies directly devoted to the geometry of frames and their subsets. Connectivity properties of some important subsets of the frame space $\mathcal{F}^{\mathbf{F}}_{k,n}$ were studied in ([7],[27]). Differential and algebro-geometric properties of these subsets were studied in ([14],[29],[**?**],[**?**]) and (chapter 4 of [9]) respectively. A fiber bundle structure with respect to the L^1 and L^∞ norms was established for continuous frames in ([1],[2]). A notion of density for general frames analogous to Beurling density was introduced and studied in [3]. Finally, connectivity and density properties were studied for Gabor ([4],[11],[21],[26]) and wavelet ([6],[17],[18],[28]) frames.

Plan of the article. This article is organized as follows. In section 2, we set some notations, introduce the definition of continuous Bessel and frame families and their basic properties in $\mathbf{F}^n$, and present Stiefel manifolds with an emphasis on their topological aspects. In section 3, after introducing and proving an essential lemma, we prove that $\bigcap_{j\in J}(U(j)+St(n,H))$ is path-connected by polygonal paths under a condition on the codimension of the span of the components of the translating J-family. Then, in section 4, we show that the topological closure of $St(n,H)\cap S$ contains all polynomial paths contained in S and passing through a point in $St(n,H)$. As a consequence, we prove that $St(n,H)$ is relatively dense in a certain class of subsets which we illustrate, in section 6, with many examples from frame theory coming from the study of the solutions of some linear and quadratic equations which are finite-dimensional continuous frames (section 5).

2. Preliminaries

2.1. Notation

The following notations are used throughout this article.
$\mathbf{N}$ denotes the set of natural numbers including 0 and $\mathbf{N}^* = \mathbf{N}\setminus\{0\}$.
We denote by n an element of $\mathbf{N}^*$ and by $\mathbf{F}$ one of the fields $\mathbf{R}$ or $\mathbf{C}$.
If K is a Hilbert space, we denote by $L(K)$ and $B(K)$ respectively the set of linear and bounded operators in K. Id_K is the identity operator of K.
If K is a Hilbert space, $m\in\mathbf{N}^*$, and $\theta_1,\cdots,\theta_m\in H$, the Gram matrix of $(\theta_1,\cdots,\theta_m)$ is the matrix $\mathrm{Gram}(\theta_1,\cdots,\theta_m)$ whose k,l-coefficient is $\mathrm{Gram}(\theta_1,\cdots,\theta_m)_{k,l}=\langle\theta_k,\theta_l\rangle$.
If $\sigma,\tau\in\mathbf{N}^*$, we denote by $M_{\sigma,\tau}(\mathbf{F})$ the algebra of matrices of size $\sigma\times\tau$ over the field $\mathbf{F}$. When $\sigma=\tau$, we denote this algebra $M_\sigma(\mathbf{F})$.
An element $x\in\mathbf{F}^n$ is a n-tuple $(x^1,\cdots,x^n)$ with $x^k\in\mathbf{F}$ for all $k\in[1,n]$.
If $S\in L(\mathbf{F}^n)$, we denote by $[S]\in M_n(\mathbf{F})$ the matrix of S in the standard basis of $\mathbf{F}^n$, and we write I_n as a shorthand for $[Id_{\mathbf{F}^n}]$.
If $U=(u_x)_{x\in X}$ is a family in $\mathbf{F}^n$ indexed by X, then for each $k\in[1,n]$, we denote by U^k the family $(u_x^k)_{x\in X}$.

2.2. Continuous frames in $\mathbf{F}^n$

Let K be a Hilbert space and (X,Σ,μ) a measure space.

Definition 2.1. *[10] We say that a family $\Phi = (\varphi_x)_{x \in X}$ with $\varphi_x \in K$ for all $x \in X$ is a continuous frame in K if*

$$\exists\, 0 < A \leq B : \forall v \in K : A\|v\|^2 \leq \int_X |\langle v, \varphi_x \rangle|^2 d\mu(x) \leq B\|v\|^2$$

A frame is tight if we can choose $A = B$ as frame bounds. A tight frame with bound $A = B = 1$ is called a Parseval frame. A Bessel family is a family satisfying only the upper inequality. A frame is discrete if Σ is the discrete σ-algebra and μ is the counting measure. We denote by $\mathcal{F}_{(X,\mu),K}$ and $\mathcal{F}^{\mathbf{F}}_{(X,\mu),n}$ respectively the set of continuous frames with values in K and the set of continuous frames with values in $\mathbf{F}^n$.

If $U = (u_x)_{x \in X}$ with $u_x \in K$ for all $x \in K$ is a continuous Bessel family in K, we define its analysis operator $T_U : K \to L^2(X, \mu; \mathbf{F})$ by

$$\forall v \in K : T_U(v) := (\langle v, u_x \rangle)_{x \in X}.$$

The adjoint of T_U is an operator $T_U^* : L^2(X, \mu; \mathbf{F}) \to K$ given by

$$\forall c \in L^2(X, \mu; \mathbf{F}) : T_U^*(c) = \int_X c(x) u_x d\mu(x).$$

The composition $S_U = T_U^* T_U : K \to K$ is given by

$$\forall v \in K : S_U(v) = \int_X \langle v, u_x \rangle u_x d\mu(x)$$

and called the frame operator of U. Since U is a Bessel family, T_U, T_U^*, and S_U are all well defined and continuous. If U is a frame in K, then S_U is a positive self-adjoint operator satisfying $0 < A \leq S_U \leq B$ and thus, it is invertible.

We now recall a proposition preventing that a frame belongs to $L^2(X, \mu; K)$ when $\dim(K) = \infty$. Here the set $L^2(X, \mu; K)$ refers to Bochner square integrable (classes) of functions in $\mathcal{M}(X; K)$, where the latter refers to the set of measurable functions from X to K. It explains why we only study the L^2 topology of frame subspaces in the finite dimensional case.

Proposition 2.1. *Let K be a Hilbert space with $\dim K = \infty$. Then $\mathcal{F}_{(X,\mu),K} \cap L^2(X, \mu; K) = \emptyset$*

Proof. Let $\Phi = (\varphi_x)_{x\in X} \in \mathcal{F}_{(X,\mu),K} \cap L^2(X,\mu;K)$. Let $\{e_m\}_{m\in M}$ be an orthonormal basis of K. We have

$$\begin{aligned}\mathrm{Tr}(S_\Phi) = \mathrm{Tr}(T_\Phi^* T_\Phi) &= \textstyle\sum_{m\in M} \|T(e_m)\|^2 \\ &= \textstyle\sum_{m\in M} \int_X |\langle e_m, \varphi_x\rangle|^2 d\mu(x) \\ &= \textstyle\int_X \left(\sum_{m\in M} |\langle e_m, \varphi_x\rangle|^2\right) d\mu(x) \\ &= \textstyle\int_X \|\varphi_x\|^2.\end{aligned}$$

Since $\Phi \in \mathcal{F}_{(X,\mu),K}$, there exists a constant $A > 0$ such that

$$S_\Phi \geq A \cdot Id,$$

so

$$\int_X \|\varphi_x\|^2 = Tr(S_\Phi) = +\infty$$

since $\dim(K) = \infty$. Hence $\Phi \notin L^2(X,\mu;K)$. □

From now on, we consider $K = \mathbf{F}^n$. In what follows, we will recall some elementary facts about Bessel sequences and frames in this setting.

Proposition 2.2. *A family $U = (u_x)_{x\in X}$ with $u_x \in \mathbf{F}^n$ for all $x \in X$ is a continuous Bessel family if and only if it belongs to $L^2(X,\mu,\mathbf{F}^n)$.*

Proof. ($\Rightarrow$) Suppose that $U = (u_x)_{x\in X}$ is a continuous Bessel family. For each $k \in [1,n]$, denote by e_k the k-th vector of the standard basis of $\mathbf{F}^n$.

Applying the definition to the vector e_k, we have for each $k \in \{1,\cdots,n\}$: $\|U^k\|^2_{L^2(X,\mu,\mathbf{F})} < \infty$, and so

$$\|U\|^2_{L^2(X,\mu,\mathbf{F}^n)} = \sum_{k=1}^{n} \|U^k\|^2_{L^2(X,\mu,\mathbf{F})} < \infty,$$

which implies $U \in L^2(X,\mu;\mathbf{F}^n)$.

