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On a maximal subgroup of the orthogonal group $O_8^+(3)$

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Abstract

The orthogonal simple group $O_8^+(3)$ has three conjugacy classes of maximal subgroups of the form $3^6{:}L_4(3)$. These groups are all isomorphic to each other and each group has order 4421589120 with index 1120 in $O_8^+(3)$. In this paper, we will compute the ordinary character table of one of these classes of maximal subgroups using the technique of Fischer-Clifford matrices. This technique is very efficient to compute the ordinary character table of an extension group $\overline{G} = N.G$ and especially where the normal subgroup N of $\overline{G}$ is an elementary abelian p-group. The said technique reduces the computation of the ordinary character table of $\overline{G}$ to find a handful of so-called Fischer-Clifford matrices of $\overline{G}$ and the ordinary or projective character tables of the inertia factor groups of the action of $\overline{G}$ on N.

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1. Introduction

The orthogonal group $O_8^+(3)$ of order $4952179814400 = 2^{12}.3^{12}.5^2.7.13$ has 27 conjugacy classes of maximal subgroups [8]. Among the maximal subgroups of $O_8^+(3)$ are three non-conjugate but isomorphic subgroups $\overline{G}_1$, $\overline{G}_2$ and $\overline{G}_3$ of the form $3^6{:}L_4(3)$ with order 4421589120 and index 1120 in $O_8^+(3)$. The aim of this paper is to compute the Fischer-Clifford matrices and hence the character table of $\overline{G}_1$, the first group of the three as they appear in the ATLAS [8]. For this purpose, the Fischer-Clifford matrices technique which is based on Clifford theory and was developed by Bernd Fischer [9] is used. The group $3^6{:}L_4(3)$ which we shall now denote by $\overline{G}$ is a split-extension of $N = 3^6$, the vector space of dimension 6 over $GF(3)$, by the linear group $G = L_4(3) \cong O_6^+(3)$.

Let $\overline{G} = N{:}G$ be a split extension of N by G, where N is a vector space of dimension n over $GF(p)$, for a prime p, on which a linear group G acts naturally. The Fischer-Clifford technique involves the construction of a non-singular matrix $M(g)$ for each conjugacy class representative g of G, which together with the fusion maps and ordinary character tables of some subgroups of G, called the inertia factor groups, are used to assemble the complete ordinary character table of $\overline{G}$.

The Fischer-Clifford matrix $M(g)$ is partitioned row-wise into blocks, where each block corresponds to an inertia group $\overline{H}_i$ of $\theta_i \in Irr(N)$ in $\overline{G}$. Using the columns of the character tables of the inertia factors $H_i \cong \frac{\overline{H}_i}{N}$ which correspond to classes of H_i which fuse to the class $[g]$ of G and multiplying these columns by the rows of the the Fischer-Clifford matrix $M(g)$ that correspond to H_i, a portion of the character table of $\overline{G}$ which is in the block corresponding to $\overline{H}_i$ for the classes that come from the coset Ng is constructed. The character table of $\overline{G}$ is thus divided row-wise into blocks, where each block corresponds to an inertia group $\overline{H}_i = N{:}H_i$. The reader is referred to [2], [15], [16], [19], [20] and [21] for more literature on this technique. A brief theoretical background of the Fischer-Clifford theory is given in Section 2.

In Section 3, the coset analysis technique [17] is used to determine the conjugacy classes of $\overline{G}$. In Sections 4 and 5 the inertia factor groups H_i and their fusion maps into $G = L_3(4)$ are computed. The Fischer-Clifford matrices of $\overline{G} = 3^6{:}L_4(3)$ are determined in Section 6 and the associated ordinary character table of $\overline{G}$ is to be found in Section 7. The technique of set intersection of characters (see [1], [17], [18]) is mainly used to compute the fusion of the conjugacy classes of $\overline{G} = 3^6{:}L_4(3)$ into $O_8^+(3)$. Most of our computations are carried out with the computer algebra systems MAGMA [7] and GAP [11] and the notation of ATLAS is mostly followed.

2. Theory of Fischer-Clifford Matrices

Let $\overline{G} = N{:}G$ be a split extension of N by G. Then for $\theta \in Irr(N)$, we define $\overline{H} = \{x \in \overline{G} | \theta^x = \theta\} = I_{\overline{G}}(\theta)$ and $H = \{x \in G | \theta^x = \theta\} = I_G(\theta)$ where $I_{\bar{G}}(\theta)$ is the stabilizer of θ in the action of $\overline{G}$ on $Irr(N)$, we have that $I_{\overline{G}}(\theta)$ is a subgroup of $\overline{G}$ and N is normal subgroup in $I_{\overline{G}}(\theta)$. Also $[\overline{G}{:}I_{\overline{G}}(\theta)]$ is the size of the orbit containing θ. Then it can be shown that $\overline{H} = N{:}H$, where $\overline{H}$ is the inertia group of θ in $\overline{G}$. The inertia factor $\overline{H}/N \cong H$ can be regarded as the inertia group of θ in the factor group $\overline{G}/N \cong G$. Define θ^g by $\theta^g(n) = \theta(gng^{-1})$ for $g \in \overline{G}, \ n \in N$, then $\theta^g \in Irr(N)$. We say that θ is extendible to $\overline{H}$ if there exists $\varphi \in Irr(\overline{H})$ such that $\varphi \downarrow N = \theta$. If θ is extendible to $\overline{H}$ then by Gallagher [10], we have $\{\varphi | \varphi \in Irr(\overline{H}), < \varphi \downarrow_N, \theta > \neq 0\} = \{\overline{\beta}\varphi | \beta \in Irr(\overline{H}/N)\}$, where $\overline{\beta} \in Irr(\overline{H})$ is a lifting for β into $\overline{H}$. Let $\overline{G}$ have the property that every irreducible character of N can be extended to its inertia group. Now let $\theta_1 = 1_N$, θ_2, ... , θ_t be representatives of the orbits of $\overline{G}$ on $Irr(N)$, $\overline{H}_i = I_{\overline{G}}(\theta_i)$, $1 \leq i \leq t$, $\varphi_i \in Irr(\overline{H}_i)$ be an extension of θ_i to $\overline{H}_i$ and $\overline{\beta} \in Irr(\overline{H}_i)$ such that $N \subseteq Ker(\overline{\beta})$. Then it can be shown that

