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Monophonic graphoidal covering number of corona product graphs

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Abstract

In a graph G, a chordless path is called a monophonic path. A collection ψ_m of monophonic paths in G is called a monophonic graphoidal cover of G if every vertex of G is an internal vertex of at most one monophonic path in ψ_m and every edge of G is in exactly one monophonic path in ψ_m. The monophonic graphoidal covering number $\eta_m(G)$ of G is the minimum cardinality of a monophonic graphoidal cover of G. In this paper, we find the monophonic graphoidal covering number of corona product of some standard graphs.

AMS Subject Classification: *05C.*

Key Words: *graphoidal cover, monophonic path, monophonic graphoidal cover, monophonic graphoidal covering number.*

1. Introduction

In a finite, undirected connected graph $G = (V, E)$, the *order* and *size* of G are denoted by $|V(G)|$ and $|E(G)|$ respectively. For basic definitions and results we refer to Harary [6]. In [2], Acharya and Sampathkumar introduced a new graph theoretical parameter called graphoidal covering number and it was further studied in [1, 3, 7, 8, 9].

A *graphoidal cover* of a graph G is a collection ψ of (not necessarily open) paths in G such that every vertex of G is an internal vertex of at most one path in ψ and every edge of G is in exactly one path in ψ. The *monophonic graphoidal covering number* $\eta_m(G)$ of G is the minimum cardinality of a monophonic graphoidal cover of G.

In a graph G, if no member of a graphoidal cover ψ is cycle, then ψ is called an *acyclic graphoidal cover* of G and the minimum cardinality of an acyclic graphoidal cover of G is called the *acyclic graphoidal covering number* $\eta_a(G)$ of G. The concept of acyclic graphoidal cover was introduced by Arumugam and Suresh Suseela [4]. If every member of a graphoidal cover ψ is a shortest path in G, then ψ is called a *geodesic graphoidal cover* of G and the minimum cardinality of a geodesic graphoidal cover of G is called the *geodesic graphoidal covering number* $\eta_g(G)$ of G. The concept of geodesic graphoidal cover was introduced by the same authors in [5].

A *chord* of a path $v_1, v_2, ..., v_n$ in G is an edge v_iv_j with $j \geq i + 2$. A chordless path is also called as a *monophonic path*. The *monophonic distance* between any two vertices u and v is the length of a longest monophonic path joining the vertices u and v, and it is denoted by $d_m(u, v)$. For any vertex x in G, the *monophonic eccentricity* of a vertex x is defined as $e_m(x) = max\{d_m(x, y) : y \in V\}$. The *monophonic radius* $rad_m(G)$ is the minimum monophonic eccentricity among the vertices of G and the *monophonic diameter* $diam_m(G)$ is the maximum monophonic eccentricity among the vertices of G. The monophonic distance was introduced by Santhakumaran and Titus [10] and further studied by the same authors in [11].

A *monophonic graphoidal cover* of a graph G is a collection ψ_m of monophonic paths in G such that every vertex of G is an internal vertex of at most one monophonic path in ψ_m and every edge of G is in exactly one monophonic path in ψ_m. The minimum cardinality of a monophonic graphoidal cover of G is called the monophonic graphoidal covering number of G and is denoted by $\eta_m(G)$. The monophonic graphoidal covering number was introduced in [12] and further studied in [13].

Product graphs have been used to generate mathematical models of complex networks which inherits properties of real networks. By using basic graphs, corona graphs are defined by taking corona product of the basic graphs.

Definition 1.1. *The corona of two graphs G and H is the graph $G \circ H$ formed from one copy of G and $|V(G)|$ copies of H, where the i^{th} vertex of G is adjacent to every vertex in the i^{th} copy of H.*

2. Monophonic Graphoidal Covering Number

Theorem 2.1. *If $G = P_r \circ P_s (r, s \geq 2)$, then $\eta_m(G) = r(s+1) - 1$.*

Proof. Let $P_r : u_1, u_2, \ldots, u_r$ and $P_s : v_1, v_2, \ldots, v_s$ be paths of lengths r and s, respectively, where $r, s \geq 2$. Let G be the corona product of P_r and P_s; and the graph G is shown in Figure 2.1.