($\Leftarrow$) Suppose that $U = (u_x)_{x\in I} \in L^2(X,\mu;\mathbf{F}^n)$. We have

$$\forall v \in \mathbf{F}^n : \int_{x\in X} |\langle v, u_x\rangle|^2 d\mu(x) \leq \|U\|^2_{L^2(X,\mu;\mathbf{F}^n)} v^2 < \infty$$

by the Cauchy-Schwarz inequality, which implies that $U = (u_x)_{x\in X}$ is a continuous Bessel family. □

Lemma 2.1. *If $U \in L^2(X,\mu;\mathbf{F}^n)$, then $[S_U] = Gram(U^1,\cdots,U^n)$.*

Proof. Let $(e_k)_{k\in[1,n]}$ the standard basis of $\mathbf{F}^n$. Let $i,j\in[1,n]$. Then

$$[S_U]_{i,j} = \langle Se_j, e_i\rangle = \int_X \langle e_j, u_x\rangle\langle u_x, e_i\rangle d\mu(x) = \int_X \overline{u_x^j} u_x^i d\mu(x) = \langle U^i, U^j\rangle.$$

□

Proposition 2.3. *[10] Suppose $\Phi = (\varphi_x)_{x\in X}$ is a family in $\mathbf{F}^n$. Then*

$$\begin{aligned}\Phi \text{ is a continuous frame } &\Leftrightarrow \Phi \in L^2(X,\mu;\mathbf{F}^n) \text{ and } S_\Phi \text{ is invertible}\\ &\Leftrightarrow \Phi \in L^2(X,\mu;\mathbf{F}^n) \text{ and } \det(Gram(\Phi^1,\cdots,\Phi^n)) > 0\\ &\Leftrightarrow \Phi \in L^2(X,\mu;\mathbf{F}^n) \text{ and } \{\Phi^1,\cdots,\Phi^n\} \text{ is free.}\end{aligned}$$

Proposition 2.4. *[10] Suppose $\Phi = \{\varphi_x\}_{x\in X}$ is a family in $\mathbf{F}^n$ and let $a > 0$. Then*

$$\begin{aligned}\Phi \text{ is a measurable a-tight frame } &\Leftrightarrow \Phi \in L^2(X,\mu;\mathbf{F}^n) \text{ and } S_\Phi = aI_n\\ &\Leftrightarrow \Phi \in L^2(X,\mu;\mathbf{F}^n) \text{ and } Gram(\Phi^1,\cdots,\Phi^n) = aI_n\\ &\Leftrightarrow \Phi \in L^2(X,\mu;\mathbf{F}^n) \text{ and } (\Phi^1,\cdots,\Phi^n)\\ &\quad \text{is an orthogonal family of } L^2(X,\mu;\mathbf{F})\\ &\quad \text{and } (\forall i\in[1,n] : \Phi^i = \sqrt{a}).\end{aligned}$$

Example 2.1. *Define $\varphi_m^1 = \frac{1}{m}e^{2\pi iam}$ and $\varphi_m^2 = \frac{1}{m}e^{2\pi ibm}$ with a,b two real numbers such that $a-b$ is not an integer. Then $\Phi^1 = (\varphi_m^1)_{m\in\mathbf{N}}$ and $\Phi^2 = (\varphi_m^2)_{m\in\mathbf{N}}$ are square summable with sum $\frac{\pi^2}{6}$. Since the sequences Φ^1 and Φ^2 are not proportional due to the constraint on a and b, it follows by 2.3 that Φ is a discrete frame in $\mathbf{C}^2$. It is not however a tight frame since Φ^1 and Φ^2 are not orthogonal.*

2.3. Basic topological properties of $St(n,H)$St(n,H) and $St_o(n,H)$

In this subsection, we introduce $St(n,H)$ and $St_o(n,H)$ as well as some of their basic topological properties. We recall that n is a fixed element of $\mathbf{N}^*$. If H is a Hilbert space, then $St(n,H)$ is non-empty exactly when $\dim(H)\geq n$. In the following, we will always suppose this condition.

Definition 2.2. *The non-compact Stiefel manifold of independent n-frames in H is defined by*

$St(n,H) := \{h=(h_1,\cdots,h_n)\in H^n : \{h_1,\cdots,h_n\}$ is free$\}$. The Stiefel manifold of orthonormal n-frames in H is defined by

$St_o(n,H) := \{h=(h_1,\cdots,h_n)\in H^n : \{h_1,\cdots,h_n\}$ is an orthonormal system$\}$.

Proposition 2.5. *We have*

1. *$St(n, L^2(X, \mu; \mathbf{F}))$ is isometric to $\mathcal{F}^{\mathbf{F}}_{(X,\mu),n}$.*

2. *$St_o(n, L^2(X, \mu; \mathbf{F}))$ is isometric to the set of continuous (X, μ)-Parseval frames with values in $\mathbf{F}^n$.*

Proof. Define

$$\textbf{Transpose} : \begin{cases} L^2(X, \mu; \mathbf{F}^n) & \to L^2(X, \mu; \mathbf{F})^n \\ F = (f_x)_{x \in X} & \mapsto ((f^1_x)_{x\in X}, \cdots, (f^n_x)_{x\in X}) \end{cases}.$$

Then **Transpose** is clearly an isometry, and it sends $\mathcal{F}^{\mathbf{F}}_{(X,\mu),n}$ to $St(n, L^2(X, \mu; \mathbf{F}))$ and $St_o(n, L^2(X, \mu; \mathbf{F}))$ to the set of continuous (X, μ)-Parseval frames with values in $\mathbf{F}^n$ by propositions 2.3 and 2.4 respectively. □

Remark 2.1. *Because of proposition 2.5, the reader should keep in mind that the following topological properties and the new results of this article are also shared, for any measure space (X, Σ, μ), by $\mathcal{F}^{\mathbf{F}}_{(X,\mu),n}$ or the set of continuous (X, μ)-Parseval frames with values in $\mathbf{F}^n$, depending on the context.*

Proposition 2.6. *We have*

1. *$St(n, H)$ is open in H^n.*

2. *$St_o(n, H)$ is closed in H^n.*

Proof.

1. $St(n, H)$ is open because $St(n, H) = (\det \circ \mathrm{Gram})^{-1}((0, \infty))$.

2. $St_o(n, H)$ is closed because $St_o(n, H) = \mathrm{Gram}^{-1}(I_n)$.

□

By joining continuously each element of $St(n, H)$ to its corresponding Gram-Schmidt orthonormalized system in $St_o(n, H)$, we can prove

Proposition 2.7. *$St_o(n, H)$ is a deformation retract of $St(n, H)$.*

Concerning the connectedness properties of Stiefel manifolds, we have the following

Definition 2.3. *Let X be a topological space and $m \in \mathbf{N}$. Then X is said to be m-connected if its homotopy groups $\pi_i(X)$ are trivial for all $i \in [0, m]$.*

Proposition 2.8. *(see pp. 382-383 of [20]) We have*

1. *$St(n, \mathbf{R}^k)$ is $(k - n - 1)$-connected.*

2. *$St(n, \mathbf{C}^k)$ is $(2k - 2n)$-connected.*

3. *If H is infinite dimensional, then $St(n, H)$ is contractible.*

Moreover, we have

Proposition 2.9. *If H is infinite dimensional, then $St(n, H)$ is contractible.*

A proof can be found in **this math.stackexchange thread**.

The following proposition asserts the density of $St(n, H)$ in H^n. A corollary of one of our results in this article (corollary 4.1) gives a generalization of this proposition.

Proposition 2.10. *$St(n, H)$ is dense in H^n.*

Proof. Consider $h = (h_1, \cdots, h_n) \in H^n$. Pick some $\theta = (\theta_1, \cdots, \theta_n) \in St(n, H)$. Let γ be the straight path connecting θ to h, i.e. for each $t \in [0, 1] : \gamma(t) = th + (1 - t)\theta \in H^n$. Let $\Gamma(t) = \det(\mathrm{Gram}((\gamma(t)_1, \cdots, \gamma(t)_n)))$. Clearly, $\Gamma(t)$ is a polynomial function in t which satisfies $\Gamma(0) \neq 0$ since $\theta \in St(n, H)$. Therefore

$$\Gamma(t) \neq 0 \text{ except for a finite number of } t's.$$

Moreover,

$$||\gamma(t) - u||_{H^n}^2 = \sum_{k=1}^{n} ||\gamma(t)_k - h_k||_H^2 = \sum_{i=1}^{n} |1 - t|^2 ||\theta_k - h_k||_H^2 \to 0 \text{ when t } \to 1$$

Hence, there exists $t \in [0, 1]$ such that $\gamma(t)$ is close to h and $\Gamma(t) \neq 0$, and so $\gamma(t) \in St(n, H)$. □

We also include the following proposition on the differential structure of the Stiefel manifolds.

Proposition 2.11. *[24] We have*

1. *$St(n, \mathbf{R}^k)$ is a real manifold of dimension nk.*

2. *$St_o(n, \mathbf{R}^k)$ is a real manifold of dimension $nk - \frac{n(n+1)}{2}$.*

3. *$St(n, \mathbf{C}^k)$ is a real manifold of dimension $2nk$.*

4. *$St_o(n, \mathbf{C}^k)$ is a real manifold of dimension $2nk - n^2$.*

5. *If $\dim(H) = \infty$, then $St(n, H)$ and $St_o(n, H)$ are Hilbert manifolds of infinite dimension.*

We ask the reader to keep in mind remark 2.1 when reading the remaining parts of this article.

3. Path-connectedness of $\bigcap_{j\in J}(U(j) + St(n, H))$

Definition 3.1. *Let E be a topological vector space and $\gamma : [0,1] \to E$ be a continuous path. We say that γ is a polygonal path if there exists $q \in \mathbf{N}^*$, $(e_k)_{k\in[1,q]}$ and $(f_k)_{k\in[1,q]}$ two finite sequences with $e_k, f_k \in E$ for all $k \in [1, q]$, and $(\gamma_k)_{k\in[1,q]}$ a finite sequence of (continuous) straight paths with $\gamma_k = \begin{cases} [\frac{k-1}{q}, \frac{k}{q}] & \to E \\ t & \mapsto q(t - \frac{k-1}{q})f_k + q(\frac{k}{q} - t)e_k \end{cases}$ such that $\gamma = \gamma_1 * \cdots * \gamma_q$, where $*$ is the path composition operation. We say that a subset $S \subseteq E$ is polygonally connected if every two points of S are connected by a polygonal path.*

In the following, when we say that $S \subseteq E$ is polygonally connected, we mean that each two points of S are connected by a polygonal path of the type $\gamma_1 * \gamma_2$ where γ_1 and γ_2 are two straight paths.