$$\begin{aligned} Irr(\overline{G}) &= \textstyle\bigcup_{i=1}^{t} \{(\overline{\beta}\varphi_i)^{\overline{G}} | \overline{\beta} \in Irr(\overline{H}_i), N \subseteq Ker(\overline{\beta})\} \\ &= \textstyle\bigcup_{i=1}^{t} \{(\overline{\beta}\varphi_i)^{\overline{G}} | \beta \in Irr(\overline{H}_i/N)\} \end{aligned}$$

Hence the irreducible characters of $\overline{G}$ will be divided into blocks, where each block corresponds to an inertia group $\overline{H}_i$. Let H_i be the inertia factor group and φ_i be an extension of θ_i to $\overline{H}_i$. Take $\theta_1 = 1_N$ as the identity character of N, then $\overline{H}_1 = \overline{G}$ and $H_1 \cong G$. Let $X(g) = \{x_1, x_2, ..., x_{c(g)}\}$ be a set of representatives of the conjugacy classes of $\overline{G}$ from the coset $N\overline{g}$ whose images under the natural homomorphism $\overline{G} \to G$ are in $[g]$ and we take $x_1 = \overline{g}$. We define,

$$R(g) = \{(i, y_k) | 1 \leq i \leq t, H_i \cap [g] \neq 0, 1 \leq k \leq r\}$$

and we note that y_k runs over representatives of the conjugacy classes of elements of H_i which fuse into $[g]$ in G. Then we define the Fischer-Clifford matrix $M(g)$ by $M(g) = (a^j_{(i,y_k)})$, where $a^j_{(i,y_k)} = \sum_l' \frac{|C_{\overline{G}}(x_j)|}{|C_{\overline{H_i}}(y_{l_k})|}\varphi_i(y_{l_k})$ with columns indexed by $X(g)$ and rows indexed by $R(g)$ and where $\sum_l'$ is the summation over all l for which $y_{l_k} \sim x_j$ in $\overline{G}$. Then the partial character table of $\overline{G}$ on the classes $\{x_1, x_2, ..., x_{c(g)}\}$ is given by

$$\begin{pmatrix} C_1(g)M_1(g) \\ C_2(g)M_2(g) \\ \vdots \\ C_t(g)M_t(g) \end{pmatrix}$$

where the Fischer-Clifford matrix

$$M(g) = \begin{pmatrix} M_1(g) \\ M_2(g) \\ \vdots \\ M_t(g) \end{pmatrix}$$

is divided into blocks $M_i(g)$ with each block corresponding to an inertia group $\overline{H}_i$ and $C_i(g)$ is the partial character table of H_i consisting of the columns corresponding to the classes that fuse into $[g]$ in G. We can also observe that the number of irreducible characters of $\overline{G}$ is the sum of the number of irreducible characters of the inertia factors H_i's. For complete information on the properties of Fischer-Clifford matrices the reader is referred to [2], [15], [16], [19], [20] and [21]. The group $\overline{G} = 3^6{:}L_4(3)$ is a split extension with 3^6 abelian and therefore by Mackey's theorem (see Theorem 5.1.15 in [18]), we have that each irreducible character of 3^6 can be extended to its inertia group in $\overline{G}$. With this theoretical assertion in mind, the character table of $\overline{G} = 3^6{:}L_4(3)$ is now going to be determined using the above outline.

3. The Conjugacy Classes of $\overline{G} = 3^6{:}L_4(3)$

In this section, the method of coset analysis is used to determine the conjugacy classes of the elements of $\overline{G} = 3^6{:}L_4(3)$. This method was developed and first used by Moori in [17] and since then, it has been used by many other researchers to compute the conjugacy classes of groups of extension type. The reader is referred to [3] and [4] for recent application of this technique. By making use of the standard generators from the online ATLAS of Group Representations [25], the groups $P = O_8^+(3)$

and $\overline{G}$ are represented as permutations on 1080 points in MAGMA. The command "*IsMaximal*$(P, \overline{G})$;" confirms that $\overline{G}$ is a maximal subgroup of P. Proceeding with the commands, "$a, b:=$ *ChiefSeries*$(\overline{G})$;", "$N:= a[2]$", "$M:=$ *GModule*$(\overline{G}, N)$;" and "M:*Maximal*;" the group $G = L_4(3)$ is constructed as matrix group of degree 6 over $GF(3)$. Note that N is the only elementary abelian 3-group of order 729 in $\overline{G}$. The following two 6×6 matrices g_1 and g_2 of orders 2 and 12, respectively, are obtained as the generators of G.

$$g_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 2 & 1, \end{pmatrix} \qquad g_2 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 2 & 1 & 1 & 1 & 0 & 0 \\ 2 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}$$

The 29 conjugacy classes of $G = \langle g_1, g_2 \rangle$ are computed within GAP.

3.1. The Action of $G = L_4(3)$ on 3^6

Let $\overline{G} = 3^6{:}L_4(3)$ be the split extension of $G = \langle g_1, g_2 \rangle$ by $N = 3^6$, where N is considered as a vector space $V_6(3)$ of dimension 6 over $GF(3)$. Also, $N \cong V_6(3)$ as a G-module of $G = \langle g_1, g_2 \rangle$ is irreducible. Using GAP, it turns out that the action of $G = L_4(3)$ on $N = 3^6$ has four orbits of lengths $1, 234, 234$ and 260 with corresponding point stabilizers P_1, P_2, P_3 and P_4.

