Proyecciones Journal of Mathematics
Vol. 41, N^o 5, pp. 1035-1050, October 2022.
Universidad Católica del Norte
Antofagasta - Chile

10.22199/issn.0717-6279-4789

Lie (Jordan) centralizers on alternative algebras

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Received : March 2021. Accepted : June 2022

Abstract

In this article, we study Lie (Jordan) centralizers on alternative algebras and prove that every multiplicative Lie centralizer has proper form on alternative algebras under certain assumptions.

Subjclass: *16W25, 15A78, 47L35.*

Keywords: *alternative algebra, Lie centralizer, centralizer.*

1. Introduction

The study of nonassociative rings has received fair amount of attention during the last few decades. Many authors studied nonassociative algebras (see [22] and references therein), in particular, alternative rings after the discovery of their connection with the theory of projective planes. Let $\mathcal{A}$ be an alternative ring unless otherwise mentioned. For any $x, y \in \mathcal{A}$, $x \circ y = xy + yx$ will denote the Jordan product on $\mathcal{A}$. We recall that a ring $\mathcal{A}$ (not necessarily associative or commutative) is called an alternative ring if $\mathcal{A}$ satisfies $x^2y = x(xy)$ and $yx^2 = (yx)x$ for all $x, y \in \mathcal{A}$ and flexible if $x(yx) = (xy)x$ holds for all $x, y \in \mathcal{A}$. It can be easily seen that all associative rings are alternative and any alternative ring is flexible. Hence the product xyx will denote the product $x(yx)$ or $(xy)x$ for all $x, y \in \mathcal{A}$. An alternative ring $\mathcal{A}$ is said to be k-torsion free if $kx = 0$ implies that $x = 0$ for $k \in \mathbf{N}$ and for all $x \in \mathcal{A}$. For any $x, y \in \mathcal{A}$, $[x, y] = xy - yx$ will denote the Lie product on $\mathcal{A}$. The commutative center of an algebra $\mathcal{A}$ is defined by $\mathrm{Z}(\mathcal{A}) = \{a \in \mathcal{A} \mid [a, x] = 0 \text{ for all } x \in \mathcal{A}\}$.

Remark 1.1. *[9, Theorem 1.1] Let $\mathcal{A}$ be a 3-torsion free alternative ring. Then $\mathcal{A}$ is a prime ring if and only if $x\mathcal{A} \cdot y = 0$ (or $x \cdot \mathcal{A}y = 0$) implies $x = 0$ or $y = 0$ for $x, y \in \mathcal{A}$.*

In the remaining part of the paper, let $\mathcal{A}$ be an alternative ring with a nontrivial idempotent e_1 and formally set $e_0 = 1 - e_1$ ($\mathcal{A}$ need not have an identity element). It can be easily seen that $(e_ix)e_j = e_i(xe_j)$, where $i, j = 0, 1$ for all $x \in \mathcal{A}$. By Pierce decomposition $\mathcal{A} = \mathcal{A}_{11} + \mathcal{A}_{10} + \mathcal{A}_{01} + \mathcal{A}_{00}$, where $\mathcal{A}_{ij} = e_i\mathcal{A}e_j$ for $i, j \in \{0, 1\}$. The symbol x_{ij} denote an arbitrary element by $\mathcal{A}_{ij}$ and any element $x \in \mathcal{A}$ can be expressed as $x = x_{11} + x_{10} + x_{01} + x_{00}$. Pierce decomposition of alternative rings satisfy the following relations:

(i) $\mathcal{A}_{ij}\mathcal{A}_{jk} \subseteq \mathcal{A}_{ik}$, when $i, j, k \in \{0, 1\}$,

(ii) $\mathcal{A}_{ij}\mathcal{A}_{ij} \subseteq \mathcal{A}_{ji}$ with $x_{ij}^2 = x_{ij}y_{ij} + y_{ij}x_{ij} = 0$,

(iii) $\mathcal{A}_{ij}\mathcal{A}_{kl} = 0,\ (j \neq k),\ (i, j) \neq (k, l)$.

for all $x_{ij}, y_{ij} \in \mathcal{A}_{ij}$.