Before we prove the main proposition of this section, let's prove a useful lemma.

Lemma 3.1. 1. *Suppose we have a family $(a(j))_{j\in J}$ indexed by J where each $a(j)$ belongs to $St(n, H)$. Then if $(a(j)_k)_{j\in J, k\in[1,n]}$ is free, we have*

$$Span(\{a(j)\}_{j\in J}) \setminus \{0\} \subseteq St(n, H)$$

.

2. *Suppose we have a family indexed by J where each $a(j)$ belongs to $St_o(n, H)$ for all $j \in J$. Then if $(a(j)_k)_{j\in J, k\in[1,n]}$ is an orthonormal system, we have*

$$\{x \in Span(\{a(j)\}_{j\in J}) : \|x\| = \sqrt{n}\} \subseteq St_o(n, H)$$

Proof.

1. Let $h = (h_1, \cdots, h_n) \in \text{Span}(\{a(j)\}_{j \in J}) \setminus \{0\}$. We can write $h = \sum_{u=1}^{r} \lambda_u a(j_u)$ with $\lambda_u \in \mathbf{F}$ for all $u \in [1, r]$ and the λ_u's are not all zeros. We need to show that $(h_1, \cdots, h_n)$ is an independent system. Suppose otherwise $\sum_{k=1}^{n} c_k h_k = 0$. This means that $\sum_{k=1}^{n} c_k (\sum_{u=1}^{r} \lambda_u a(j_u)_k) = 0$, and so $\sum_{k=1}^{n} \sum_{u=1}^{r} (\lambda_u c_k) a(j_u)_k = 0$. Since $\bigcup_{j \in J} \bigcup_{k \in [1,n]} \{a(j)_k\}$ is free, we deduce that $\lambda_u c_k = 0$ for all $u \in [1, r]$ and $k \in [1, n]$, which implies that $c_k = 0$ for all $k \in [1, n]$ since the λ_u's are not all zeros. Therefore, $h \in St(n, H)$.

2. Let $h = (h_1, \cdots, h_n) \in \text{Span}(\{a(j)\}_{j \in J})$ such that $\|h\| = \sqrt{n}$. We can write $h = \sum_{u=1}^{r} \lambda_u a(j_u)$ with $\lambda_u \in \mathbf{F}$ for all $u \in [1, r]$. We need to show that $\langle h_k, h_l \rangle = \delta_{k,l}$ for all $k, l \in [1, n]$. For $k \neq l$, we have : $\langle h_k, h_l \rangle = \langle \sum_{u=1}^{r} \lambda_u a(j_u)_k, \sum_{u=1}^{r} \lambda_u a(j_u)_l \rangle$ $= \sum_{u=1}^{r} \sum_{v=1}^{r} \lambda_u \overline{\lambda_v} \langle a(j_u)_k, a(j_v)_l \rangle = 0$ since $\bigcup_{j \in J} \bigcup_{k \in [1,n]} \{a(j)_k\}$ is an orthogonal system. Moreover, $\|h_k\|^2$ $= \sum_{u=1}^{r} \sum_{v=1}^{r} \lambda_u \overline{\lambda_v} \langle a(j_u)_k, a(j_v)_k \rangle = \sum_{u=1}^{r} |\lambda_u|^2 \|a(j_u)_k\|^2 = \sum_{u=1}^{r} |\lambda_u|^2$ since $\bigcup_{j \in J} \bigcup_{k \in [1,n]} \{a(j)_k\}$ is an orthonormal system. By hypothesis, $n = \|h\|^2 = \sum_{k=1}^{n} \|h_k\|^2 = n(\sum_{u=1}^{r} |\lambda_u|^2)$, so $\|h_k\|^2 = \sum_{u=1}^{r} |\lambda_u|^2 = 1$ for all $k \in [1, n]$ as desired.

$\square$

Proposition 3.1. *Let H be a Hilbert space with $\dim(H) \geq n$, J an index set and $(U(j))_{j \in J}$ a family with $U(j) \in H^n$ for all $j \in J$. If ${}_H(Span(\{u(j)_k : j \in J, k \in [1, n]\})) \geq 3n$, then $\bigcap_{j \in J} (U(j) + St(n, H))$ is polygonally-connected.*

Proof. Let $X = (x_1, \cdots, x_n)$ and $Y = (y_1, \cdots, y_n)$ in $\bigcap_{j \in J} (U(j) + St(n, H))$.
Let $(z_1, \cdots, z_n)$ be an independent family in H such that

$$\text{Span}(\{z_k : k \in [1, n]\}) \bigcap \text{Span}\left(\{x_k : k \in [1, n]\} \bigcup \{y_k : k \in [1, n]\} \bigcup \{u(j)_k : j \in J, k \in [1, n]\}\right) = \{0\}$$

This is possible since ${}_H(\text{Span}(\{u(j)_k : j \in J, k \in [1, n]\}) \geq 3n$.
This ensures that we have for all $j \in J$

$$\text{Span}(\{-u(j)_k + z_k : k \in [1, n]\}) \cap \text{Span}(\{-u(j)_k + x_k : k \in [1, n]\}) = \{0\},$$

$$\text{Span}(\{-u(j)_k + z_k : k \in [1, n]\}) \cap \text{Span}(\{-u(j)_k + y_k : k \in [1, n]\}) = \{0\},$$

and

$$(-u(j)_k + z_k)_{k\in[1,n]} \text{ is independent.}$$

We define the straight paths

$$\begin{cases} \gamma_1 & : [0,1] \to H^n \\ \gamma_2 & : [0,1] \to H^n \end{cases}$$

by $\gamma_1(t) = tZ + (1-t)X$ and $\gamma_2(t) = tY + (1-t)Z$ respectively.

We have $\gamma_1(0) = X$, $\gamma_1(1) = \gamma_2(0) = Z$, and $\gamma_2(1) = Y$.

Since for all $j \in J$

$$\mathrm{Span}(\{-u(j)_k + z_k : k \in [1,n]\}) \cap \mathrm{Span}(\{-u(j)_k + x_k : k \in [1,n]\}) = \{0\},$$

we have for all $t \in [0,1]$ and $j \in J$

$$-U(j) + tZ + (1-t)X = t(-U(j) + Z) + (1-t)(-U(j) + X) \in St(n,H)$$

by lemma 3.1, and so $\gamma_1(t) \in \bigcap_{j\in J}(U(j) + St(n,H))$.
Similarly, $\gamma_2(t) \in \bigcap_{j\in J}(U(j) + St(n,H))$ for all $t \in [0,1]$.
Composing γ_1 with γ_2, we see that $\bigcap_{j\in J}(U(j) + St(n,H))$ is polygonally connected. $\square$

4. Topological closure of $St(n,H) \cap S$

Before moving on, we need a definition and a small lemma.

Definition 4.1. *Let V be a $\mathbf{F}$-vector space, $q \in \mathbf{N}$, and $v, v' \in V$. We say that $\gamma : [0,1] \to V$ is a polynomial path up to reparametrization joining v and v' if there exist $q \in \mathbf{N}$ and a finite sequence of vectors $(v^k)_{k\in[0,q]}$ with $v^k \in V$ for all $k \in [0,q]$ and a homeomorphism $\phi : [0,1] \to [a,b] \subseteq \mathbf{R}$ such that $\forall t \in [a,b] : \gamma(\phi^{-1}(t)) = \sum_{k=0}^{q} t^k v^k$, $\gamma(0) = v$ and $\gamma(1) = v'$. If V is equipped with a topology, then we say that γ is a continuous polynomial path when it is continuous as a map from $[0,1]$ to V.*

Remark 4.1. *Every expression of the form $\sum_{k=0}^{q} P_k(t)v^k$ where $v^k \in V$ and $P_k \in \mathbf{F}[X]$ for all $k \in [0,q]$ can be written in the form $\sum_{k=0}^{q'} t^k w^k$ where $q' \in \mathbf{N}$ and $w^k \in V$ for all $k \in [0,q']$ (group by increasing powers of t). Therefore there is no difference whether we define polynomial paths using the first expression or the second.*

Lemma 4.1. *Let E be a normed vector space, and $v, v' \in E$. Then each polynomial path up to reparametrization $\gamma : [0,1] \to E$ joining v and v' is continuous.*

Proof. Let $\gamma : [0,1] \to E$ be a polynomial path up to reparametrization joining v and v'. Hence we can write $\forall t \in [0,1] : \gamma(t) = \sum_{k=0}^{q} \phi(t)^k v^k$. The continuity of γ follows from

$$\gamma(t) - \gamma(t') \leq \sum_{k=0}^{q} v_k |\phi(t)^k - \phi(t')^k)|$$

which goes to 0 when t goes to t' by continuity of ϕ^k for all $k \in [0,q]$. □

The following is our first original proposition in this section.