3.2. Permutation Character of $G = L_4(3)$ on 3^6

Checking the indices of maximal subgroups of $G = L_4(3)$ in the ATLAS [8], P_2 and P_3 sit maximally inside the maximal subgroups with the structure $U_4(2){:}2$ while P_4 sits maximal inside the maximal subgroup $3^4{:}2(A_4 \times A_4).2$ of G. It follows that $P_1 = L_4(3), P_2 = U_4(2), P_3 = U_4(2)$ and $P_4 = 3^4{:}2(A_4 \times A_4)$ of indices $1, 234, 234$ and 260 respectively in $L_4(3)$. We will now determine, with the use of the permutation character $\chi(L_4(3)|3^6)$ of G on N whether P_2 and P_3 are sitting separately inside one of the two classes of $U_4(2){:}2$ or both are in one of the classes of $U_4(2){:}2$. The permutation character $\chi(L_4(3)|3^6) = 1 + I_{P_2}^{L_4(3)} + I_{P_3}^{L_4(3)} + I_{P_4}^{L_4(3)}$ of G acting on N is now going to be computed, where $I_{P_2}^{L_4(3)}, I_{P_3}^{L_4(3)}$ and $I_{P_4}^{L_4(3)}$ are the identity characters of P_2, P_3 and P_4 induced to G respectively. To determine

$I_{P_2}^{L_4(3)}$, the fusion of conjugacy classes of P_2 into $L_4(3)$ and the restrictions of $\chi_i \in Irr(L_4(3))$ to P_2, where $deg(\chi_i) < 234$ are used. Thus restricting $\chi_i \in Irr(L_4(3))$ to P_2 where $i \in 1, 2, 3, ..., 8$, and computing the inner product $\langle \chi_i, \psi_1 \rangle$ of each χ_i, $i \in 1, 2, 3, ..., 8$, with the identity character ψ_1 of P_2, the values below are obtained.

	χ_1	χ_2	χ_3	χ_4	χ_5	χ_6	χ_7	χ_8
$\langle \chi_i, \psi_1 \rangle$	1	0	1	0	1	0	1	1

From the above table and taking into consideration the Frobenius-Reciprocity theorem [12], the permutation character $I_{P_2}^{L_4(3)} = 1a + 26b + 52a + 65b + 90a$ is obtained. Similarly, $I_{P_4}^{L_4(3)}$ is determined. In this case, we restrict $\chi_i \in Irr(L_4(3)$, where $i \in 1, 2, 3, ..., 10$ to P_4 and let ψ_1 be the identity character P_4. Computing the inner product $\langle \chi_i, \psi_1 \rangle$ of each χ_i for $i \in 1, 2, 3, ..., 10$ with ψ_1, the values below are found.

	χ_1	χ_2	χ_3	χ_4	χ_5	χ_6	χ_7	χ_8	χ_9	χ_{10}
$\langle \chi_i, \psi_1 \rangle$	1	0	0	1	0	1	1	1	0	0

The identity character $I_{P_4}^{L_4(3)}$ of P_4 induced to $L_4(3)$ is therefore given by, $I_{P_4}^{L_4(3)} = 1a + 39a + 65a + 65b + 90a$. It follows that the permutation character $\chi(L_4(3)|3^6)$ is given as,

$$\begin{aligned}
\chi(L_4(3)|3^6) &= 1 + 2I_{P_2}^{L_4(3)} + I_{P_4}^{L_4(3)} \\
&= 1a + 2(1a + 26b + 52a + 65b + 90a) + 1a + 39a + 65a + 65b + 90a \\
&= 4 \times 1a + 2 \times 26b + 39a + 2 \times 52a + 65a + 3 \times 65b + 3 \times 90a.
\end{aligned}$$

The permutation characters $\chi(L_4(3)|P_i)$ are written in terms of the ordinary irreducible characters of G and are computed directly using the character table of G. The permutation character $\chi(L_4(3)|3^6)$ on the different conjugacy classes of G determines the number k of fixed points of each $g \in G$ in 3^6. The values of k obtained by the above permutation character are listed in Table 1.

Table 1: Permutation Character of $G = L_4(3)$ on 3^6

$[g]_G$	$1A$	$2A$	$2B$	$3A$	$3B$	$3C$	$3D$	$4A$	$4B$	$4C$	$5A$	$6A$	$6B$	$6C$	$6D$
$\chi(G\|P_1)$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$\chi(G\|P_2)$	234	30	2	18	0	18	36	0	2	4	4	0	6	2	2
$\chi(G\|P_3)$	234	30	2	18	0	18	36	0	2	4	4	0	6	2	2
$\chi(G\|P_4)$	260	20	4	44	8	26	26	0	4	0	0	2	2	4	4
k	729	81	9	81	9	63	99	1	9	9	9	3	15	9	9
	$6E$	$8A$	$9A$	$9B$	$10A$	$12A$	$12B$	$12C$	$13A$	$13B$	$13C$	$13D$	$20A$	$20B$	
$\chi(G\|P_1)$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	
$\chi(G\|P_2)$	2	0	0	6	0	0	0	2	0	0	0	0	0	0	
$\chi(G\|P_3)$	2	0	0	6	0	0	0	2	0	0	0	0	0	0	
$\chi(G\|P_4)$	4	0	2	2	0	0	0	4	0	0	0	0	0	0	
k	9	1	3	15	1	1	1	9	1	1	1	1	1	1	