For $x, y \in \mathcal{A}$ $[x, y]$ (resp. $x \circ y$) will denote the Lie product $xy - yx$ (resp. Jordan product $xy + yx$). A map (not necessarily linear) $L : \mathcal{A} \to \mathcal{A}$ is called multiplicative left centralizer (resp. multiplicative right centralizer) if $L(xy) = L(x)y$ (resp. $L(xy) = xL(y)$) for all $x, y \in \mathcal{A}$. Further, L is called a multiplicative centralizer if it is both multiplicative left centralizer as well as multiplicative right centralizer. A map (not necessarily linear) $L : \mathcal{A} \to \mathcal{A}$ is called a multiplicative Jordan centralizer if $L(x \circ y) = L(x) \circ y = x \circ L(y)$ for all $x, y \in \mathcal{A}$. A map (not necessarily linear) $L : \mathcal{A} \to \mathcal{A}$ is called a multiplicative Lie centralizer if $L([x, y]) = [L(x), y] = [x, L(y)]$ for all $x, y \in \mathcal{A}$.

Characterizing the interrelation between the multiplicative and additive maps on algebraic structures is an interesting topic and has received fair amount of attention of many mathematicians (see for reference [29, 9, 10] where further references can be found). It was Martindale [29], who first studied this problem and raised the question : When is a multiplicative map additive? He answered this question for a multiplicative isomorphism of an associative ring with a family of idempotents under certain assumptions. More precisely, he proved the following result:

Theorem 1.2. *[29, Theorem 1] Let $\mathcal{A}$ be a ring (not necessarily with identity element) containing a family $\{e_\alpha : \alpha \in \Lambda\}$ of idempotents which satisfies :*

(*i*) *$x\mathcal{A} = \{0\}$ implies $x = 0$,*

(*ii*) *if $e_\alpha \mathcal{A} x = \{0\}$ for each $\alpha \in \Lambda$, then $x = 0$ (and hence $\mathcal{A}x = \{0\}$ implies $x = 0$),*

(*iii*) *for each $\alpha \in \Lambda$, $e_\alpha x e_\alpha \mathcal{A}(1 - e_\alpha) = \{0\}$ implies $e_\alpha x e_\alpha = \{0\}$.*

Then any multiplicative bijective map from $\mathcal{A}$ onto an arbitrary ring $\mathcal{A}'$ is additive.

Ferreira and Nascimento [19] initiated the study of this problem for nonassociative rings named as alternative rings for derivable maps. Further this problem was studied by Ferreira and Ferreira [10, 9] for Jordan (triple) derivable map on alternative rings. Later on many authors studied the different maps on alternative rings or algebras see [32, 18, 17, 31, 30] and references therein. Centralizers on rings as well as algebras have been extensively investigated by many mathematicians see [6, 5, 4, 1, 2, 3] and references therein. In this paper, we obtain the necessary and sufficient conditions for a Lie centralizer map to be proper on alternative algebras. Further, we prove that every Jordan centralizer is a centralizer on alternative algebras under certain assumptions.

Lemma 1.1. *[33, Lemma 8] For $z_{ii} \in \mathrm{Z}(\mathcal{A}_{ii}), i = 1, 2$, there exists an element $z \in \mathrm{Z}(\mathcal{A})$ such that $z_{ii} = z e_i$.*

2. Lie Centralizer

Theorem 2.1. *Let $\mathcal{A}$ be a 2-torsion free unital alternative algebra with nontrivial idempotent and $\mathrm{L} : \mathcal{A} \to \mathcal{A}$ be a multiplicative Lie centralizer satisfying the following for $i \neq j = 1, 2$:*

1. $e_i \mathrm{L}(\mathcal{A}_{jj}) e_i \subseteq \mathrm{Z}(\mathcal{A}) e_i$,
2. $x_{ii}\mathcal{A}_{ij} = 0$ *or* $\mathcal{A}_{ji}x_{ii} = 0$ *then* $x_{ii} = 0$.

Then L *has the form* $\mathrm{L} = \delta + \tau$ *where* $\delta : \mathcal{A} \to \mathcal{A}$ *is an additive centralizer and* $\tau : \mathcal{A} \to \mathrm{Z}(\mathcal{A})$ *maps commutators into the zero.*

We prove the above theorem via series of lemmas as follows:

Lemma 2.1. $\mathrm{L}(0) = 0$.

Proof. For any $x \in \mathcal{A}$, we have $\mathrm{L}(0) = \mathrm{L}([0, x]) = [0, \mathrm{L}(x)] = 0$. □

Lemma 2.2. L *is almost additive.*

Proof. For any $x, y, z \in \mathcal{A}$, it follows that

$$
\begin{array}{rcl}
[\mathrm{L}(x+y), z] & = & \mathrm{L}([(x+y), z]) \\
 & = & [x+y, \mathrm{L}(z)] \\
 & = & [x, \mathrm{L}(z)] + [y, \mathrm{L}(z)] \\
 & = & \mathrm{L}([(x), z]) + \mathrm{L}([(y), z]) \\
 & = & [\mathrm{L}(x), z] + [\mathrm{L}(y), z] \\
\mathrm{L}(x+y) - & \mathrm{L}(x) & -\mathrm{L}(y) \in \mathrm{Z}(\mathcal{A}).
\end{array}
$$

Hence L is almost additive on $\mathcal{A}$. □

Lemma 2.3. $\mathrm{L}(e_i) \in \mathrm{Z}(\mathcal{A})$ *for* $i \neq j = 1, 2$.