Proposition 4.1. *Let $S \subseteq H^n$ and*
$E := \{\gamma : [0,1] \to S$ *such that* γ *is a polynomial path up to reparametrization and* $(\exists a_\gamma \in [0,1]) : \gamma(a_\gamma) \in St(n,H) \cap S\}$.
Then $\bigcup_{\gamma \in E} Range(\gamma) \subseteq \overline{St(n,H) \cap S}$

Proof. Let $\gamma \in E$. We have $\forall t \in [a,b] : \gamma(\phi^{-1}(t)) = \sum_{k=0}^{q} t^k V^k$.
Let $\Gamma(t) := \det(\mathrm{Gram}((\gamma(\phi^{-1}(t)))_1, \cdots, \gamma(\phi^{-1}(t))_n)))$ for all $t \in [a,b]$.
Since for all $i, j \in [1,n]$ and $t \in [a,b]$

$$\begin{aligned} \langle \gamma(\phi^{-1}(t))_i, \gamma(\phi^{-1}(t))_j \rangle &= \left\langle \sum_{k=0}^{q} t^k v_i^k, \sum_{k=0}^{q} t^k v_j^k \right\rangle \\ &= \sum_{k,k'=0}^{q} \left\langle v_i^k, v_j^{k'} \right\rangle t^{k+k'} \end{aligned}$$

is a polynomial function in $t \in [a,b]$, and the determinant of a matrix in $M_{n,n}(\mathbf{F})$ is a polynomial function in its coefficients, $\Gamma(t)$ is a polynomial function in $t \in [a,b]$ which satisfies $\Gamma(\phi(a_\gamma)) \neq 0$ since $\gamma(a_\gamma) \in St(n,H)$. Therefore

$$\Gamma(t) \neq 0 \text{ for t in a cofinite set } L \subseteq [a,b].$$

Hence $Range(\gamma) \setminus \{\gamma(\phi^{-1}(t))\}_{t \in [a,b] \setminus L} \subseteq St(n,H) \cap S \subseteq \overline{St(n,H) \cap S}$. Since $[a,b] \setminus L$ is finite, the continuity of γ (see lemma 4.1) at $\{\phi^{-1}(t))\}_{t \in [a,b] \setminus L}$ implies

$$Range(\gamma) = \overline{Range(\gamma)} = \overline{Range(\gamma) \setminus \{\gamma(\phi^{-1}(t))\}_{t \in [a,b] \setminus L}} \subseteq \overline{St(n,H) \cap S}.$$

This being true for all $\gamma \in E$, the result follows. □

The following is a corollary.

Corollary 4.1. *Let $S \subseteq H^n$ such that for all $U \in S$ there exists a polynomial path up to reparametrization connecting U to some $\Theta(U) \in St(n,H) \cap S$ and contained in S. Then $St(n,H) \cap S$ is dense in S.*

Proof. For all $U \in S$, there exists by the hypothesis $\gamma \in E$ such that $\gamma(0) = U$. By proposition 4.1, we have $Range(\gamma) \subseteq \overline{St(n, H) \cap S}$. It follows that $U \in \overline{St(n, H) \cap S}$ since $U \in Range(\gamma)$. $\square$

This result shows the abundance of independent n-frames not only in H^n but also in many subsets S of the form of corollary 4.1. Importantly, notice that for $S \neq \emptyset$, there should exist at least one $\Theta \in St(n, H) \cap S$ (i.e. $St(n, H) \cap S \neq \emptyset$) for the result to follow.

As an example, this is true when S is a star domain of H^n with respect to some $\Theta \in St(n, H) \cap S$; if it is a convex subset of H^n and contains some $\Theta \in St(n, H) \cap S$; and if it is in particular an affine subspace containing some $\Theta \in St(n, H) \cap S$.

In the next section, we will find sufficient conditions under which some sets of the form $f^{-1}(\{d\}) \subseteq L^2(X, \mu; \mathbf{F}^n)$ where f is some linear or quadratic function contain a continuous frame. This will allow us to apply corollary 4.1 to these examples, which will be done in section 6.

5. Existence of solutions of some linear and quadratic equations which are finite-dimensional continuous frames

In this section, we show how to construct continuous finite-dimensional frames that are mapped to a given element by a linear operator or a quadratic function. In other words, we show the existence of frames in the inverse image of singletons by these functions.

5.1. Linear equations

Proposition 5.1. *Let $n \in \mathbf{N}^*$, V an $\mathbf{F}$-vector field of dimension $\geq n$ and $T : V^n \to \mathbf{F}$ a non-zero linear form. Then for all $d \neq 0$, there exists $(a_1, \cdots, a_n) \in V^n$ such that the system $(a_1, \cdots, a_n)$ is free and $T(a_1, \cdots, a_n) = d$.*

The proof relies on the following lemma, which may be of independent interest.

Lemma 5.1. *Let $n \in \mathbf{N}^*$ and V be a vector space of dimension $\geq n$. Let $(x_1, \cdots, x_n) \in V^n \setminus \{(0, \cdots, 0)\}$. Then there exist $e \in \mathbf{N}^*$ with*
$e = \begin{cases} 1 \text{ if } k = n \\ 2 \text{ if } k \in [1, n-1] \end{cases}$ *where $k = \dim(Span\{x_1, \cdots, x_n\})$, and e independent systems $(a_1^u, \cdots, a_n^u)$ in V for $u \in [1, e]$ such that $(x_1, \cdots, x_n) = \sum_{u=1}^{e} (a_1^u, \cdots, a_n^u)$.*

Proof. Without loss of generality, suppose that $(x_1, \cdots, x_k)$ is free where $k = \dim(\mathrm{Span}\{x_1, \cdots, x_n\}) \geq 1$. Let $(b_i)_{i \in [1,n-k]}$ be an $n-k$-tuple of vectors of V such that the system $(x_{j_1}, \cdots, x_{j_k}, b_1, \cdots, b_{n-k})$ is free.

- If $k = n$, then $(x_1, \cdots, x_n)$ is already free and so we can choose $e = 1$ and $a_i^1 = x_i$ for all $i \in [1, n]$.
- If $k \in [1, n-1]$, we can write

$$(x_1, \cdots, x_n) = (\frac{x_1}{2}, \cdots, \frac{x_k}{2}, b_1, \cdots, b_{n-k})$$

$$+(\frac{x_1}{2}, \cdots, \frac{x_k}{2}, x_{k+1} - b_1, \cdots, x_n - b_{n-k}).$$

$(\frac{x_1}{2}, \cdots, \frac{x_k}{2}, b_1, \cdots, b_{n-k})$ is free by assumption, and we can easily show that $(\frac{x_1}{2}, \cdots, \frac{x_k}{2}, x_{k+1} - b_1, \cdots, x_n - b_{n-k})$ is free by expressing $x_{k+1}, \cdots, x_n$ in terms of $x_1, \cdots, x_k$. Hence we can choose $e = 2$ and $a_i^1 = a_i^2 = \frac{x_i}{2}$ for all $i \in [1, k]$, $a_i^1 = b_{i-k}$, and $a_i^2 = x_i - b_{i-k}$ for all $i \in [k+1, n]$.

□

Proof. (of proposition 5.1)
Let's show that there exists a free system $(a_1, \cdots, a_n)$ such that $T(a_1, \cdots, a_n) \neq 0$. Suppose to the contrary that $T(a_1, \cdots, a_n) = 0$ for all free systems $(a_1, \cdots, a_n)$ in V. Let $(x_1, \cdots, x_n) \in V^n \setminus \{(0, \cdots, 0)\}$. By lemma 5.1, $(x_1, \cdots, x_n)$ decomposes as a finite sum of free systems. By linearity of T, we thus have $T(x_1, \cdots, x_n) = 0$. Hence T is the zero form, a contradiction. Hence there exists a free system $(a_1, \cdots, a_n)$ such that $T(a_1, \cdots, a_n) \neq 0$. Then $\frac{d}{T(a_1, \cdots, a_n)}(a_1, \cdots, a_n)$ satisfies the requirement. □

Corollary 5.1. *Let $n \in \mathbf{N}^*$ and $T : L^2(X, \mu; \mathbf{F}^n) \to \mathbf{F}$ be a non-zero linear form.*
Suppose that $\dim(L^2(X, \mu; \mathbf{F})) \geq n$.
Then for all $d \neq 0$, there exists a continuous frame $\Phi = (\varphi_x)_{x \in X}$ such that $T(\Phi) = d$.

Proposition 5.2. *Let $n \in \mathbf{N}^*$, V be a vector space over $\mathbf{F}$, and $S : V \to \mathbf{F}$ be a non-zero linear form.*

Define the linear operator

$$T : \left\{ \begin{array}{ll} V^n & \to \mathbf{F}^n \\ (x_1, \cdots, x_n) & \mapsto (S(x_1), \cdots, S(x_n)) \end{array} \right. .$$

For all $d \in \mathbf{F}^n$, suppose that

- *if $d \neq 0$, then* $\dim(V) \geq n$,
- *if $d = 0$ then* $\dim(V) \geq n + 1$.

Then $T^{-1}(\{d\})$ contains an independent n-tuple $(a_1, \cdots, a_n)$.

Proof.

- Suppose that $d \neq 0$ and $\dim(V) \geq n$. Let $h \in V$ such that $S(h) = 1$, $(h_{(2)}, \cdots, h_{(n)})$ be an independent system in $Ker(S)$ (this is possible since $(Ker(S)) = 1$), and $(d, d_{(2)}, \cdots, d_{(n)})$ be an independent system in $\mathbf{F}^n$. Consider the V-valued column matrix $\mathbf{H} = \begin{bmatrix} h \\ h_{(2)} \\ \vdots \\ h_{(n)} \end{bmatrix}$ and the $\mathbf{F}$-valued square matrix $\mathbf{D} = \left[\begin{array}{c|ccc} d^1 & d^1_{(2)} & \cdots & d^1_{(n)} \\ \vdots & \vdots & \vdots & \vdots \\ d^n & d^n_{(2)} & \cdots & d^n_{(n)} \end{array} \right]$. Moreover, we set $\mathbf{A} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \mathbf{DH}$.