Clearly, $\chi(L_4(3)|3^6) = 4 \times 1a + 2 \times 26b + 39a + 2 \times 52a + 65a + 3 \times 65b + 3 \times 90a$ is not the required permutation character of G acting on N since $k \neq 3^n$, $n \in \{0,1,2,3,4,5,6\}$ for all the conjugacy classes of G. In order to obtain values of k such that $k = 3^n$, $n \in \{0,1,2,3,4,5,6\}$, another possible fusion of conjugacy classes of $P_2 \cong P_3$ into $L_4(3)$ is considered. We restrict $\chi_i \in Irr(L_4(3))$ to P_2, and then compute the inner product $\langle \chi_i, \psi_1 \rangle$ of each χ_i, $i \in 1,2,3,...,8$ with the identity character ψ_1 of P_2. The values of $\langle \chi_i, \psi_1 \rangle$ are listed below,

	χ_1	χ_2	χ_3	χ_4	χ_5	χ_6	χ_7	χ_8
$\langle \chi_i, \psi_1 \rangle$	1	1	0	0	1	1	0	1

From the above table and taking into consideration the Frobenius-Reciprocity theorem, the permutation character $I_{P_2}^{L_4(3)}$ now assumes the following form,

$$I_{P_2}^{L_4(3)} = 1a + 26a + 52a + 65a + 90a.$$

Using $I_{P_2}^{L_4(3)} = 1a + 26a + 52a + 65a + 90a$ and $I_{P_3}^{L_4(3)} = 1a + 26b + 52a + 65b + 90a$ we obtain that

$$\begin{aligned}
\chi(L_4(3)|3^6) &= 1 + I_{P_2}^{L_4(3)} + I_{P_3}^{L_4(3)} + I_{P_4}^{L_4(3)} \\
&= 4 \times 1a + 1 \times 26a + 1 \times 26b + 1 \times 39a + 2 \times 52a \\
&\quad +2 \times 65a + 2 \times 65b + 3 \times 90a.
\end{aligned}$$

Using this result, the correct number k of fixed points of each $g \in G$ in 3^6 is obtained and is listed in Table 2. This confirms that P_2 and P_3 sit separately inside the two maximal subgroups of G of the form $U_4(2){:}2$.

Table 2: Permutation Character of $G = L_4(3)$ on 3^6

$[g]_G$	$1A$	$2A$	$2B$	$3A$	$3B$	$3C$	$3D$	$4A$	$4B$	$4C$	$5A$	$6A$	$6B$	$6C$	$6D$
$\chi(G\|P_1)$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$\chi(G\|P_2)$	234	30	2	18	0	18	36	0	2	4	4	0	6	2	2
$\chi(G\|P_3)$	234	30	2	18	0	36	18	0	2	4	4	6	0	2	2
$\chi(G\|P_4)$	260	20	4	44	8	26	26	0	4	0	0	2	2	4	4
k	729	81	9	81	9	81	81	1	9	9	9	9	9	9	9
	$6E$	$8A$	$9A$	$9B$	$10A$	$12A$	$12B$	$12C$	$13A$	$13B$	$13C$	$13D$	$20A$	$20B$	
$\chi(G\|P_1)$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	
$\chi(G\|P_2)$	2	0	0	6	0	0	0	2	0	0	0	0	0	0	
$\chi(G\|P_3)$	2	0	6	0	0	0	0	2	0	0	0	0	0	0	
$\chi(G\|P_4)$	4	0	2	2	0	0	0	4	0	0	0	0	0	0	
k	9	1	9	9	1	1	1	9	1	1	1	1	1	1	

3.3. The Classes of $\overline{G} = 3^6{:}L_4(3)$

The values of k enable us to determine the number f_j of orbits Q_i's, $1 \leq i \leq k$ that fuse together under the action of $C_G(g)$ to form one orbit $\triangle_j$ (see [18]). To determine the values of these f_j's and the orders of class representatives $dg \in \overline{G}$, Programmes A and B in [24] written in GAP are used, respectively. If $o(g) = m$ and $w = 1_N$ then $o(dg) = m$ and if $w \neq 1_N$ then $o(dg) = 3m$ (see Theorem 2.3.10 in [18]). The formula $|C_{\overline{G}}(x)| = \frac{k}{f_j}|C_G(g)|$ is then used to calculate the order of the centralizer of each class of $\overline{G}$ with representative x and a constant $m_j = \frac{f_j}{k}|N|$ is also calculated for each value of f_j. This constant plays a very crucial role in determination of the entries of the Fischer Clifford matrices. The group $\overline{G} = 3^6{:}L_4(3)$ is found to have 111 conjugacy classes of elements. Table 3 below gives a detailed information on the conjugacy classes of $\overline{G} = 3^6{:}L_4(3)$. The power maps of elements of $\overline{G}$ are given in the second last column of Table 3 whereas the fusion of $\overline{G}$ into $O_8^+(3)$, as determined in Section 8 of this paper, is found in the last column of Table 3.