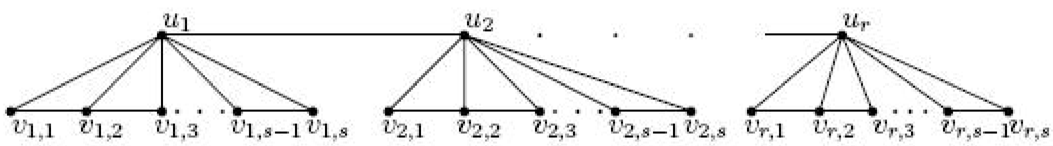

Figure 2.1: G

Let $M_1 : v_{1,1}, u_1, u_2, \ldots, u_r, v_{r,s}$; $M_{i+1} : v_{i,1}, v_{i,2}, \ldots, v_{i,s}$ $(1 \leq i \leq r)$ and $S = \bigcup_{i=1}^{r} \bigcup_{j=1}^{s} (u_i, v_{i,j})$. It is clear that every $M_i (1 \leq i \leq r+1)$ is a monophonic path and every element in S is a monophonic path. Hence $\psi_m = \{M_1, M_2, \ldots, M_{r+1}\} \cup (S - \{(u_1, v_{1,1}), (u_r, v_{r,s})\})$ is a minimum monophonic graphoidal cover of G and so $\eta_m(G) = r + 1 + rs - 2 = r(s+1) - 1$. □

Result 2.2. *If $G = P_s \circ P_r (r, s \geq 2)$, then $\eta_m(G) = s(r+1) - 1$.*

Result 2.3. *If $2 \leq r < s$, then $\eta_m(P_r \circ P_s) < \eta_m(P_s \circ P_r)$.*

Theorem 2.4. *If $G = C_n \circ P_r$, then $\eta_m(G) = n(r+1)$.*

Proof. Let $C_n : u_1, u_2, \ldots, u_n, u_1$ be a cycle of order n and let P_r : $v_1, v_2, \ldots, v_r$ be a path of order r. Let G be the corona product of C_n and P_r.

Case 1. $n = 3$. The graph G is shown in Figure 2.2.

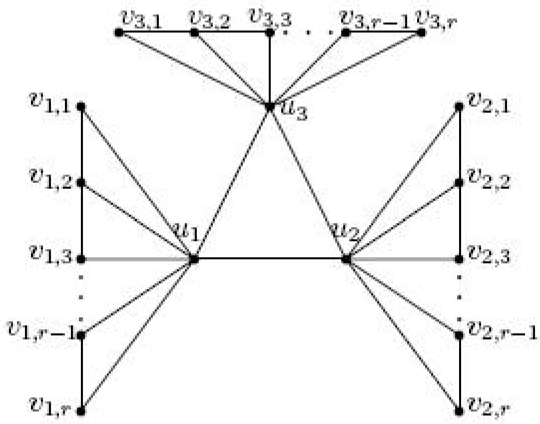

Figure 2.2: G

Let $M_1 : v_{1,1}, u_1, u_2, v_{2,1}$;
$M_2 : u_2, u_3, v_{3,1}$;
$M_3 : u_1, u_3$;
$M_4 : v_{1,1}, v_{1,2}, \ldots, v_{1,r}$;
$M_5 : v_{2,1}, v_{2,2}, \ldots, v_{2,r}$;
$M_6 : v_{3,1}, v_{3,2}, \ldots, v_{3,r}$;
and $S = \bigcup_{i=1}^{3} \bigcup_{j=2}^{r} (u_i, v_{i,j})$.

It is clear that every $M_i (1 \leq i \leq 6)$ is a monophonic path and every element in S is a monophonic path. Hence $\psi_m = S \cup \{M_1, M_2, \ldots, M_6\}$ is a minimum monophonic graphoidal cover of G and so $\eta_m(G) = 3(r-1)+6 = 3(r+1)$.