Proof. By definition of Lie centralizer, we have

$$\begin{aligned} \mathrm{L}([e_1,e_2]) &= [\mathrm{L}(e_1),e_2] = [e_1,\mathrm{L}(e_2)] \\ 0 &= \mathrm{L}(e_1)e_2 - e_2\mathrm{L}(e_1) = e_1\mathrm{L}(e_2) - \mathrm{L}(e_2)e_1. \end{aligned}$$

This implies that $e_2\mathrm{L}(e_1)e_1 = 0 = e_1\mathrm{L}(e_1)e_2$ and $e_2\mathrm{L}(e_2)e_1 = 0 = e_1\mathrm{L}(e_2)e_2$. Now for any $a_{11} \in \mathcal{A}_{11}$, we arrive at

$$\begin{aligned} \mathrm{L}([a_{11},e_1]) &= [a_{11},\mathrm{L}(e_1)] = [\mathrm{L}(a_{11}),e_1] \\ 0 &= a_{11}\mathrm{L}(e_1)e_1 - e_1\mathrm{L}(e_1)a_{11} = e_2\mathrm{L}(a_{11})e_1 - e_1\mathrm{L}(a_{11})e_2. \end{aligned}$$

This leads to $e_1\mathrm{L}(e_1)e_1 \in \mathrm{Z}(\mathcal{A}_{11})$ and $e_2\mathrm{L}(a_{11})e_1 = 0 = e_1\mathrm{L}(a_{11})e_2$. Likewise, we have $e_1\mathrm{L}(e_2)e_1 \in \mathrm{Z}(\mathcal{A}_{11})$. Therefore, in view of Lemma 1.1, we conclude that $\mathrm{L}(e_1) \in \mathrm{Z}(\mathcal{A})$. With similar arguments, we can have $\mathrm{L}(e_2) \in \mathrm{Z}(\mathcal{A})$. □

Lemma 2.4. $\mathrm{L}(\mathcal{A}_{ij}) \subseteq \mathcal{A}_{ij}$ *for* $i \neq j,\ i,j = 1,2$.

Proof. For any $a_{12} \in \mathcal{A}_{12}$, we find that

$$\begin{aligned} \mathrm{L}([e_1,a_{12}]) &= [e_1,\mathrm{L}(a_{12})] \\ \mathrm{L}(a_{12}) &= e_1\mathrm{L}(a_{12}) - \mathrm{L}(a_{12})e_1 \\ \mathrm{L}(a_{12}) &= e_1\mathrm{L}(a_{12})e_2 - e_2\mathrm{L}(a_{12})e_1. \end{aligned}$$

Then we see that $e_1\mathrm{L}(a_{12})e_1 = 0 = e_2\mathrm{L}(a_{12})e_2$. Also, on using 2-torsion freeness, we have $e_2\mathrm{L}(a_{12})e_1 = 0$. Hence $\mathrm{L}(a_{12}) = e_1\mathrm{L}(a_{12})e_2 \in \mathcal{A}_{12}$ for all $a_{12} \in \mathcal{A}_{12}$. With similar calculations, we get that $\mathrm{L}(a_{21}) \in \mathcal{A}_{21}$ for all $a_{21} \in \mathcal{A}_{21}$. □

Lemma 2.5. $\mathrm{L}(\mathcal{A}_{ii}) \subseteq \mathcal{A}_{ii} + \mathrm{Z}(\mathcal{A})$ *for* $i = 1,2$.

Proof. Consider $i = 1$. For any $a_{11} \in \mathcal{A}_{11}$, we have

$$\begin{aligned} \mathrm{L}([a_{11},e_1]) &= [a_{11},\mathrm{L}(e_1)] = [\mathrm{L}(a_{11}),e_1] \\ 0 &= a_{11}\mathrm{L}(e_1)e_1 - e_1\mathrm{L}(e_1)a_{11} = e_2\mathrm{L}(a_{11})e_1 - e_1\mathrm{L}(a_{11})e_2. \end{aligned}$$