Table 3: The Congujacy Classes of $\overline{G} = 3^6{:}L_4(3)$

$[g]_G$	k	f_j	m_j	d_j	w	$[x]_{\overline{G}}$	$\lvert C_{\overline{G}}(x)\rvert$	2	3	5	7	13	$\mapsto O_8^+(3)$
$1A$	729	1	1	$(0,0,0,0,0,0)$	$(0,0,0,0,0,0)$	$1A$	4421589120						$1A$
		234	234	$(0,0,0,1,0,1)$	$(0,0,0,1,0,1)$	$3A$	18895680		$1A$				$3B$
		234	234	$(0,0,0,1,0,2)$	$(0,0,0,1,0,2)$	$3B$	18895680		$1A$				$3E$
		260	260	$(0,0,0,0,0,1)$	$(0,0,0,0,0,1)$	$3C$	17006112		$1A$				$3A$
$2A$	81	1	9	$(0,0,0,0,0,0)$	$(0,0,0,0,0,0)$	$2A$	233280	$1A$					$2A$
		20	180	$(0,0,0,0,0,1)$	$(0,0,0,0,0,2)$	$6A$	11664	$3C$	$2A$				$6A$
		30	270	$(0,0,0,0,1,0)$	$(1,1,1,2,1,2)$	$6B$	7776	$3A$	$2A$				$6E$
		30	270	$(0,0,0,1,0,2)$	$(0,0,0,2,0,1)$	$6C$	7776	$3B$	$2A$				$6H$
$2B$	9	1	81	$(0,0,0,0,0,0)$	$(0,0,0,0,0,0)$	$2B$	10368	$1A$					$2D$
		2	162	$(0,0,0,0,0,1)$	$(2,1,2,1,2,1)$	$6D$	5184	$3B$	$2B$				$6N$
		2	162	$(0,0,0,0,1,1)$	$(0,0,2,1,2,2)$	$6E$	5184	$3A$	$2B$				$6K$
		4	324	$(0,0,0,0,1,0)$	$(1,2,0,0,0,1)$	$6F$	2592	$3C$	$2B$				$6D$
$3A$	81	1	9	$(0,0,0,0,0,0)$	$(0,0,0,0,0,0)$	$3D$	472392		$1A$				$3A$
		1	9	$(0,0,0,1,0,1)$	$(0,0,0,0,0,0)$	$3E$	472392		$1A$				$3K$
		1	9	$(0,0,0,2,0,2)$	$(0,0,0,0,0,0)$	$3F$	472392		$1A$				$3J$
		1	9	$(0,1,0,0,2,0)$	$(0,0,0,0,0,0)$	$3G$	472392		$1A$				$3H$
		1	9	$(0,1,0,1,2,1)$	$(0,0,0,0,0,0)$	$3H$	472392		$1A$				$3G$
		1	9	$(0,1,0,2,2,2)$	$(0,0,0,0,0,0)$	$3I$	472392		$1A$				$3F$
		1	9	$(0,2,0,0,1,0)$	$(0,0,0,0,0,0)$	$3J$	472392		$1A$				$3I$
		1	9	$(0,1,0,1,1,1)$	$(0,0,0,0,0,0)$	$3K$	472392		$1A$				$3C$
		1	9	$(0,2,0,2,1,2)$	$(0,0,0,0,0,0)$	$3L$	472392		$1A$				$3D$
		72	648	$(0,0,0,0,0,1)$	$(0,0,0,0,0,0)$	$3M$	6561		$1A$				$3M$
$3B$	9	1	81	$(0,0,0,0,0,0)$	$(0,0,0,0,0,0)$	$3N$	729		$1A$				$3M$
		1	81	$(0,0,0,0,0,1)$	$(2,2,0,1,0,1)$	$9A$	729		$3C$				$9E$
		1	81	$(0,0,0,0,0,2)$	$(1,1,0,2,0,2)$	$9B$	729		$3C$				$9B$
		1	81	$(0,0,0,0,1,0)$	$(0,1,0,2,2,1)$	$9C$	729		$3C$				$9H$
		1	81	$(0,0,0,0,1,1)$	$(2,0,0,0,2,2)$	$9D$	729		$3C$				$9C$
		1	81	$(0,0,0,0,1,2)$	$(1,2,0,1,2,0)$	$9E$	729		$3C$				$9I$
		1	81	$(0,0,0,0,2,0)$	$(0,2,0,1,1,2)$	$9F$	729		$3C$				$9G$
		1	81	$(0,0,0,0,2,1)$	$(2,1,0,2,1,0)$	$9G$	729		$3C$				$9J$
		1	81	$(0,0,0,0,2,2)$	$(1,0,0,0,1,1)$	$9H$	729		$3C$				$9F$
$3C$	81	1	9	$(0,0,0,0,0,0)$	$(0,0,0,0,0,0)$	$3O$	157464		$1A$				$3B$
		4	36	$(0,0,0,0,1,0)$	$(0,0,0,0,0,0)$	$3P$	39366		$1A$				$3H$
		4	36	$(0,0,0,0,2,0)$	$(0,0,0,0,0,0)$	$3Q$	39366		$1A$				$3I$
		6	54	$(0,0,0,1,2,1)$	$(0,0,0,0,0,0)$	$3R$	26244		$1A$				$3L$
		12	108	$(0,0,0,1,1,1)$	$(0,0,0,0,0,0)$	$3S$	13122		$1A$				$3M$
		54	486	$(0,0,0,0,0,1)$	$(2,2,0,1,0,1)$	$9I$	2916		$3C$				$9A$
$3D$	81	1	9	$(0,0,0,0,0,0)$	$(0,0,0,0,0,0)$	$3T$	157464		$1A$				$3E$
		4	36	$(0,0,1,0,0,0)$	$(0,0,0,0,0,0)$	$3U$	39366		$1A$				$3J$
		4	36	$(0,0,1,1,2,1)$	$(0,0,0,0,0,0)$	$3V$	39366		$1A$				$3K$
		6	54	$(0,0,0,1,0,1)$	$(0,0,0,0,0,0)$	$3W$	26244		$1A$				$3L$
		12	108	$(0,0,1,2,1,2)$	$(0,0,0,0,0,0)$	$3X$	13122		$1A$				$3M$
		54	486	$(0,0,0,0,0,1)$	$(1,1,0,2,0,2)$	$9J$	2916		$3C$				$9D$
$4A$	1	1	729	$(0,0,0,0,0,0)$	$(0,0,0,0,0,0)$	$4A$	1440	$2A$					$4B$
$4B$	9	1	81	$(0,0,0,0,0,0)$	$(0,0,0,0,0,0)$	$4B$	864	$2B$					$4A$
		2	162	$(0,0,0,0,0,1)$	$(1,2,1,2,1,2)$	$12A$	432	$6D$	$4B$				$12G$
		2	162	$(0,0,0,0,1,0)$	$(2,1,0,0,0,2)$	$12B$	432	$6F$	$4B$				$12B$
		2	162	$(0,0,0,0,1,1)$	$(0,0,1,2,1,1)$	$12C$	432	$6E$	$4B$				$12D$
		2	162	$(0,0,0,0,1,2)$	$(1,2,2,1,2,0)$	$12D$	432	$6F$	$4B$				$12C$
$4C$	9	1	81	$(0,0,0,0,0,0)$	$(0,0,0,0,0,0)$	$4C$	288	$2A$					$4B$
		4	324	$(0,0,0,0,0,1)$	$(1,0,1,1,2,0)$	$12E$	72	$6C$	$4C$				$12R$
		4	324	$(0,0,0,0,1,0)$	$(0,1,2,1,2,1)$	$12F$	72	$6B$	$4C$				$12O$
$5A$	9	1	81	$(0,0,0,0,0,0)$	$(0,0,0,0,0,0)$	$5A$	180			$1A$			$5A$
		4	324	$(0,0,0,0,1,0)$	$(2,2,2,1,1,1)$	$15A$	45		$5A$	$3A$			$15A$
		4	324	$(0,0,1,0,0,0)$	$(1,0,2,2,2,2)$	$15B$	45		$5A$	$3B$			$15D$
$6A$	9	1	81	$(0,0,0,0,0,0)$	$(0,0,0,0,0,0)$	$6G$	648	$3T$	$2A$				$6H$
		2	162	$(0,0,0,1,2,1)$	$(0,0,0,0,0,0)$	$6H$	324	$3W$	$2A$				$6Q$
		6	486	$(0,0,0,1,0,2)$	$(2,2,0,1,0,1)$	$18A$	108	$9J$	$6A$				$18D$