Case 2. $n > 3$. The graph G is shown in Figure 2.3

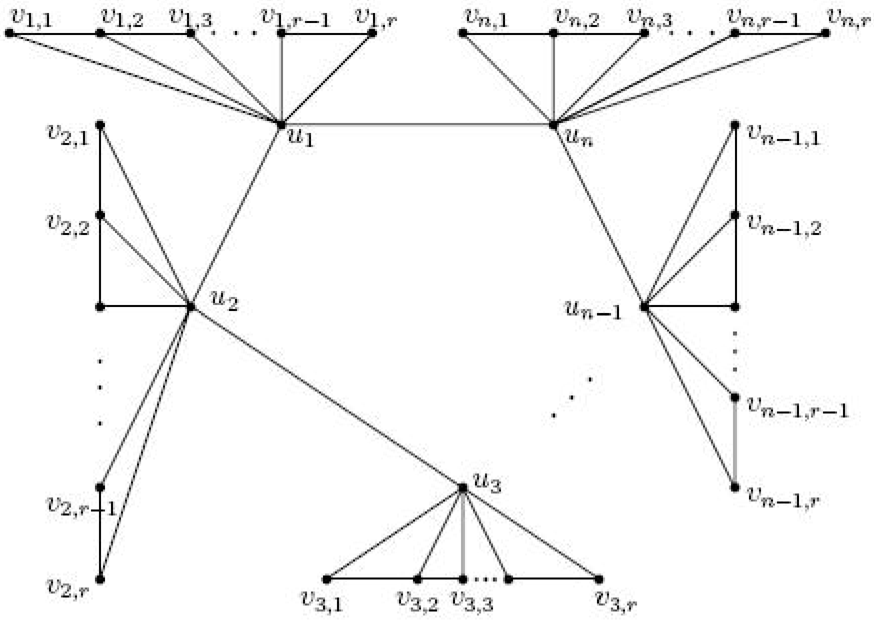

Figure 2.3: G

Let $M_1 : v_{1,1}, u_1, u_2, \ldots, u_{n-1}, v_{n-1,1}$;
$M_2 : u_1, u_n, u_{n-1}$;
$M_{i+2} : v_{i,1}, v_{i,2}, \ldots, v_{i,r} (1 \leq i \leq n)$;
and $S = \bigcup_{j=1}^{n} \bigcup_{k=1}^{r} (u_j, v_{j,k})$.

It is clear that every $M_i (1 \leq i \leq n+2)$ is a monophonic path and every element in S is a monophonic path. Hence $\psi_m = \{M_1, M_2, \ldots, M_{n+2}\} \cup (S - \{(u_1, v_{1,1}), (u_{n-1}, v_{n-1,1})\})$ is a minimum monophonic graphoidal cover of G and so $\eta_m(G) = n + 2 + nr - 2 = n(r + 1)$. □

Theorem 2.5. *If* $G = P_r \circ C_n$, *then* $\eta_m(G) = \begin{cases} 6r - 1 & if \quad n = 3 \\ r(n + 2) - 1 & if \quad n > 3. \end{cases}$

Proof. Let $P_r : v_1, v_2, \ldots, v_r$ be a path of order r and let $C_n : u_1, u_2, \ldots, u_n, u_1$ be a cycle of order n. Let G be the corona product of P_r and C_n. The graph G is shown in Figure 2.4.

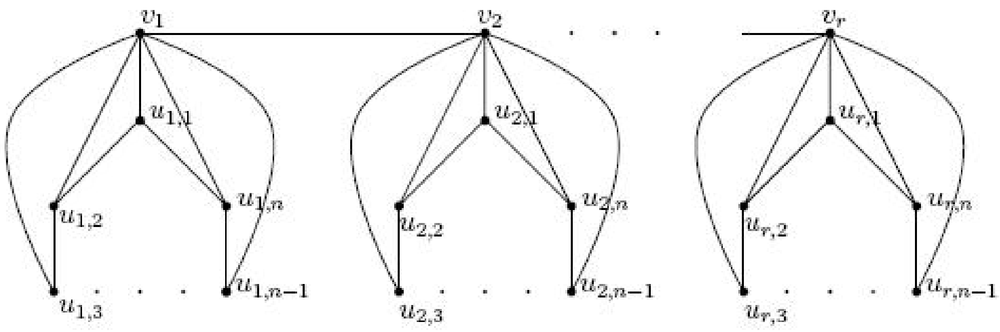

Figure 2.4 : G

Case 1. $n = 3$. Let $M : u_{1,1}, v_1, v_2, \ldots, v_r, u_{r,1}$. Then $\psi_m = (E(G) - E(M)) \cup \{M\}$ is a minimum monophonic graphoidal cover of G and so $\eta_m(G) = (7r - 1) - (r + 1) + 1 = 6r - 1$.

Case 2. $n > 3$.
Let $M : u_{1,1}, v_1, v_2, \ldots, v_r, u_{r,1}$;
$M_i : u_{i,1}, u_{i,2}, u_{i,3} (1 \leq i \leq r)$;
$M_i' : u_{i,3}, u_{i,4}, \ldots, u_{i,n}, u_{i,1} (1 \leq i \leq r)$;
and $S = \bigcup_{j=1}^{r} \bigcup_{k=1}^{n} (v_j, u_{j,k})$.

It is clear that M, M_i and $M_i' (1 \leq i \leq r)$ are monophonic paths, and every element in S is a monophonic path. Hence

$$\psi_m = \{M, M_1, M_2, \ldots, M_r, M_1', M_2', \ldots, M_r'\} \cup (S - \{(v_1, u_{1,1}), (v_r, u_{r,1})\})$$

is a minimum monophonic graphoidal cover of G and hence
$\eta_m(G) = (2r + 1) + rn - 2 = r(n + 2) - 1$. □

Theorem 2.6. *If $G = C_r \circ C_s$, then*

$$\eta_m(G) = \begin{cases} 6r & if \quad r \geq 3 \ and \ s = 3 \\ r(s+2) & if \quad r \geq 3 \ and \ s > 3. \end{cases}$$

Proof. Let $C_r : u_1, u_2, \ldots, u_r, u_1$ and $C_s : v_1, v_2, \ldots, v_s, v_1$ be two cycles of orders r and s respectively.