It follows that $e_2\mathrm{L}(a_{11})e_1 = 0 = e_1\mathrm{L}(a_{11})e_2$ for all $a_{11} \in \mathrm{A}_{11}$. Similarly, we get $e_2\mathrm{L}(a_{22})e_1 = 0 = e_1\mathrm{L}(a_{22})e_2$ for all $a_{22} \in \mathrm{A}_{22}$. Also for any $a_{11} \in \mathcal{A}_{11}$ and $a_{22} \in \mathcal{A}_{22}$, we have

$$\begin{aligned} \mathrm{L}([a_{11}, a_{22}]) &= [\mathrm{L}(a_{11}), a_{22}] = [a_{11}, \mathrm{L}(a_{22})] \\ 0 &= [e_2\mathrm{L}(a_{11})e_2, a_{22}] = [a_{11}, e_1\mathrm{L}(a_{22})e_1]. \end{aligned}$$

Hence we obtain that $e_2\mathrm{L}(a_{11})e_2 \in \mathrm{Z}(\mathcal{A}_{22})$ and $e_1\mathrm{L}(a_{22})e_1 \in \mathrm{Z}(\mathcal{A}_{11})$. In view of Lemma 1.1, we get

$$\begin{aligned} \mathrm{L}(a_{11}) &= e_1\mathrm{L}(a_{11})e_1 + e_2\mathrm{L}(a_{11})e_2 \\ &= e_1\mathrm{L}(a_{11})e_1 + z_{22} \\ &= e_1\mathrm{L}(a_{11})e_1 + ze_2 \\ &= e_1\mathrm{L}(a_{11})e_1 - ze_1 + z \\ &\in \mathcal{A}_{11} + \mathrm{Z}(\mathcal{A}). \end{aligned}$$

for all $a_{11} \in \mathcal{A}_{11}$. Likewise, we can find for $i = 2$. □

Remark 2.2. *In view of Lemmas 2.1-2.5, we conclude that* $\mathrm{L}(a_{ij}) = b_{ij}$ *and* $\mathrm{L}(a_{ii}) = b_{ii} + z_i$ *for each* $b_{ij}, a_{ij} \in \mathcal{A}_{ij}$ *and* $z_i \in \mathrm{Z}(\mathcal{A})$. *Now let us define a mapping* $\delta : \mathcal{A} \to \mathcal{A}$ *by* $\delta(a_{ij}) = b_{ij},\ a_{ij} \in \mathcal{A}_{ij}$. *For each* $a = a_{11} + a_{12} + a_{21} + a_{22} \in \mathcal{A}$, *we define* $\delta(a) = \sum \delta(a_{ij})$. *Again, define a mapping* $\tau : \mathcal{A} \to \mathrm{Z}(\mathcal{A})$ *by*

$$\tau(a) = \mathrm{L}(a) - \delta(a) \text{ for all } a \in \mathcal{A}.$$

Lemma 2.6. *For any* $a, b \in \mathcal{A}$, *we have*

1. $\tau(a + b) - \tau(a) - \tau(b) \in \mathrm{Z}(\mathcal{A})$,
2. $\delta(a + b) = \delta(a) + \delta(b)$.

Proof. (1) For any $a_{12}, b_{12} \in \mathcal{A}_{12}$, it follows for $i = 1,\ j = 2$,

$$\begin{aligned} \tau(a_{12} + b_{12}) &= \mathrm{L}(a_{12} + b_{12}) - \delta(a_{12} + b_{12}) \\ &= \mathrm{L}(a_{12}) + \mathrm{L}(b_{12}) + z_{a_{12},b_{12}} - \delta(a_{12}) - \delta(b_{12}) \\ &= \tau(a_{12}) + \tau(b_{12}) + z_{a_{12},b_{12}} \\ \tau(a_{12} + b_{12}) - \tau(a_{12}) - \tau(b_{12}) &\in \mathrm{Z}(\mathcal{A}). \end{aligned}$$

Likewise, we can prove other cases and in the end, we get $\tau(a+b)-\tau(a)-\tau(b)\in \mathrm{Z}(\mathcal{A})$ for all $a,b\in\mathcal{A}$.
(2) In view of part (1) it is easy to verify that δ is an additive mapping. □

Lemma 2.7. *For every $a_{ij}, b_{ij}\in\mathcal{A}_{ij}$ and for $i,j=1,2$ we have*

1. $\delta(a_{ii}b_{ij})=\delta(a_{ii})b_{ij}=a_{ii}\delta(b_{ij})$,
2. $\delta(a_{ij}b_{jj})=\delta(a_{ij})b_{jj}=a_{ij}\delta(b_{jj})$,
3. $\delta(a_{ii}b_{ii})=\delta(a_{ii})b_{ii}=a_{ii}\delta(b_{ii})$,
4. $\delta(a_{ij}b_{ij})=\delta(a_{ij})b_{ij}=a_{ij}\delta(b_{ij})$,
5. $\delta(a_{ij}b_{ji})=\delta(a_{ij})b_{ji}=a_{ij}\delta(b_{ji})$.