Table 3: (continued)

$[g]_G$	k	f_j	m_j	d_j	w	$[x]_{\overline{G}}$	$\lvert C_{\overline{G}}(x)\rvert$	2	3	5	7	13	$\mapsto O_8^+(3)$
6B	9	1	81	(0,0,0,0,0,0)	(0,0,0,0,0,0)	6I	648	3O	2A				6E
		2	162	(0,0,0,1,1,1)	(0,0,0,0,0,0)	6J	324	3R	2A				6Q
		6	486	(0,1,0,0,1,0)	(1,1,0,2,0,2)	18B	108	9I	6A				18A
6C	9	1	81	(0,0,0,0,0,0)	(0,0,0,0,0,0)	6K	648	3D	2B				6D
		1	81	(0,0,0,0,0,1)	(0,0,0,0,0,0)	6L	648	3F	2B				6V
		1	81	(0,0,0,0,0,2)	(0,0,0,0,0,0)	6M	648	3E	2B				6W
		1	81	(0,0,0,0,1,0)	(0,0,0,0,0,0)	6N	648	3K	2B				6L
		1	81	(0,0,0,0,1,1)	(0,0,0,0,0,0)	6O	648	3J	2B				6U
		1	81	(0,0,0,0,1,2)	(0,0,0,0,0,0)	6P	648	3L	2B				6M
		1	81	(0,0,0,0,2,0)	(0,0,0,0,0,0)	6Q	648	3I	2B				6O
		1	81	(0,0,0,0,2,1)	(0,0,0,0,0,0)	6R	648	3H	2B				6P
		1	81	(0,0,0,0,2,2)	(0,0,0,0,0,0)	6S	648	3G	2B				6T
6D	9	1	81	(0,0,0,0,0,0)	(0,0,0,0,0,0)	6T	324	3O	2B				6K
		1	81	(0,0,0,0,1,1)	(0,0,0,0,0,0)	6U	324	3R	2B				6Y
		1	81	(0,0,0,0,2,2)	(0,0,0,0,0,0)	6V	324	3R	2B				6X
		2	162	(0,0,0,0,0,1)	(0,0,0,0,0,0)	6W	162	3P	2B				6T
		2	162	(0,0,0,0,1,0)	(0,0,0,0,0,0)	6X	162	3Q	2B				6U
		2	162	(0,0,0,0,2,0)	(0,0,0,0,0,0)	6Y	162	3S	2B				6AB
6E	9	1	81	(0,0,0,0,0,0)	(0,0,0,0,0,0)	6Z	324	3T	2B				6N
		1	81	(0,0,0,0,1,1)	(0,0,0,0,0,0)	6AA	324	3W	2B				6AA
		1	81	(0,0,0,0,2,2)	(0,0,0,0,0,0)	6AB	324	3W	2B				6Z
		2	162	(0,0,0,0,0,1)	(0,0,0,0,0,0)	6AC	162	3U	2B				6V
		2	162	(0,0,0,0,1,0)	(0,0,0,0,0,0)	6AD	162	3V	2B				6W
		2	162	(0,0,0,0,2,0)	(0,0,0,0,0,0)	6AE	162	3X	2B				6AB
8A	1	1	729	(0,0,0,0,0,0)	(0,0,0,0,0,0)	8A	8	4B					8A
9A	9	1	81	(0,0,0,0,0,0)	(0,0,0,0,0,0)	9K	243		3D				9A
		1	81	(0,0,0,0,1,1)	(0,0,0,0,0,0)	9L	243		3D				9T
		1	81	(0,1,0,0,2,0)	(0,0,0,0,0,0)	9M	243		3D				9J
		3	243	(0,0,0,0,0,1)	(0,0,0,0,0,0)	9N	81		3J				9L
		3	243	(0,0,0,0,0,2)	(0,0,0,0,0,0)	9O	81		3G				9K
9B	9	1	81	(0,0,0,0,0,0)	(0,0,0,0,0,0)	9P	243		3D				9D
		1	81	(1,0,0,0,1,1)	(0,0,0,0,0,0)	9Q	243		3D				9H
		1	81	(2,0,0,0,2,0)	(0,0,0,0,0,0)	9R	243		3D				9G
		3	243	(0,0,0,0,1,0)	(0,0,0,0,0,0)	9S	81		3E				9N
		3	243	(0,0,0,0,2,0)	(0,0,0,0,0,0)	9T	81		3F				9M
10A	1	1	729	(0,0,0,0,0,0)	(0,0,0,0,0,0)	10A	20	5A					10A
12A	1	1	729	(0,0,0,0,0,0)	(0,0,0,0,0,0)	12G	36	6G	4A				12R
12B	1	1	729	(0,0,0,0,0,0)	(0,0,0,0,0,0)	12H	36	6I	4A				12O
12C	9	1	81	(0,0,0,0,0,0)	(0,0,0,0,0,0)	12I	108	6K	4B				12A
		1	81	(0,0,0,0,0,1)	(0,0,0,0,0,0)	12J	108	6M	4B				12N
		1	81	(0,0,0,0,0,2)	(0,0,0,0,0,0)	12K	108	6L	4B				12M
		1	81	(0,0,0,1,0,0)	(0,0,0,0,0,0)	12L	108	6Q	4B				12H
		1	81	(0,0,0,1,0,1)	(0,0,0,0,0,0)	12M	108	6S	4B				12K
		1	81	(0,0,0,1,0,2)	(0,0,0,0,0,0)	12N	108	6R	4B				12I
		1	81	(0,0,0,2,0,0)	(0,0,0,0,0,0)	12O	108	6N	4B				12E
		1	81	(0,0,0,2,0,1)	(0,0,0,0,0,0)	12P	108	6P	4B				12F
		1	81	(0,0,0,2,0,2)	(0,0,0,0,0,0)	12Q	108	6O	4B				12L
13A	1	1	729	(0,0,0,0,0,0)	(0,0,0,0,0,0)	13A	13	13D		13D		1A	13B
13B	1	1	729	(0,0,0,0,0,0)	(0,0,0,0,0,0)	13B	13	13A		13A		1A	13A
13C	1	1	729	(0,0,0,0,0,0)	(0,0,0,0,0,0)	13C	13	13B		13B		1A	13B
13D	1	1	729	(0,0,0,0,0,0)	(0,0,0,0,0,0)	13D	13	13C		13C		1A	13A
20A	1	1	729	(0,0,0,0,0,0)	(0,0,0,0,0,0)	20A	20	10A		4A			20A
20B	1	1	729	(0,0,0,0,0,0)	(0,0,0,0,0,0)	20B	20	10A		4A			20A