Case 1. $r = s = 3$. The graph G is shown in Figure 2.5.

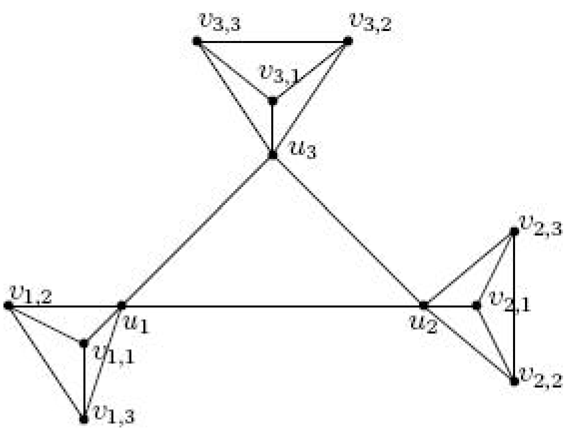

Figure 2.5: G

Let $M_1 : v_{1,1}, u_1, u_2, v_{2,1}$;
$M_2 : u_2, u_3, v_{3,1}$;
and $M_3 : u_1, u_3$.

Then $\psi_m = (E(G) - \{E(M_1) \cup E(M_2) \cup E(M_3)\}) \cup \{M_1, M_2, M_3\}$ is a minimum monophonic graphoidal cover of G and so $\eta_m(G) = 21 - 6 + 3 = 18 = 6r$.

Case 2. $r > 3$ and $s = 3$. The graph G is shown in Figure 2.6.

Let $M_1 : v_{1,1}, u_1, u_2, \ldots, u_{r-1}, v_{r-1,1}$; and $M_2 : u_1, u_r, u_{r-1}$.

Then $\psi_m = (E(G) - \{E(M_1) \cup E(M_2)\}) \cup \{M_1, M_2\}$ is a minimum monophonic graphoidal cover of G and so $\eta_m(G) = (7r - (r + 2)) + 2 = 6r$.

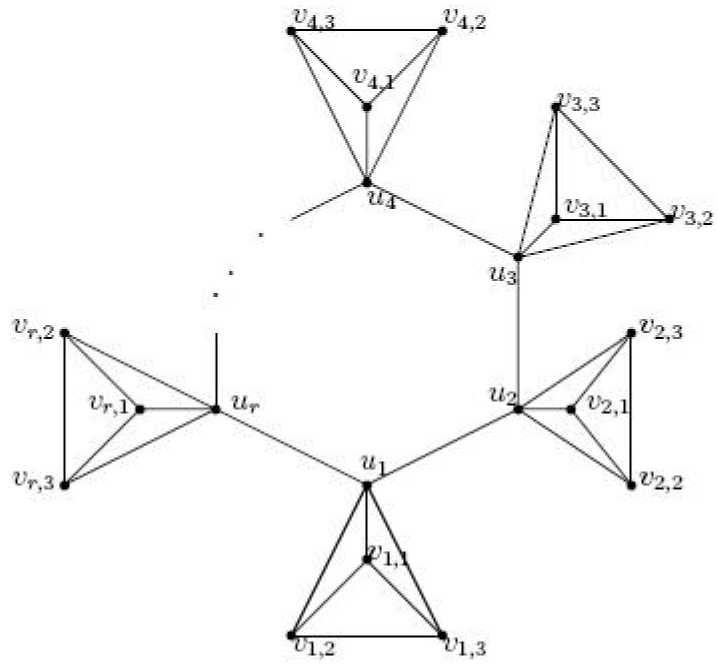

Figure 2.6: G

Case 3 . $r = 3$ and $s > 3$. The graph is G is shown in Figure 2.7.

Let $M_1 : v_{1,1}, u_1, u_2, v_{2,1}$;

$M_2 : u_2, u_3, v_{3,1}$;

$M_3 : u_1, u_3$;

$N_i : v_{i,1}, v_{i,2}, \ldots, v_{i,s-1} (1 \leq i \leq 3)$;

$N'_i : v_{i,1}, v_{i,s}, v_{i,s-1} (1 \leq i \leq 3)$;

and $S = \bigcup_{j=1}^{3} \bigcup_{k=1}^{s} (u_j, v_{j,k})$.