  We have $T(a_1, \cdots, a_n) = (S(a_1), \cdots, S(a_n))$
  $= (S(d^i h + \sum_{k=2}^n d^i_{(k)} h_{(k)}))_{i \in [1,n]} = (d_i)_{i \in [1,n]} = d$ since $h_{(2)}, \cdots, h_{(n)}$ belong to $Ker(S)$.
  Let's show that $(a_1, \cdots, a_n)$ is free. Let $\lambda = (\lambda^1, \cdots, \lambda^n) \in \mathbf{F}^n$ such that $\sum_{i=1}^n \lambda^i a_i = 0$. Consider $\mathbf{\Lambda} = \begin{bmatrix} \lambda^1 \\ \vdots \\ \lambda^n \end{bmatrix}$. Therefore we have $0 = \mathbf{\Lambda}^\top \mathbf{A} = \mathbf{\Lambda}^\top \mathbf{DH} \in V$. At this point, we can complete $(h, h_{(2)}, \cdots, h_{(n)})$ into a Hamel basis of V, and denote by $(h^*, h^*_{(2)}, \cdots, h^*_{(n)})$ the first n linear forms of its dual basis. Applying these linear forms

to $\mathbf{\Lambda}^\top \mathbf{D}\mathbf{H}$, we see that $\mathbf{\Lambda}^\top \mathbf{D} = \begin{bmatrix} [0] & \cdots & [0] \end{bmatrix}$, therefore $\lambda = 0$ as $\mathbf{D}^\top$ is obviously invertible since $(h, d_{(2)}, \cdots, d_{(n)})$ is an independent system in $\mathbf{F}^n$.

- Suppose that $d = 0$ and $\dim(V) \geq n+1$. Let $h \in V$ such that $S(h) = 1$ and $(a_1, \cdots, a_n)$ be an independent system in $Ker(S)$.

  We have $T(a_1, \cdots, a_n) = (S(a_1), \cdots, S(a_n)) = 0 \in \mathbf{F}^n$.

  Moreover, we have by construction that $(a_1, \cdots, a_n)$ is free.

□

Remark 5.1. *In the previous proposition, if $d \neq 0$, the condition $\dim(V) \geq n$ is necessary for the existence of an independent n-tuple because the existence of an independent n-tuple implies that $\dim(V) \geq n$, and if $d = 0$, the condition $\dim(V) \geq n+1$ is also necessary for the existence of an independent n-tuple in $T^{-1}(\{0\}$ because the existence of an independent n-tuple $(a_1, \cdots, a_n)$ in $T^{-1}(\{0\}$ implies that $S(a_i) = 0$ for all $i \in [1, n]$, and since there exists $h \in V$ such that $S(h) = 1$ because S is non-zero, then $(a_1, \cdots, a_n, h)$ is free which implies $\dim(V) \geq n+1$.*

Corollary 5.2. *Let $h \in L^2(X, \mu; \mathbf{F}) \neq 0$. Define the linear operator*

$$T : \begin{cases} L^2(X, \mu; \mathbf{F}^n) & \to \mathbf{F}^n \\ F = (f_x)_{x \in X} & \mapsto \int_X h(x) f_x d\mu(x) \end{cases}.$$

For all $d \in \mathbf{F}^n$, suppose that

- *if $d \neq 0$, then $\dim(L^2(X, \mu; \mathbf{F})) \geq n$,*
- *if $d = 0$ then $\dim(L^2(X, \mu; \mathbf{F})) \geq n+1$.*

Then $T^{-1}(\{d\})$ contains a continuous frame $\Phi = (\varphi_x)_{x \in X}$.

Proposition 5.3. *Let $h \in L^2(X, \mu; \mathbf{F})$. Define the linear operator*

$$T : \begin{cases} L^2(X, \mu; \mathbf{F}^n) & \to \mathbf{F}^n \\ F = (f_x)_{x \in X} & \mapsto \int_X h(x) f_x d\mu(x) \end{cases}.$$

If there exists a measurable subset $Y \subset X$ such that $\dim(L^2(Y, \mu; \mathbf{F})) \geq n$ and $\mu((X \setminus Y) \cap h^{-1}(\mathbf{F}^*)) > 0$, then for all $d \in \mathbf{F}^n$, $T^{-1}(\{d\})$ contains a continuous frame $\Phi = (\varphi_x)_{x \in X}$.

Proof. Since $\dim(L^2(Y,\mu;\mathbf{F})) \geq n$, there exists a continuous frame $(\varphi_y)_{y\in Y} \in \mathcal{F}^{\mathbf{F}}_{(Y,\mu),n}$. We extend $(\varphi_y)_{y\in Y}$ by setting

$$\varphi_x := \frac{\overline{h(x)}}{h^2_{L^2(X\setminus Y,\mu;\mathbf{F})}}\left(d - \int_Y h(y)\varphi_y d\mu(y)\right) \qquad \text{for all } x \in X \setminus Y$$

Let $\Phi = (\varphi_x)_{x\in X}$. We have

$$\begin{aligned} T(\Phi) &= \textstyle\int_X h(x)\varphi_x d\mu(x) \\ &= \textstyle\int_Y h(y)\varphi_y d\mu(y) + \left(\int_{X\setminus Y} h(x)\frac{\overline{h(x)}}{h^2_{L^2(X\setminus Y,\mu;\mathbf{F})}} d\mu(x)\right) \\ &\quad (d - \textstyle\int_Y h(y)\varphi_y d\mu(y)) \\ &= d. \end{aligned}$$

Moreover, $\Phi \in \mathcal{F}^{\mathbf{F}}_{(X,\mu),n}$ since we have only completed $(\varphi_y)_{y\in Y}$ by a function in $L^2(X \setminus Y,\mu;\mathbf{F}^n)$. $\square$

Proposition 5.4. *Let (X,Σ,μ) be a measure space, $l \in \mathbf{N}^*$, $(X_j)_{j\in[1,l]}$ a partition of X by measurable subsets, and $h \in L^2(X,\mu;\mathbf{F})$ such that there exist a family $(Y_j)_{j\in[1,l]}$ with Y_j a measurable subset of X_j for all $j \in [1,l]$, $\mu((X_j \setminus Y_j) \cap h^{-1}(\mathbf{F}^*)) > 0$ for all $j \in [1,l]$, and $\sum\limits_{j=1}^{l} \dim(L^2(Y_j,\mu;\mathbf{F})) \geq n$. Define the operator*

$$W : \left\{ \begin{array}{ll} L^2(X,\mu;\mathbf{F}^n) & \to \prod_{j\in[1,l]} \mathbf{F}^n \\ F = (f_x)_{x\in X} & \mapsto (\int_{X_j} h(x) f_x d\mu(x))_{j\in[1,l]} \end{array} \right. .$$

Then for all $D = (d_j)_{j\in[1,l]} \in \prod_{j\in[1,l]} \mathbf{F}^n$, $W^{-1}(\{D\})$ contains at least one continuous frame $\Phi \in \mathcal{F}^{\mathbf{F}}_{(X,\mu),n}$.

Remark 5.2. *Proposition 5.3 results from proposition 5.4 by taking $l = 1$.*

Remark 5.3. *Proposition 5.4 can be generalized to $l = +\infty$ or to partitions indexed by a general index set J if we restrict to $D = 0$ (due to convergence issues).*

Proof. For each $i \in [1,n]$, let e_i be the i-th vector of the standard basis of $\mathbf{F}^n$. Since $\sum\limits_{j=1}^{l} \dim(L^2(Y_j,\mu;\mathbf{F})) \geq n$, we can find distinct $j_1,\cdots,j_r \in [1,l]$ such that for each $u \in [1,r]$, $\dim(L^2(Y_{j_u},\mu;\mathbf{F})) \geq 1$ and $\sum_{u=1}^{r} \dim(L^2(Y_{j_u},\mu;\mathbf{F})) \geq n$. Take a partition $P_1,\cdots,P_r$ of $\{e_1,\cdots,e_n\}$ with $|P_u| \leq \dim(L^2(Y_{j_u},\mu;\mathbf{F}))$

for all $u \in [1, r]$.
For all $u \in [1, r]$, let $(g_u^p)_{p \in P_u}$ be an orthonormal family in $L^2(Y_{j_u}, \mu; \mathbf{F})$ and define $(\varphi_x)_{x \in X_{j_u}}$ by

$$\varphi_y = \sum_{p \in P_u} g_u^p(y) p \qquad \text{for all y } \in Y_{j_u}$$

and

$$\varphi_x := \frac{\overline{h(x)}}{h^2_{L^2(X_{j_u} \setminus Y_{j_u}, \mu; \mathbf{F})}} \left(d_{j_u} - \int_Y h(y) \varphi_y d\mu(y) \right) \qquad \text{for all } x \in X_{j_u} \setminus Y_{j_u}.$$

For all $j \notin \{j_u : u \in [1, r]\}$, define $(\varphi_x)_{x \in X_j}$ by

$$\varphi_y = 0 \qquad \text{for all y } \in Y_j$$

and

$$\varphi_x := \frac{\overline{h(x)}}{h^2_{L^2(X_j \setminus Y_j, \mu; \mathbf{F})}} d_j \qquad \text{for all } x \in X_j \setminus Y_j.$$