Proof. (1) Consider the case for $i=1,\ j=2$, we have

$$\begin{aligned}\delta(a_{11}b_{12}) &= \delta([a_{11},b_{12}])\\ &= \mathrm{L}([a_{11},b_{12}])\\ &= [\mathrm{L}(a_{11}),b_{12}]=[a_{11},\mathrm{L}(b_{12})]\\ &= \delta(a_{11})b_{12}=a_{11}\delta(b_{12}).\end{aligned}$$

On similar pattern, we can prove other parts and (2).
(3) For $i=1$ with (1), we have $\delta(a_{11}b_{11}b_{12})=\delta(a_{11}b_{11})b_{12}=a_{11}b_{11}\delta(b_{12})$. On the other hand, we get

$$\begin{aligned}\delta(a_{11}b_{11}b_{12}) &= \delta(a_{11})b_{11}b_{12}=a_{11}\delta(b_{11}b_{12})\\ &= \delta(a_{11})b_{11}b_{12}=a_{11}b_{11}\delta(b_{12})=a_{11}\delta(b_{11})b_{12}.\end{aligned}$$

Now combining last two expressions, we obtain

$$\begin{aligned}(\delta(a_{11}b_{11})-\delta(a_{11})b_{11})b_{12} &= 0,\\ (\delta(a_{11}b_{11})-a_{11}\delta(b_{11}))b_{12} &= 0.\end{aligned}$$

With application of assumption (2), we obtain the result. Likewise we can obtain other cases.
(4) For $i=1,\ j=2$, it follows by 2-torsion freeness

$$\begin{aligned}
2\delta(a_{12}b_{12}) &= \delta([a_{12}, b_{12}]) \\
&= \mathrm{L}([a_{12}, b_{12}]) \\
&= [\mathrm{L}(a_{12}), b_{12}] = [a_{12}, \mathrm{L}(b_{12})] \\
&= \delta(a_{12})b_{12} - b_{12}\delta(a_{12}) = a_{12}\delta(b_{12}) - \delta(b_{12})a_{12} \\
&= 2\delta(a_{12})b_{12} = 2a_{12}\delta(b_{12}) \\
\delta(a_{12}b_{12}) &= \delta(a_{12})b_{12} = a_{12}\delta(b_{12}).
\end{aligned}$$

(5) Again, for $i = 1,\ j = 2$, we have

$$\begin{aligned}
\tau([a_{12}, b_{21}]) &= \mathrm{L}([a_{12}, b_{21}]) - \delta([a_{12}, b_{21}]) \\
&= [\mathrm{L}(a_{12}), b_{21}] - \delta([a_{12}, b_{21}]) = [a_{12}, \mathrm{L}(b_{21})] - \delta([a_{12}, b_{21}]) \\
&= [\delta(a_{12}), b_{21}] - \delta([a_{12}, b_{21}]) = [a_{12}, \delta(b_{21})] - \delta([a_{12}, b_{21}]).
\end{aligned}$$

This implies that

$$\begin{aligned}
\delta(a_{12})b_{21} - b_{21}\delta(a_{12}) - \delta(a_{12}b_{21}) + \delta(b_{21}a_{12}) &= z \in \mathrm{Z}(\mathcal{A}), \\
a_{12}\delta(b_{21}) - \delta(b_{21})a_{12} - \delta(a_{12}b_{21}) + \delta(b_{21}a_{12}) &= z \in \mathrm{Z}(\mathcal{A}).
\end{aligned}$$

Now multiplying a_{12} by left side and b_{21} by right side in above expressions respectively, we arrive at

$$\begin{aligned}
a_{12}(\delta(a_{12})b_{21} - b_{21}\delta(a_{12}) - \delta(a_{12}b_{21}) + \delta(b_{21}a_{12})) &= a_{12}z, \\
(a_{12}\delta(b_{21}) - \delta(b_{21})a_{12} - \delta(a_{12}b_{21}) + \delta(b_{21}a_{12}))b_{21} &= zb_{21}.
\end{aligned}$$

Hence we find that

$$\begin{aligned}
-a_{12}b_{21}\delta(a_{12}) + a_{12}\delta(b_{21}a_{12}) &= a_{12}z = 0, \\
-\delta(b_{21})a_{12}b_{21} + \delta(b_{21}a_{12})b_{21} &= zb_{21} = 0.
\end{aligned}$$