4. Inertia Factor Groups of $\overline{G} = 3^6{:}L_4(3)$

We have already seen in Section 3 that the action of $G = L_4(3)$ on $N = 3^6$ has four orbits of lengths 1, 234, 234 and 260. By Brauer's theorem (see Theorem 5.1.5 in [18]), the action of G on $Irr(N)$ will also have four orbits of lengths $1, r, s$ and t with $1+r+s+t = 729$ such that $[G{:}H_1] = 1, [G{:}H_2] = r, [G{:}H_3] = s$ and $[G{:}H_4] = t$, where H_1, H_2, H_3 and H_4 are the inertia factor groups of $\overline{G}$. When N is an elementary abelian p-group, then it can be regarded as a vector space V over $F = GF(p)$. Any vector space has a dual (the set of all linear functional maps from V into F) denoted by V^*. Although V and V^* are isomorphic as vector spaces (so they have the same dimension), they may not be equivalent as G-modules. It is easy to show that $N^* = Irr(N)$ and hence the action of G on $Irr(N)$ is the same as action of G on N^*. Seretlo [24] developed a programme for the action of G on V^*. It is found in the Brauer ATLAS that $G \cong O_6^+(3)$ has only one irreducible module of dimension 6 over $GF(3)$ and thus N and N^* are equivalent as G-modules. Hence the actions of G on N and N^* are isomorphic, and so the point stabilizers P_i and stabilizers on N^* (inertia factor groups) are the same. We now have that $r = 234$, $s = 234$ and $t = 260$ and that $H_1 = L_4(3)$, $H_2 \cong H_3 = U_4(2)$ and $H_4 = 3^4{:}2(A_4 \times A_4)$. Using GAP, the number of conjugacy classes of the inertia factor groups are determined and it turns out that

$$|Irr(H_1)| + |Irr(H_2)| + |Irr(H_3)| + |Irr(H_4)| = 29 + 20 + 20 + 42 = 111.$$

This shows that the total contribution of irreducible characters from the four inertia groups is 111 and is equal to the number of classes of $\overline{G}$ as determined in Section 3. The inertia factor groups of $\overline{G}$ are constructed from elements within $G = L_4(3)$ and their generators are as follows:

$H_2 = U_4(2) = \langle \alpha_1, \alpha_2 \rangle$, $\alpha_1 \in 6A$, $\alpha_2 \in 4B$, where

$$\alpha_1 = \begin{pmatrix} 1 & 0 & 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 0 & 0 & 2 \\ 2 & 0 & 0 & 2 & 2 & 2 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \alpha_2 = \begin{pmatrix} 2 & 2 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 2 & 0 & 1 \\ 2 & 2 & 2 & 2 & 1 & 2 \\ 2 & 2 & 1 & 0 & 0 & 1 \\ 2 & 0 & 2 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$H_3 = U_4(2) = \langle \beta_1, \beta_2 \rangle$, $\beta_1 \in 6B$, $\beta_2 \in 4C$, where

$$\beta_1 = \begin{pmatrix} 2 & 2 & 2 & 2 & 0 & 0 \\ 1 & 2 & 2 & 1 & 0 & 1 \\ 0 & 1 & 0 & 2 & 1 & 0 \\ 0 & 1 & 1 & 1 & 2 & 1 \\ 0 & 2 & 2 & 0 & 2 & 2 \\ 2 & 0 & 1 & 1 & 2 & 1 \end{pmatrix}, \quad \beta_2 = \begin{pmatrix} 2 & 2 & 1 & 0 & 1 & 1 \\ 1 & 2 & 2 & 0 & 1 & 1 \\ 2 & 0 & 0 & 1 & 0 & 2 \\ 2 & 2 & 2 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 2 \\ 0 & 2 & 1 & 1 & 2 & 0 \end{pmatrix}$$