Let $\Phi = (\varphi_x)_{x \in X}$. We have

$$\begin{aligned} W(\Phi) &= \left(\int_{X_j} h(x) \varphi_x d\mu(x) \right)_{j \in [1, l]} \\ &= \Bigg(\int_{Y_j} h(y) \varphi_y d\mu(y) \\ &\quad + \left(\int_{X_j \setminus Y_j} h(x) \frac{\overline{h(x)}}{h^2_{L^2(X_j \setminus Y_j, \mu; \mathbf{F})}} d\mu(x) \right) \left(d_j - \int_{Y_j} h(y) \varphi_y d\mu(y) \right) \Bigg)_{j \in [1, l]} \\ &= (d_j)_{j \in [1, l]} = D. \end{aligned}$$

Moreover, $\Phi \in \mathcal{F}^{\mathbf{F}}_{(X, \mu), n}$ since

$$\forall v \in \mathbf{F}^n : v^2 = \sum_{u=1}^{r} \int_{Y_{j_u}} |\langle v, \varphi_x \rangle|^2 d\mu(x) \leq \int_X |\langle v, \varphi_x \rangle|^2 d\mu(x)$$

and

$$\begin{aligned} \int_X |\langle v, \varphi_x \rangle|^2 d\mu(x) &\leq v^2 + \left(\sum_{u=1}^{r} \frac{d_{j_u} - \int_Y h(y) \varphi_y d\mu(y)^2}{h^2_{L^2(X_{j_u} \setminus Y_{j_u}, \mu; \mathbf{F})}} \right) v^2 \\ &\quad + \left(\sum_{j \notin \{j_u : u \in [1, r]\}} \frac{d_j^2}{h^2_{L^2(X_j \setminus Y_j, \mu; \mathbf{F})}} \right) v^2 \end{aligned}$$

$\square$

5.2. A quadratic equation

Proposition 5.5. *Let (X, Σ, μ) be a σ-finite measure space, $b \in \mathbf{C}^n \setminus \{0\}$, $\epsilon > 0$ and $h \in L^\infty(X, \mu; \mathbf{C})$.*
Consider

$$q : \begin{cases} L^2(X, \mu; \mathbf{C}^n) & \to \mathbf{C}^n \\ F = (f_x)_{x \in X} & \mapsto \int_X h(x)\langle b, f_x\rangle f_x d\mu(x) \end{cases} .$$

Then for all $d \in \mathbf{C}^n$ such that

- *if $d \neq 0$, then there exists a measurable subset $Y \subseteq X$, two measurable subsets $B_1 \subseteq \{z \in \mathbf{C} : Re(\langle b, d\rangle z) > \epsilon$ and $Im(\langle b, d\rangle z) < -\epsilon\}$ and $B_2 \subseteq \{z \in \mathbf{C} : Re(\langle b, d\rangle z) > \epsilon$ and $Im(\langle b, d\rangle z) > \epsilon\}$ such that $\dim(L^2(Y, \mu; \mathbf{C})) \geq n$ and $\mu((X \setminus Y) \cap h^{-1}(B_1)), \mu((X \setminus Y) \cap h^{-1}(B_2)) > 0$,*

- *if $d = 0$, then there exist a measurable subset $Y \subseteq X$ such that $\dim(L^2(Y, \mu; \mathbf{C})) \geq n$ and $h(x) < 0$ μ-almost everywhere on Y, and [(two measurable subsets $B_1 \subseteq \{z \in \mathbf{C} : Re(z) > 0$ and $Im(z) < 0\}$ and $B_2 \subseteq \{z \in \mathbf{C} : Re(z) > 0$ and $Im(z) > 0\}$ such that $\mu((X \setminus Y) \cap h^{-1}(B_1)), \mu((X \setminus Y) \cap h^{-1}(B_2)) > 0$) or (a measurable subset $B_3 \subseteq \{z \in \mathbf{C} : Re(z) > 0$ and $Im(z) = 0\}$ such that $\mu((X \setminus Y) \cap h^{-1}(B_3)) > 0$)],*

there exists a continuous frame $\Phi = (\varphi_x)_{x \in X} \in q^{-1}(\{d\})$.

Proof.

- Suppose $d \neq 0$. Let $\widetilde{B_1} = (X \setminus Y) \cap h^{-1}(B_1)$ and $\widetilde{B_2} = (X \setminus Y) \cap h^{-1}(B_2)$. There is no loss in generality in assuming that $\mu(\widetilde{B_1})$ and $\mu(\widetilde{B_2})$ are finite since μ is σ-finite. Let $0 < a < \frac{\epsilon}{h_\infty^2 b^2}$. Since $\dim(L^2(Y, \mu; \mathbf{C})) \geq n$, we can pick an a-tight frame $(\varphi_y)_{y \in Y} \in \mathcal{F}^{\mathbf{C}}_{(Y,\mu),n)}$. Let $\widetilde{h}(x) = (\langle b, d\rangle - \int_Y h(y)|\langle b, \varphi_y\rangle|^2 d\mu(y))\, h(x)$ for all $x \in X$. Notice that we have

$$Re(\widetilde{h}(x)) > 0 \qquad \mu - \text{almost everywhere on } \widetilde{B_1},$$

$$Im(\widetilde{h}(x)) < 0 \qquad \mu - \text{almost everywhere on } \widetilde{B_1},$$

$$Re(\widetilde{h}(x)) > 0 \qquad \mu - \text{almost everywhere on } \widetilde{B_2},$$

and

$$Im(\widetilde{h}(x)) > 0 \qquad \mu - \text{almost everywhere on } \widetilde{B_2}.$$

Let

$$A = \frac{1}{\frac{\langle -Im(\tilde{h}), Re(\tilde{h})\rangle_{L^2(\tilde{B_1},\mu;\mathbf{C})}}{Im(\tilde{h})^2_{L^2(\tilde{B}1,\mu;\mathbf{C})}} + \frac{\langle Im(\tilde{h}), Re(\tilde{h})\rangle_{L^2(\tilde{B_2},\mu;\mathbf{C})}}{Im(\tilde{h})^2_{L^2(\tilde{B}2,\mu;\mathbf{C})}}} > 0$$

and

$$g(x) = \sqrt{A}\frac{\sqrt{-Im(\widetilde{h}(x))}}{Im(\widetilde{h})_{L^2(\widetilde{B_1},\mu;\mathbf{C})}} 1_{\widetilde{B_1}}(x) + \sqrt{A}\frac{\sqrt{Im(\widetilde{h}(x))}}{Im(\widetilde{h})_{L^2(\widetilde{B_2},\mu;\mathbf{C})}} 1_{\widetilde{B_2}}(x)$$

for all $x \in X \setminus Y$.

Then it is easily seen that

$$\left(\int_{X\setminus Y} h(x)|g(x)|^2 d\mu(x)\right)\left(-\langle b,d\rangle + \int_Y h(y)|\langle b,\varphi_y\rangle|^2 d\mu(y)\right) = -1. \tag{5.1}$$

Consider $(\varphi_x)_{x\in X\setminus Y}$ defined by $\varphi_x = g(x)\left(-d + \int_Y h(y)\langle b,\varphi_y\rangle\varphi_y d\mu(y)\right)$ for all $x \in X \setminus Y$. Then $\Phi = (\varphi_x)_{x\in X} \in \mathcal{F}^{\mathbf{C}}_{(X,\mu),n}$ since we have only completed $(\varphi_y)_{y\in Y}$ by a function in $L^2(X \setminus Y,\mu;\mathbf{C}^n)$. Moreover

$$\begin{aligned}
q(\Phi) &= \textstyle\int_X h(x)\langle b,\varphi_x\rangle\varphi_x d\mu(x)\\
&= \textstyle\int_Y h(y)\langle b,\varphi_y\rangle\varphi_y d\mu(y)\\
&\quad + \textstyle\int_{X\setminus Y} h(x)\langle b, g(x)\left(-d + \int_Y h(y)\langle b,\varphi_y\rangle\varphi_y d\mu(y)\right)\rangle\\
&\quad .g(x)\left(-d + \textstyle\int_Y h(y)\langle b,\varphi_y\rangle\varphi_y d\mu(y)\right) d\mu(x)\\
&= \textstyle\int_Y h(y)\langle b,\varphi_y\rangle\varphi_y d\mu(y)\\
&\quad - \underbrace{\left[\int_{X\setminus Y} h(x)|g(x)|^2\left(-\langle b,d\rangle + \int_Y h(y)|\langle b,\varphi_y\rangle|^2 d\mu(y)\right) d\mu(x)\right]}_{=-1} d\\
&\quad + \underbrace{\left[\int_{X\setminus Y} h(x)|g(x)|^2\left(-\langle b,d\rangle + \int_Y h(y)|\langle b,\varphi_y\rangle|^2 d\mu(y)\right) d\mu(x)\right]}_{=-1}\\
&\quad .\left(\textstyle\int_Y h(y)\langle b,\varphi_y\rangle\varphi_y d\mu(y)\right)\\
&= d
\end{aligned}$$

using equality 1.

- Suppose $d = 0$. Let $\widetilde{B_1} = (X \setminus Y) \cap h^{-1}(B_1)$ and $\widetilde{B_2} = (X \setminus Y) \cap h^{-1}(B_2)$.