Now in view of (2) we have

$$\begin{aligned}
-\delta(a_{12}b_{21})a_{12} + \delta(a_{12})b_{21}a_{12} &= 0, \\
-b_{21}\delta(a_{12}b_{21}) + b_{21}a_{12}\delta(b_{21}) &= 0.
\end{aligned}$$

On applying assumption (2), we get $\delta(a_{12}b_{21}) = \delta(a_{12})b_{21} = a_{12}\delta(b_{21})$ and follow similarly for other cases. □

Lemma 2.8. *δ is a centralizer.*

Proof. Suppose that $a, b \in \mathcal{A}$, then in view of Lemmas 2.6 and 2.7, we have

$$\begin{aligned}\delta(ab) &= \delta((a_{11}+a_{12}+a_{21}+a_{22})(b_{11}+b_{12}+b_{21}+b_{22}))\\ &= \delta(a_{11}b_{11})+\delta(a_{11}b_{12})+\delta(a_{12}b_{12})+\delta(a_{12}b_{21})+\delta(a_{12}b_{22})\\ &\quad +\delta(a_{21}b_{11})+\delta(a_{21}b_{12})+\delta(a_{21}b_{21})+\delta(a_{22}b_{21})+\delta(a_{22}b_{22})\\ &= \delta(a)b.\end{aligned}$$

Similarly, we can have $\delta(ab) = a\delta(b)$ for all $a, b \in \mathcal{A}$. □

Lemma 2.9. *For any $a, b \in \mathcal{A}$, $\tau([a, b]) = 0$.*

Proof. For any $a, b \in \mathcal{A}$, we have

$$\begin{aligned}\tau([a,b]) &= \mathrm{L}([a,b]) - \delta([a,b])\\ &= [\mathrm{L}(a), b] - \delta(ab) + \delta(ba)\\ &= [\tau(a)+\delta(a), b] - \delta(ab) + \delta(ba)\\ &= [\delta(a), b] - \delta(a)b + b\delta(a)\\ &= 0.\end{aligned}$$

□

Proof. [Proof of Theorem 2.1] In view of Remark 2.2 and Lemmas 2.6-2.9, we conclude that multiplicative Lie centralizer can be written as a sum of additive centralizer and a central map vanishing at commutators on $\mathcal{A}$. □

3. Jordan Centralizer

Theorem 3.1. *Let $\mathcal{A}$ be a 2-torsion free unital alternative algebra with nontrivial idempotent and $\mathrm{J} : \mathcal{A} \to \mathcal{A}$ be a multiplicative Jordan centralizer satisfying for $i, j, k = 1, 2$; $x_{ij}\mathcal{A}_{jk} = 0$ or $\mathcal{A}_{ki}x_{ij} = 0$ then $x_{ij} = 0$. Then J is an additive centralizer.*

We prove the above theorem via series of lemmas as follows:

Lemma 3.1. $\mathrm{J}(0) = 0$.

Proof. For any $x \in \mathcal{A}$, we have $\mathrm{J}(0) = \mathrm{J}(0 \circ x) = 0 \circ \mathrm{J}(x) = 0$. $\square$

Lemma 3.2. *$\mathrm{J}(e_i) \in \mathcal{A}_{ii}$ for $i = 1, 2$.*

Proof. By definition of Jordan centralizer, we find that

$$\begin{aligned}
\mathrm{J}(e_1 \circ e_2) &= \mathrm{J}(e_1) \circ e_2 = e_1 \circ \mathrm{J}(e_2)\\
0 &= \mathrm{J}(e_1)e_2 + e_2\mathrm{J}(e_1) = e_1\mathrm{J}(e_2) + \mathrm{J}(e_2)e_1\\
&= e_1\mathrm{J}(e_1)e_2 + 2e_2\mathrm{J}(e_1)e_2 + e_2\mathrm{J}(e_1)e_1 = e_1\mathrm{J}(e_2)e_2 + 2e_1\mathrm{J}(e_2)e_1 + e_2\mathrm{J}(e_2)e_1.
\end{aligned}$$

This implies that $e_2\mathrm{J}(e_1)e_1 = e_1\mathrm{J}(e_1)e_2 = e_2\mathrm{J}(e_1)e_2 = 0$ and $e_2\mathrm{J}(e_2)e_1 = e_1\mathrm{J}(e_2)e_2 = e_1\mathrm{J}(e_2)e_1 = 0$. $\square$

Lemma 3.3. *$\mathrm{J}(\mathcal{A}_{ij}) \subseteq \mathcal{A}_{ij}$ for $i \neq j,\ i, j = 1, 2$.*