Then $\psi_m = \{M_1, M_2, M_3, N_1, N_2, N_3, N'_1, N'_2, N'_3\} \cup (S - \{(u_1, v_{1,1}), (u_2, v_{2,1}), (u_3, v_{3,1})\})$ is a minimum monophonic graphoidal cover of G. Hence $\eta_m(G) = 9 + (3s - 3) = 3(s + 2) = r(s + 2)$.

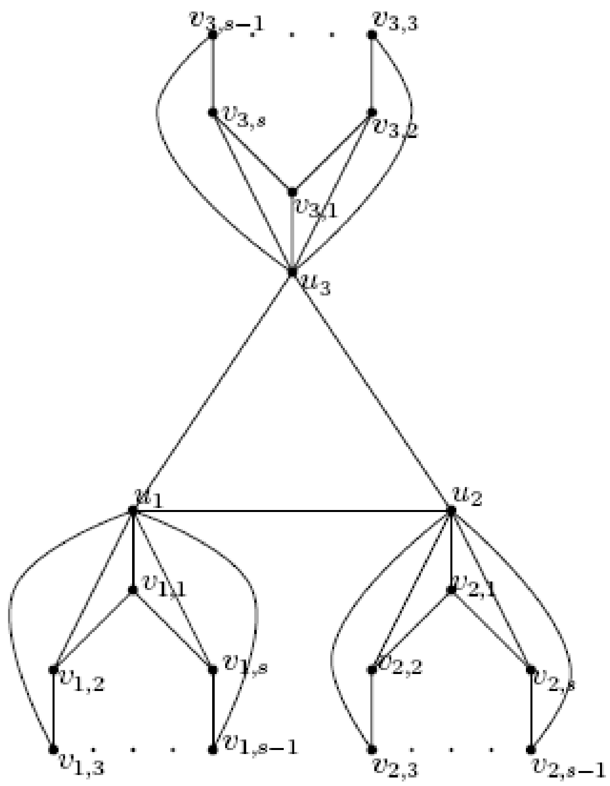

Figure 2.7: G

Case 4. $r > 3$ and $s > 3$. The graph G is shown in Figure 2.8.
Let $M_1 : v_{1,1}, u_1, u_2, \ldots, u_{r-1}, v_{r-1,1}$;
$M_2 : u_1, u_r, u_{r-1}$;
$N_i : v_{i,1}, v_{i,2}, \ldots, v_{i,s-1} (1 \leq i \leq r)$;
$N_i' : v_{i,1}, v_{i,s}, v_{i,s-1} (1 \leq i \leq r)$;
and $S = \bigcup_{j=1}^{r} \bigcup_{k=1}^{s} (u_j, v_{j,k})$.

Then $\psi_m = \{M_1, M_2, N_1, N_2, \ldots, N_r, N_1', N_2', \ldots, N_r'\} \cup (S - \{(u_1, v_{1,1}), (u_{r-1}, v_{r-1,1})\})$ is a minimum monophonic graphoidal cover of G and so $\eta_m(G) = (2r + 2) + (rs - 2) = r(s + 2)$. □

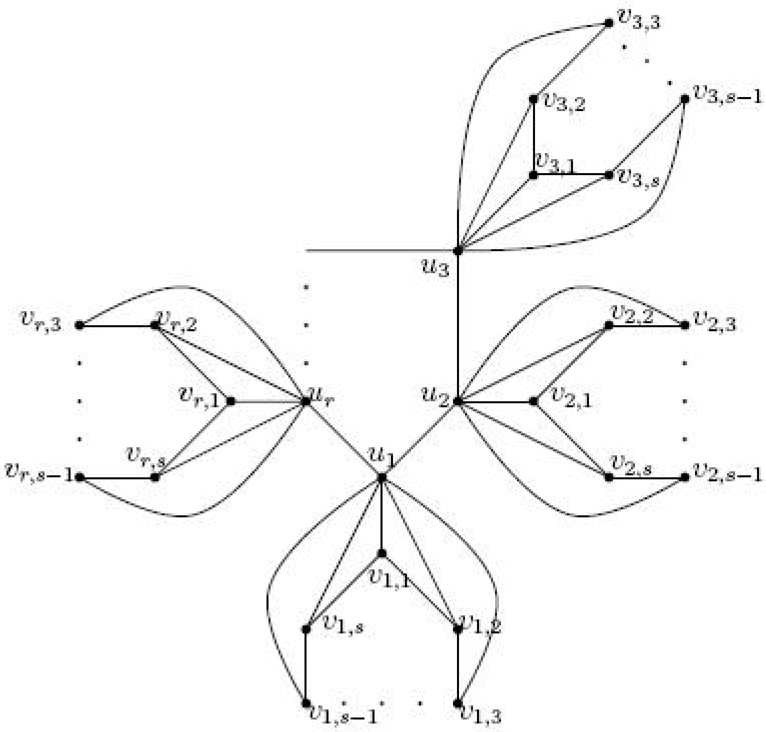

Figure 2.8: G

Theorem 2.7. *(i) If* $G = P_r \circ K_n$*, then* $\eta_m(G) = \frac{r}{2}(n^2 + n) - 1$.
(ii) If $G = K_n \circ P_r$*, then* $\eta_m(G) = \frac{n}{2}(n + 2r - 1)$.