Suppose first that $\mu(\widetilde{B_1}), \mu(\widetilde{B_2}) > 0$. There is no loss in generality in assuming that $\mu(\widetilde{B_1})$ and $\mu(\widetilde{B_2})$ are finite since μ is σ-finite. Since $\dim(L^2(Y, \mu; \mathbf{C})) \geq n$, we can pick a frame $(\varphi_y)_{y \in Y} \in \mathcal{F}^{\mathbf{C}}_{(Y,\mu),n}$. Let $\widetilde{h}(x) = -\left(\int_Y h(y)|\langle b, \varphi_y\rangle|^2 d\mu(y)\right) h(x)$ for all $x \in X$. Notice that we have

$$\begin{aligned} Re(\widetilde{h}(x)) > 0 &\qquad \mu-\text{almost everywhere on } \widetilde{B_1}, \\ Im(\widetilde{h}(x)) < 0 &\qquad \mu-\text{almost everywhere on } \widetilde{B_1}, \\ Re(\widetilde{h}(x)) > 0 &\qquad \mu-\text{almost everywhere on } \widetilde{B_2}, \end{aligned}$$

and

$$Im(\widetilde{h}(x)) > 0 \qquad \mu-\text{almost everywhere on } \widetilde{B_2}.$$

Let

$$A = \frac{1}{\frac{\langle -Im(\tilde{h}), Re(\tilde{h})\rangle_{L^2(\tilde{B}_1,\mu;\mathbf{C})}}{Im(\tilde{h})^2_{L^2(\tilde{B}1,\mu;\mathbf{C})}} + \frac{\langle Im(\tilde{h}), Re(\tilde{h})\rangle_{L^2(\tilde{B}_2,\mu;\mathbf{C})}}{Im(\tilde{h})^2_{L^2(\tilde{B}2,\mu;\mathbf{C})}}} > 0$$

and

$$g(x) = \sqrt{A}\frac{\sqrt{-Im(\widetilde{h}(x))}}{Im(\widetilde{h})_{L^2(\widetilde{B_1},\mu;\mathbf{C})}} 1_{\widetilde{B_1}}(x) + \sqrt{A}\frac{\sqrt{Im(\widetilde{h}(x))}}{Im(\widetilde{h})_{L^2(\widetilde{B_2},\mu;\mathbf{C})}} 1_{\widetilde{B_2}}(x)$$

for all $x \in X \setminus Y$.
Then it is easily seen that

$$\left(\int_{X\setminus Y} h(x)|g(x)|^2 d\mu(x)\right)\left(\int_Y h(y)|\langle b, \varphi_y\rangle|^2 d\mu(y)\right) = -1. \tag{5.2}$$

Consider $(\varphi_x)_{x\in X\setminus Y}$ defined by $\varphi_x = g(x)\int_Y h(y)\langle b, \varphi_y\rangle\varphi_y d\mu(y)$ for all $x \in X \setminus Y$. Then $\Phi = (\varphi_x)_{x\in X} \in \mathcal{F}^{\mathbf{C}}_{(X,\mu),n}$ since we have only completed $(\varphi_y)_{y\in Y}$ by a function in $L^2(X \setminus Y, \mu; \mathbf{C}^n)$. Moreover

$$\begin{aligned} q(\Phi) &= \textstyle\int_X h(x)\langle b, \varphi_x\rangle\varphi_x d\mu(x) \\ &= \textstyle\int_Y h(y)\langle b, \varphi_y\rangle\varphi_y d\mu(y) \\ &\quad + \textstyle\int_{X\setminus Y} h(x)\langle b, g(x)\left(\int_Y h(y)\langle b, \varphi_y\rangle\varphi_y d\mu(y)\right)\rangle \\ &\quad .g(x)\left(\textstyle\int_Y h(y)\langle b, \varphi_y\rangle\varphi_y d\mu(y)\right) d\mu(x) \\ &= \textstyle\int_Y h(y)\langle b, \varphi_y\rangle\varphi_y d\mu(y) \\ &\quad + \underbrace{\left[\int_{X\setminus Y} h(x)|g(x)|^2\left(\int_Y h(y)|\langle b, \varphi_y\rangle|^2 d\mu(y)\right) d\mu(x)\right]}_{=-1} \\ &\quad .\left(\textstyle\int_Y h(y)\langle b, \varphi_y\rangle\varphi_y d\mu(y)\right) \\ &= 0 \end{aligned}$$

using equality 2.

Now let $\widetilde{B_3} = (X \setminus Y) \cap h^{-1}(B_3)$ and suppose instead that $\mu(\widetilde{B_3}) > 0$. There is no loss in generality in assuming that $\mu(\widetilde{B_3})$ is finite since μ is σ-finite. Since $\dim(L^2(Y, \mu; \mathbf{C})) \geq n$, we can pick a frame $(\varphi_y)_{y \in Y} \in \mathcal{F}^{\mathbf{C}}_{(Y,\mu),n}$. Let $\widetilde{h}(x) = -\left(\int_Y h(y)|\langle b, \varphi_y\rangle|^2 d\mu(y)\right) h(x)$ for all $x \in X$. Notice that we have

$$\widetilde{h}(x) > 0 \qquad \mu - \text{almost everywhere on } \widetilde{B_3}.$$

Let

$$g(x) = \frac{\sqrt{\widetilde{h}(x)}}{\widetilde{h}_{L^2(\widetilde{B_3},\mu;\mathbf{C})}} 1_{\widetilde{B_3}}(x) \text{ for all } x \in X \setminus Y$$

Then it is easily seen that

$$\left(\int_{X \setminus Y} h(x)|g(x)|^2 d\mu(x)\right)\left(\int_Y h(y)|\langle b, \varphi_y\rangle|^2 d\mu(y)\right) = -1. \tag{5.3}$$

Consider $(\varphi_x)_{x \in X \setminus Y}$ defined by $\varphi_x = g(x) \int_Y h(y)\langle b, \varphi_y\rangle \varphi_y d\mu(y)$ for all $x \in X \setminus Y$. Then $\Phi = (\varphi_x)_{x \in X} \in \mathcal{F}^{\mathbf{C}}_{(X,\mu),n}$ since we have only completed $(\varphi_y)_{y \in Y}$ by a function in $L^2(X \setminus Y, \mu; \mathbf{C}^n)$. Moreover we can prove that $q(\Phi) = 0$ as before using equality 3.

□

6. Examples of subspsaces in which the space of continuous frames is relatively dense

Corollary 6.1. *Let $n \in \mathbf{N}^*$ and $T : L^2(X, \mu; \mathbf{F}^n) \to \mathbf{F}$ be a non-zero linear form.*
Suppose that $\dim(L^2(X, \mu; \mathbf{F})) \geq n$.

Then since $T^{-1}(\{d\})$ is affine and by corollaries 4.1 and 5.1, for all $d \neq 0$, $\mathcal{F}^{\mathbf{F}}_{(X,\mu),n} \cap T^{-1}(\{d\})$ is dense in $T^{-1}(\{d\})$

Corollary 6.2. *Let (X, Σ, μ) be a measure space, $d \in \mathbf{C}^n$, and $h \in L^2(X, \mu; \mathbf{F})$ such that $T^{-1}(\{d\})$ contains a continuous frame $\Phi = (\varphi_x)_{x \in X}$ (see for instance corollary 5.2 and proposition 5.3), where*

$$T : \begin{cases} L^2(X, \mu; \mathbf{F}^n) & \to \mathbf{F}^n \\ F = (f_x)_{x \in X} & \mapsto \int_X h(x) f_x d\mu(x) \end{cases}.$$

Then since $T^{-1}(\{d\})$ is affine, by corollary 4.1, $\mathcal{F}^{\mathbf{F}}_{(X,\mu),n} \cap T^{-1}(\{d\})$ is dense in $T^{-1}(\{d\})$.

Corollary 6.3. *Let (X, Σ, μ) be a measure space, $l \in \mathbf{N}^*$, $(X_j)_{j\in[1,l]}$ a partition of X by measurable subsets, $h \in L^2(X, \mu; \mathbf{F})$, and $D \in \mathbf{F}^n$ such that $W^{-1}(\{D\})$ contains a continuous frame $\Phi = (\varphi_x)_{x\in X}$ (see for instance proposition 5.4), where*

$$W : \begin{cases} L^2(X, \mu; \mathbf{F}^n) & \to \prod_{j\in[1,l]} \mathbf{F}^n \\ F = (f_x)_{x\in X} & \mapsto (\int_{X_j} h(x) f_x)_{j\in[1,l]} \end{cases} .$$

Then since since $W^{-1}(\{d\})$ is affine, and by corollary 4.1, $\mathcal{F}^{\mathbf{F}}_{(X,\mu),n} \cap W^{-1}(\{D\})$ is dense in $W^{-1}(\{D\})$.

Remark 6.1. *Consider the function q of proposition 5.5. If $\Phi = (\varphi_x)_{x\in X} \in q^{-1}(\{0\})$, then we also have $\Phi \in q^{-1}(\{0\}) \cap (q^{-1}(\{0\}) - \Phi)$ since $q(2\Phi) = 0$.*

Proposition 6.1. *Consider the function q of proposition 5.5. Let $\Phi = (\varphi_x)_{x\in X} \in q^{-1}(\{0\})$ and $U = (u_x)_{x\in X} \in q^{-1}(\{0\}) \cap (q^{-1}(\{0\}) - \Phi)$. Then for all $\lambda, \mu \in \mathbf{R}$, $\lambda\Phi + \mu U \in q^{-1}(\{0\}) \cap (q^{-1}(\{0\}) - \Phi)$.*
In particular, $q^{-1}(\{0\}) \cap (q^{-1}(\{0\}) - \Phi)$ is a star domain relatively to Φ.