Proof. For any $a_{12} \in \mathcal{A}_{12}$, we find that

$$\begin{aligned}
\mathrm{J}(e_1 \circ a_{12}) &= e_1 \circ \mathrm{J}(a_{12})\\
\mathrm{J}(a_{12}) &= e_1\mathrm{J}(a_{12}) + \mathrm{J}(a_{12})e_1\\
\mathrm{J}(a_{12}) &= 2e_1\mathrm{J}(a_{12})e_1 + e_1\mathrm{J}(a_{12})e_2 + e_2\mathrm{J}(a_{12})e_1.
\end{aligned}$$

Then we see that $e_1\mathrm{J}(a_{12})e_1 = 0 = e_2\mathrm{J}(a_{12})e_2$. Hence $\mathrm{J}(a_{12}) = e_1\mathrm{J}(a_{12})e_2 + e_2\mathrm{J}(a_{12})e_1 \in \mathcal{A}_{12} + \mathcal{A}_{21}$ for all $a_{12} \in \mathcal{A}_{12}$. Since

$$\begin{aligned}
\mathrm{J}(e_1 \circ a_{12}) &= \mathrm{J}(e_1) \circ a_{12}\\
\mathrm{J}(a_{12}) &= \mathrm{J}(e_1)a_{12} + a_{12}\mathrm{J}(e_1)\\
&= e_1\mathrm{J}(e_1)a_{12} + a_{12}\mathrm{J}(e_1)e_2.
\end{aligned}$$

So $e_2\mathrm{J}(a_{12})e_1 = 0$ and $\mathrm{J}(a_{12}) \in \mathcal{A}_{12}$, for all $a_{12} \in \mathcal{A}_{12}$. With similar calculations, we get that $\mathrm{J}(a_{21}) \in \mathcal{A}_{21}$ for all $a_{21} \in \mathcal{A}_{21}$. $\square$

Lemma 3.4. *$\mathrm{J}(\mathcal{A}_{ii}) \subseteq \mathcal{A}_{ii}$ for $i = 1, 2$.*

Proof. Consider $i = 1$. For any $a_{11} \in \mathcal{A}_{11}$, we have

$$\begin{aligned} \mathrm{J}(a_{11} \circ e_2) &= \mathrm{J}(a_{11}) \circ e_2 \\ &= \mathrm{J}(a_{11})e_2 + e_2\mathrm{J}(a_{11}) \\ 0 &= e_2\mathrm{J}(a_{11})e_1 + e_1\mathrm{J}(a_{11})e_2 + 2e_2\mathrm{J}(a_{11})e_2. \end{aligned}$$

It follows that $e_2\mathrm{J}(a_{11})e_1 = e_1\mathrm{J}(a_{11})e_2 = e_2\mathrm{J}(a_{11})e_2 = 0$ for all $a_{11} \in \mathrm{A}_{11}$. Similarly, we get $e_2\mathrm{J}(a_{22})e_1 = e_1\mathrm{J}(a_{22})e_2 = e_1\mathrm{J}(a_{22})e_1 = 0$ for all $a_{22} \in \mathrm{A}_{22}$. □

Lemma 3.5. *For every $a_{ij}, b_{ij} \in \mathcal{A}_{ij}$ and for $i, j = 1, 2$, we have*

1. $\mathrm{J}(a_{ii}b_{ij}) = \mathrm{J}(a_{ii})b_{ij} = a_{ii}\mathrm{J}(b_{ij})$,
2. $\mathrm{J}(a_{ij}b_{jj}) = \mathrm{J}(a_{ij})b_{jj} = a_{ij}\mathrm{J}(b_{jj})$,
3. $\mathrm{J}(a_{ii}b_{ii}) = \mathrm{J}(a_{ii})b_{ii} = a_{ii}\mathrm{J}(b_{ii})$,
4. $\mathrm{J}(a_{ij}b_{ij}) = \mathrm{J}(a_{ij})b_{ij} = a_{ij}\mathrm{J}(b_{ij})$,
5. $\mathrm{J}(a_{ij}b_{ji}) = \mathrm{J}(a_{ij})b_{ji} = a_{ij}\mathrm{J}(b_{ji})$.