Proof. Let $P_r : u_1, u_2, \ldots, u_r$ be a path of order r and let K_n be a complete graph of order n with the vertex set $\{v_1, v_2, \ldots, v_n\}$.
(i) Let G be the corona product of P_r and K_n; the graph G is shown in Figure 2.9.

Let $M : v_{1,1}, u_1, u_2, \ldots, u_r, v_{r,1}$. Then $\psi_m = (E(G) - E(M)) \cup \{M\}$ is a minimum monophonic graphoidal cover of G and so
$\eta_m(G) = r(\frac{n(n-1)}{2} + n) - 1 = \frac{r}{2}(n^2 + n) - 1$.

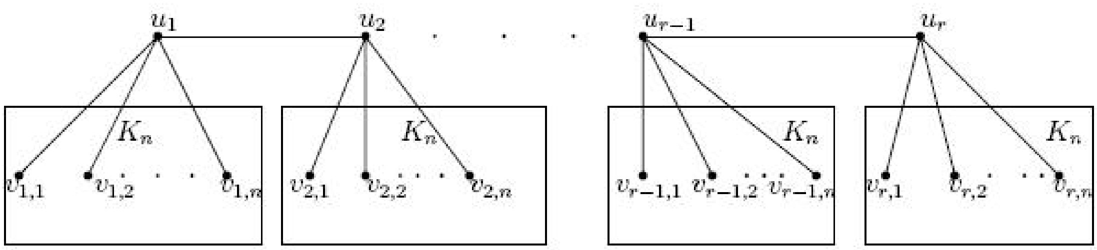

Figure 2.9: G

(ii) Let G be the corona product of K_n and P_r; and it is shown in Figure 2.10.

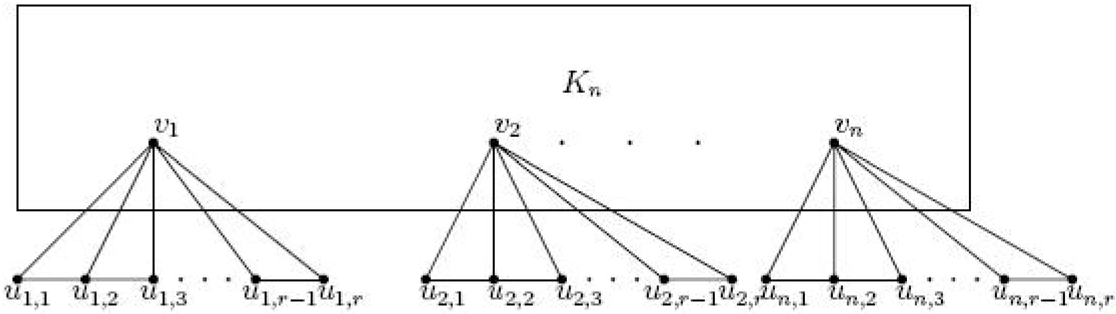

Figure 2.10: G

Let $M_i : u_{i,1}, v_i, v_{i+1} (1 \leq i \leq n-1)$;
$M_n : u_{n,1}, v_n, v_1$;
$N_i : u_{i,1}, u_{i,2}, \ldots, u_{i,r} (1 \leq i \leq n)$;
and $M_{j,k} : v_j, u_{j,k} (1 \leq j \leq n \textit{ and } 1 \leq k \leq r)$.

Then $\psi_m = (\bigcup_{j=1}^{n} \bigcup_{k=1}^{r} M_{j,k}) \cup (\bigcup_{i=1}^{n} (M_i, N_i)) \cup E(K_n) - \{(v_1, v_2), (v_2, v_3), \ldots, (v_{n-1}, v_n), (v_n, v_1), (u_{1,1}, v_1), (u_{2,1}, v_2), \ldots, (u_{n,1}, v_n)\}$ is a minimum monophonic graphoidal cover of G. Hence
$\eta_m(G) = nr + 2n + \frac{n(n-1)}{2} - 2n = \frac{n}{2}(n + 2r - 1)$. □

Theorem 2.8. *(i) If $G = C_r \circ K_n$, then $\eta_m(G) = \frac{nr}{2}(n+1)$.*

(ii) If $G = K_n \circ C_r$, then $\eta_m(G) = \begin{cases} \frac{3n}{2}(n+1) & if\, r = 3 \\ \frac{n}{2}(2r+n+1) & if r > 3. \end{cases}$

Proof. Let $C_r : u_1, u_2, \ldots, u_r, u_1$ be a cycle of order r and let K_n be the complete graph with the vertex set $\{v_1, v_2, \ldots, v_n\}$.
(i) Let G be the corona product of C_r and K_n; the graph G is shown in Figure 2.11.