Proof. Let $\lambda, \mu \in \mathbf{R}$. Let s be the sesquilinear form

$$\begin{array}{ll} L^2(X, \mu; \mathbf{C}^n) & \to \mathbf{C}^n \\ (F, G) & \mapsto \int_X \langle b, g_x \rangle f_x d\mu(x) \end{array}$$

. We have

$$\begin{aligned} q(\lambda\Phi + \mu U) &= \lambda^2 \underbrace{q(\Phi)}_{0} + \lambda\mu(s(\Phi, U) + s(U, \Phi)) + \mu^2 \underbrace{q(U)}_{0} \\ &= \lambda\mu(\underbrace{q(\Phi + U)}_{0} - \underbrace{q(\Phi)}_{0} - \underbrace{q(U)}_{0}) \\ &= 0. \end{aligned}$$

We can show similarly that $q((\lambda + 1)\Phi + \mu U) = 0$. □

Corollary 6.4. *Let (X, Σ, μ) be a σ-finite measure space, $b \in \mathbf{C}^n \setminus \{0\}$, and $h \in L^\infty(X, \mu; \mathbf{C})$ such that $q^{-1}(\{0\})$ contains a continuous frame $\Phi = (\varphi_x)_{x\in X}$ (see for instance proposition 5.5), where*

$$q : \begin{cases} L^2(X, \mu; \mathbf{C}^n) & \to \mathbf{C}^n \\ F = (f_x)_{x\in X} & \mapsto \int_X h(x) \langle b, f_x \rangle f_x d\mu(x) \end{cases} .$$

Then by proposition 6.1 and corollary 4.1, $\mathcal{F}^{\mathbf{C}}_{(X,\mu),n} \cap q^{-1}(\{0\}) \cap (q^{-1}(\{0\}) - \Phi)$ *is dense in* $q^{-1}(\{0\}) \cap (q^{-1}(\{0\}) - \Phi)$.

Acknowledgement

The first author is financially supported by the *Centre National pour la Recherche Scientifique et Technique* of Morocco.

References

[1] D. Agrawal, *The complete structure of linear and nonlinear deformations of frames on a Hilbert space.* Master thesis, 2016.

[2] D. Agrawal and J. Knisley, “Fiber Bundles and Parseval Frames”, 2015. *arXiv*:1512.03989

[3] R. Balan, P. G. Casazza, C. Heil and Z. Landau, “Density, Overcompleteness, and Localization of Frames. I. Theory”, *Journal of Fourier Analysis and Applications,* vol. 12, pp. 105-143, 2006. doi: 10.1007/s00041-006-6022-0

[4] R. Balan, P. G. Casazza, C. Heil and Z. Landau, “Density, Overcompleteness, and Localization of Frames. II. Gabor Systems”, *Journal of Fourier Analysis and Applications,* vol. 12, pp. 307-344, 2006. doi: 10.1007/s00041-005-5035-4

[5] E. Bardelli and A. C. G. Mennucci, “Probability measures on infinite-dimensional Stiefel manifolds”, *Journal of Geometric Mechanics,* vol. 9, pp. 291-316, 2017. doi: 10.3934/jgm.2017012

[6] M. Bownik, “Connectivity and density in the set of framelets”, *Mathematical Research Letters,* vol. 14, no. 2, pp. 285-293, 2017. doi: 10.4310/MRL.2007.v14.n2.a10

[7] J. Cahill, D. Mixon and N. Strawn, “Connectivity and irreducibility of algebraic varieties of finite unit norm tight frames”, *SIAM Journal on Applied Algebra and Geometry,* vol. 1, no. 1, pp. 38-72, 2017. doi: 10.48550/arXiv.1311.4748

[8] P. G. Casazza, “The art of frame theory”, *Taiwanese Journal of Mathematics,* vol. 4, no. 2, pp. 129-201, 2000. doi: 10.11650/twjm/1500407227

[9] P. G. Casazza and G. Kutyniok (eds.). *Finite frames, theory and applications. Applied and Numerical Harmonic Analysis.* New York: Birkhäuser, 2013.

[10] O. Christensen, *An Introduction to frames and Riesz Bases.* 2nd ed. Switzerland: Birkhäuser, 2016.

[11] O. Christensen, B. Deng and C. Heil, “Density of Gabor Frames”, *Applied and Computational Harmonic Analysis,* vol. 7, pp. 292-304, 1999. doi: 10.1006/acha.1999.0271

[12] I. Daubechies, A. Grossmann, and Y. Meyer, “Painless nonorthogonal expansions”, *Journal of Mathematical Physics*, vol. 27, pp. 1271-1283, 1986. doi: 10.1063/1.527388

[13] R. Duffin and A. Schaeffer, ”A class of non-harmonic Fourier series”, *Transactions of the American Mathematical Society,* vol. 72, pp. 341-366, 1952. doi: 10.2307/1990760

[14] K. Dykema and N. Strawn, “Manifold structure of spaces of spherical tight frames”, *International Journal of Pure and Applied Mathematics,* vol. 28, no. 2, pp. 217-256, 2006.

[15] M. Frank and D. R. Larson, “Frames in Hilbert C*-Modules and C*-Algebras”, *Journal of Operator Theory,* vol. 48, no. 2, pp. 273-314, 2002.

[16] D. Gabor, “Theory of communications”, *Journal of the Institution of Electrical Engineers,* vol. 93, pp. 429-457, 1946.

[17] G. Garrigos, E. Hernandez, H. iki, F. Soria, G. Weiss, and E. Wilson, “Connectivity in the set of Tight Frame Wavelets (TFW)”, *Glasnik matematicki,* vol. 38, no. 1, 2003.

[18] D. Han and D. R. Larson, “On the Orthogonality of Frames and the Density and Connectivity of Wavelet Frames”, *Acta Applicandae Mathematicae,* vol. 107, pp. 211-222, 2009. doi: 10.1007/s10440-008-9420-2

[19] P. Harms and A. C. G. Mennucci, “Geodesics in infinite dimensional Stiefel and Grassmann manifolds”, *Comptes Rendus Mathematique,* vol. 350, nos. 15-16, pp. 773-776, 2012. doi: 10.1016/j.crma.2012.08.010

[20] A. Hatcher, *Algebraic topology.* Cambridge University Press, 2005.

[21] C. Heil, “History and Evolution of the Density Theorem for Gabor Frames”, *Journal of Fourier Analysis and Applications,* vol. 13, pp. 113-166, 2007. doi: 10.1007/s00041-006-6073-2

[22] O. Henkel, “Sphere-packing bounds in the Grassmann and Stiefel manifolds”, *IEEE Transactions on Information Theory*, vol. 51, no. 10, pp. 3445-3456, 2005. doi: 10.1109/TIT.2005.855594

[23] D. Husemoller, *Fibre Bundles.* 3rd ed. New York: Springer-Verlag, 1994.

[24] I. M. James, *The topology of Stiefel manifolds.* London Mathematical Society Lecture Note Series. Cambridge: University Press, 1976.

[25] V. Jurdjevic, I. Markina, and F. Silva Leite, "Extremal Curves on Stiefel and Grassmann Manifolds", *The Journal of Geometric Analysis,* vol. 30, pp. 3948-3978, 2020. doi: 10.1007/s12220-019-00223-1

[26] D. Labate and E. Wilson, "Connectivity in the set of Gabor frames", *Applied and Computational Harmonic Analysis,* vol. 18, no. 1, pp. 123-136, 2005. doi: 10.1016/j.acha.2004.09.003

[27] T. Needham and C. Shonkwiler, "Symplectic geometry and connectivity of spaces of frames", 2019. *arXiv:*1804.05899v2

[28] D. M. Speegle, "The s-elementary wavelets are path-connected", *Proceedings of the American Mathematical Society,* vol. 127, pp. 223-233, 1999.

[29] N. Strawn, "Finite frame varieties: nonsingular points, tangent spaces, and explicit local parameterizations", *Journal of Fourier Analysis and Applications,* vol. 17, pp. 821-853, 2011. doi: 10.1007/s00041-010-9164-z

[30] N. Strawn, *Geometric structures and optimization on spaces of finite frames.* PhD thesis, 2011.

[31] N. Strawn, *Geometry and constructions of finite frames.* Master thesis, 2007.

[32] S. F. D Waldron, *An introduction to finite tight frames. Applied and Numerical Harmonic Analysis.* New York: Birkhäuser, 2018.

Nizar El Idrissi
Laboratoire : Equations aux dèrivèes partielles,
Algèbre et Gèomètrie spectrales.
Dèpartement de mathèmatiques,
Facultè des sciences,
Universitè Ibn Tofail,
14000 Kènitra
Maroc
e-mail: nizar.elidrissi@uit.ac.ma
Corresponding author

Pr. Samir Kabbaj
Laboratoire : Equations aux dèrivèes partielles,
Algèbre et Gèomètrie spectrales.
Dèpartement de Mathèmatiques,
Facultè des Sciences,
Universitè Ibn Tofail,
14000 Kènitra
Maroc
e-mail: samir.kabbaj@uit.ac.ma

and

Pr. Brahim Moalige
Laboratoire : Equations aux dèrivèes partielles,
Algèbre et Gèomètrie spectrales
Dèpartement de Mathèmatiques,
Facultè des Sciences,
Universitè Ibn Tofail,
14000 Kènitra
Maroc
e-mail: brahim.moalige@uit.ac.ma