Proof. (1) Consider the case for $i = 1,\ j = 2$, we have

$$\begin{aligned} \mathrm{J}(a_{11}b_{12}) &= \mathrm{J}(a_{11} \circ b_{12}) \\ &= \mathrm{J}(a_{11}) \circ b_{12} = a_{11} \circ \mathrm{J}(b_{12}) \\ &= \mathrm{J}(a_{11})b_{12} = a_{11}\mathrm{J}(b_{12}). \end{aligned}$$

On similar pattern, we can prove other parts and (2).
(3) For $i = 1$ with (1), we have $\mathrm{J}(a_{11}b_{11}b_{12}) = \mathrm{J}(a_{11}b_{11})b_{12} = a_{11}b_{11}\mathrm{J}(b_{12})$.
On the other hand, we get

$$\begin{aligned} \mathrm{J}(a_{11}b_{11}b_{12}) &= \mathrm{J}(a_{11})b_{11}b_{12} = a_{11}\mathrm{J}(b_{11}b_{12}) \\ &= \mathrm{J}(a_{11})b_{11}b_{12} = a_{11}b_{11}\mathrm{J}(b_{12}) = a_{11}\mathrm{J}(b_{11})b_{12}. \end{aligned}$$

Now combining last two expressions, we obtain

$$\begin{aligned} (\mathrm{J}(a_{11}b_{11}) - \mathrm{J}(a_{11})b_{11})b_{12} &= 0, \\ (\mathrm{J}(a_{11}b_{11}) - a_{11}\mathrm{J}(b_{11}))b_{12} &= 0. \end{aligned}$$

With assumption we obtain the result. Likewise we can obtain the other cases. □

Remark 3.2. *In view of above lemma we can conclude that* J *is a multiplicative centralizer, that is,* $J(xy) = J(x)y = xJ(y)$ *for all* $x, y \in \mathcal{A}$.

Lemma 3.6. *For every* $a_{ij}, b_{ij} \in \mathcal{A}_{ij}$ *and for* $i, j = 1, 2$, *we have*

1. $J(a_{ii} + b_{ij}) = J(a_{ii}) + J(b_{ij})$,
2. $J(a_{ii} + b_{ii}) = J(a_{ii}) + J(b_{ii})$,
3. $J(a_{ii} + b_{jj}) = J(a_{ii}) + J(b_{ii})$,
4. $J(a_{ij} + b_{ij}) = J(a_{ij}) + J(b_{ij})$,
5. $J(a_{ij} + b_{ji}) = J(a_{ij}) + J(b_{ji})$.

Proof. For any $a_{ij}, b_{ij} \in \mathcal{A}_{ij}$, we obtain that

$$\begin{aligned} J(a_{ii} + b_{ij})b_{jk} &= (a_{ii} + b_{ij})J(b_{jk}) \\ &= a_{ii}J(b_{jk}) + b_{ij}J(b_{jk}) \\ &= J(a_{ii}b_{jk}) + J(b_{ij}b_{jk}) \\ &= J(a_{ii})b_{jk} + J(b_{ij})b_{jk} \\ (J(a_{ii} + b_{ij}) - J(a_{ii}) - J(b_{ij}))_{ij}\, b_{jk} &= 0. \end{aligned}$$

With assumption, we obtain the result. Likewise, we can obtain other cases. □

Proof. [Proof of Theorem 3.1] In view of Lemma 3.5 and 3.6, we can say that a multiplicative Jordan centralizer is an additive centralizer on alternative algebras. □

4. Applications

Clearly, using Remark 1.1, any alternative algebra over a basic field of characteristic not 3 satisfies

$$If \quad x_{ij}\mathcal{A}_{jk} = 0 \quad or \quad \mathcal{A}_{ki}x_{ij} = 0 \quad then \quad x_{ij} = 0.$$

Consequently, we have the following applications on prime alternative algebras.

Corollary 4.1. *Let* $\mathcal{A}$ *be a* $2,3$*-torsion free unital prime alternative algebra with nontrivial idempotent and* $\mathrm{L} : \mathcal{A} \to \mathcal{A}$ *be a multiplicative Lie centralizer satisfying* (1) *condition of the Theorem 2.1. Then* L *has the form* $\mathrm{L} = \delta + \tau$*, where* $\delta : \mathcal{A} \to \mathcal{A}$ *is an additive centralizer and* $\tau : \mathcal{A} \to \mathrm{Z}(\mathcal{A})$ *maps commutators into the zero.*

Corollary 4.2. *Let* $\mathcal{A}$ *be a* $2,3$*-torsion free unital prime alternative algebra with nontrivial idempotent and* $\mathrm{J} : \mathcal{A} \to \mathcal{A}$ *be a multiplicative Jordan centralizer. Then* J *is an additive centralizer.*

5. Acknowledgments

This research is supported by Dr. D. S. Kothari Postdoctoral Fellowship under University Grants Commission (Grant No. F.4-2/2006 (BSR)/MA/18-19/0014), awarded to the author.

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