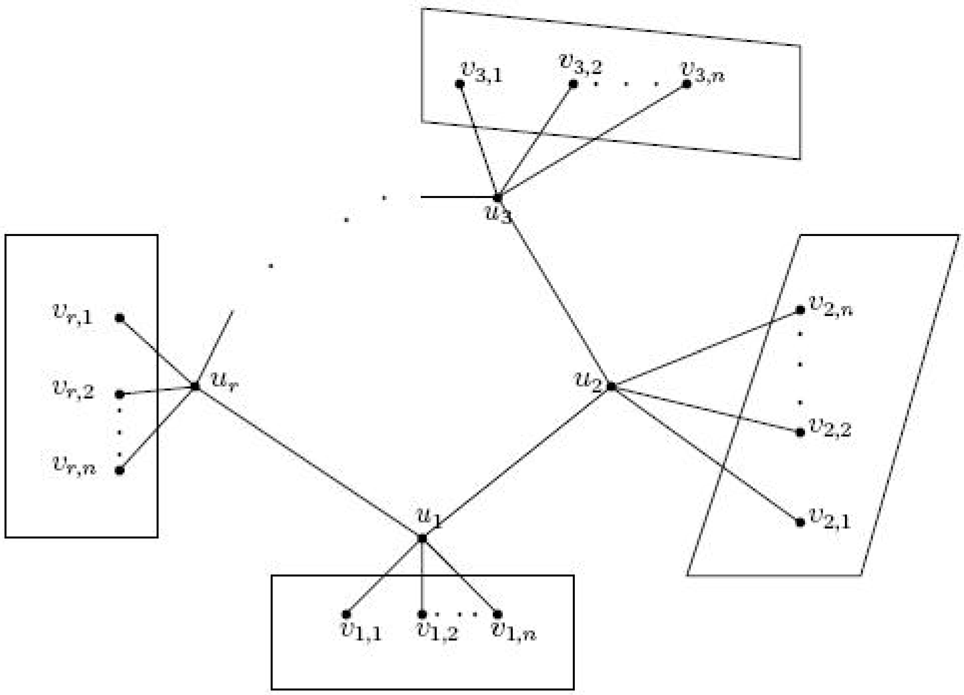

Figure 2.11: G

Case 1. $r = 3$.
Let $M_1 : v_{1,1}, u_1, u_2$;
$M_2 : v_{2,1}, u_2, u_3$; and $M_3 : v_{3,1}, u_3, u_1$.

Then $\psi_m = (E(G) - \bigcup_{i=1}^{3} E(M_i)) \bigcup \{M_1, M_2, M_3\}$ is a minimum monophonic graphoidal cover of G. Hence $\eta_m(G) = 3(\frac{n(n-1)}{2} + n + 1) - 6 + 3 = \frac{3n}{2}(n+1) = \frac{nr}{2}(n+1)$.

Case 2. $r > 3$. Let $M_1 : v_{1,1}, u_1, u_2, \ldots, u_{r-1}, v_{r-1,1}$ and $M_2 : u_1, u_r, u_{r-1}$. Then $\psi_m = (E(G) - \bigcup_{i=1}^{2} E(M_i)) \cup \{M_1, M_2\}$ is a minimum monophonic graphoidal cover of G. Hence $\eta_m(G) = r(\frac{n(n-1)}{2} + n + 1) - (r + 2) + 2 = \frac{nr}{2}(n + 1)$.

(ii) Let G be the corona product of K_n and C_r, and the graph G is shown in Figure 2.12.

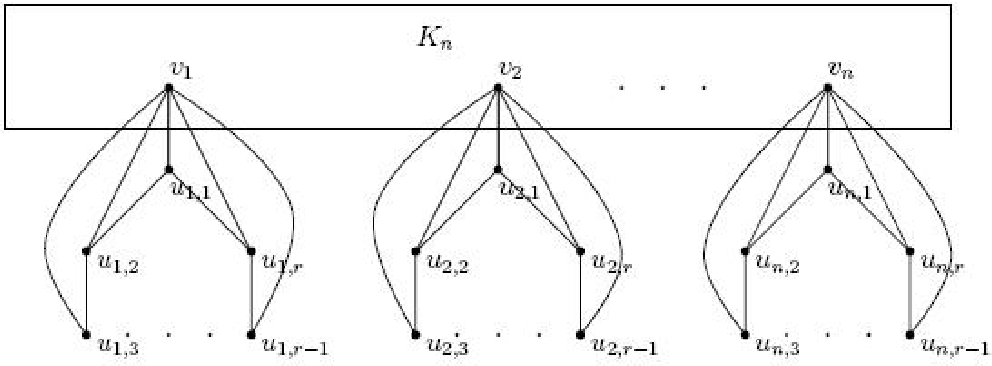

Figure 2.12: G

Case 1. $r = 3$.

Let $M_1 : u_{1,1}, v_1, v_2$;
$M_2 : u_{2,1}, v_2, v_3$;
and $M_3 : u_{3,1}, v_3, v_1$.

Then $\psi_m = (E(G) - \bigcup_{i=1}^{3} E(M_i)) \cup \{M_1, M_2, M_3\}$ is a minimum monophonic graphoidal cover of G. Hence $\eta_m(G) = 3 + \frac{3n(n-1)}{2} + 3n - 3 = \frac{3n}{2}(n + 1)$.

Case 2. $r > 3$.

Let $M_i : u_{i,1}, v_i, v_{i+1} (1 \leq i \leq n-1)$;
$M_n : u_{n,1}, v_n, v_1$;
$N_i : u_{i,1}, u_{i,2}, \ldots, u_{i,r-1} (1 \leq i \leq n)$;
$N'_i : u_{i,1}, u_{i,r}, u_{i,r-1} (1 \leq i \leq n)$;
$M_{j,k} : v_j, u_{j,k}$ $(1 \leq j \leq n$ *and* $2 \leq k \leq r)$;
and $S = V(K_n) - \{(v_1, v_2), (v_2, v_3), \ldots, (v_{n-1}, v_n), (v_n, v_1)\}$.

Then $\psi_m = \bigcup_{i=1}^{n} \{M_i, N_i, N'_i\} \cup (\bigcup_{j=1}^{n} \bigcup_{k=2}^{r} M_{j,k}) \cup S$ is a minimum monophonic graphoidal cover of G. Hence $\eta_m(G) = 3n + n(r-1) + \frac{n(n-1)}{2} - n = \frac{n}{2}(n + 2r + 1)$. □

Theorem 2.9. *If $G = K_r \circ K_s$, then $\eta_m(G) = \frac{r}{2}(s^2 + s + r - 3)$, where $r, s \geq 3$.*

Proof. Let K_r and K_s be the complete graphs with $V(K_r) = \{u_1, u_2, \ldots, u_r\}$ and $V(K_s) = \{v_1, v_2, \ldots, v_s\}$. The graph G is shown in Figure 2.13.

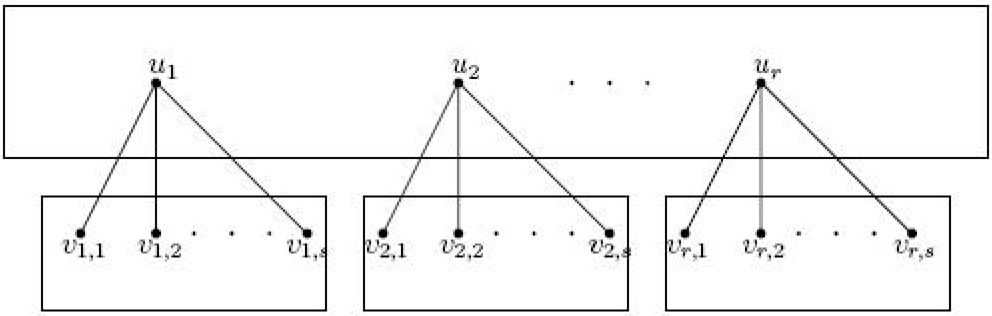

Figure 2.13: G

Let $M_i : v_{i,1}, u_i, u_{i+1} (1 \leq i \leq r-1)$ and $M_r : v_{r,1}, u_r, u_1$. Then $\psi_m = (E(G) - \bigcup_{i=1}^{r} E(M_i)) \cup \{M_1, M_2, \ldots, M_r\}$ is a minimum monophonic graphoidal cover of G. Hence $\eta_m(G) = (\frac{r(r-1)}{2} + r.\frac{s(s-1)}{2} + rs) - 2r + r = \frac{r}{2}(s^2 + s + r - 3)$. □

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