$H_4 = 3^4{:}2(A_4 \times A_4) = \langle \gamma_1, \gamma_2, \gamma_3 \rangle$, $\gamma_1, \gamma_2, \gamma_3 \in 2A$, where

$$\gamma_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 & 2 & 1 \\ 1 & 2 & 2 & 1 & 2 & 2 \\ 2 & 1 & 1 & 2 & 2 & 2 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 & 2 & 1 \\ 1 & 2 & 2 & 1 & 2 & 2 \\ 2 & 1 & 1 & 2 & 2 & 2 \end{pmatrix},$$

$$\gamma_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 & 2 & 1 \\ 1 & 2 & 2 & 1 & 2 & 2 \\ 2 & 1 & 1 & 2 & 2 & 2 \end{pmatrix}$$

5. The fusion of H_2, H_3 and H_4 into $G = L_4(3)$

The fusion maps of the inertia factor groups H_2, H_3 and H_4 into $G = L_4(3)$ are obtained, by using the generators of the H_i's and the GAP command "*FusionConjugacyClasses*(H_i, G)". The complete fusion maps of H_2, H_3 and H_4 into $G = L_4(3)$ are shown in Tables 4, 5 and 6 below.

Table 4: The fusion of H_2 into G

$[h]_{H_2}$	$\longrightarrow [g]_{L_4(3)}$	$[h]_{H_2}$	$\longrightarrow [g]_{L_4(3)}$	$[h]_{H_2}$	$\longrightarrow [g]_{L_4(3)}$	$[h]_{H_2}$	$\longrightarrow [g]_{L_4(3)}$
$1A$	$1A$	$3C$	$3D$	$6A$	$6C$	$6F$	$6B$
$2A$	$2B$	$3D$	$3C$	$6B$	$6C$	$9A$	$9B$
$2B$	$2A$	$4A$	$4B$	$6C$	$6E$	$9B$	$9B$
$3A$	$3A$	$4B$	$4C$	$6D$	$6D$	$12A$	$12C$
$3B$	$3A$	$5A$	$5A$	$6E$	$6D$	$12B$	$12C$

Table 5: The fusion of H_3 into G

$[h]_{H_3}$	$\longrightarrow [g]_{L_4(3)}$	$[h]_{H_3}$	$\longrightarrow [g]_{L_4(3)}$	$[h]_{H_3}$	$\longrightarrow [g]_{L_4(3)}$	$[h]_{H_4}$	$\longrightarrow [g]_{L_4(3)}$
$1A$	$1A$	$3C$	$3A$	$6A$	$6D$	$6F$	$6A$
$2A$	$2B$	$3D$	$3D$	$6B$	$6C$	$9A$	$9A$
$2B$	$2A$	$4A$	$4B$	$6C$	$6C$	$9B$	$9A$
$3A$	$3C$	$4B$	$4C$	$6D$	$6E$	$12A$	$12C$
$3B$	$3A$	$5A$	$5A$	$6E$	$6E$	$12B$	$12C$

Table 6: The fusion of H_4 into G

$[h]_{H_4}$	$\longrightarrow [g]_{L_4(3)}$	$[h]_{H_4}$	$\longrightarrow [g]_{L_4(3)}$	$[h]_{H_4}$	$\longrightarrow [g]_{L_4(3)}$	$[h]_{H_4}$	$\longrightarrow [g]_{L_4(3)}$
$1A$	$1A$	$3I$	$3B$	$4A$	$4B$	$6I$	$6C$
$2A$	$2B$	$3J$	$3C$	$4B$	$4B$	$6J$	$6D$
$2B$	$2A$	$3K$	$3B$	$6A$	$6A$	$9A$	$9B$
$3A$	$3C$	$3L$	$3A$	$6B$	$6B$	$9B$	$9AC$
$3B$	$3A$	$3M$	$3B$	$6C$	$6C$	$9C$	$9B$
$3C$	$3D$	$3N$	$3D$	$6D$	$6C$	$9D$	$9A$
$3D$	$3A$	$3O$	$3B$	$6E$	$6E$	$12A$	$12C$
$3E$	$3B$	$3P$	$3A$	$6F$	$6D$	$12B$	$12C$
$3F$	$3A$	$3Q$	$3B$	$6G$	$6C$	$12C$	$12C$
$3G$	$3B$	$3R$	$3C$	$6H$	$6E$	$12D$	$12C$
$3H$	$3D$	$3S$	$3B$				

6. The Fischer-Matrices of $\overline{G} = 3^6{:}L_4(3)$

Having obtained the conjugacy classes of $\overline{G}$ in coset-analysis format and the fusion maps of the inertia factor groups H_2, H_3 and H_4 into G, the Fischer-Clifford matrices of the group $\overline{G} = 3^6{:}L_4(3)$ will be now computed. Programme D in [5] is largely used for automatic determination of a possible candidate for each Fischer-Clifford matrix $M(g)$, $g \in G$, of $\overline{G}$. Then the properties of Fischer-Clifford matrices discussed in detail in [6], [13], [14] and [23] are used to rearrange the rows and columns of this candidate in order to get the unique matrix $M(g)$ for $\overline{G} = 3^6{:}L_4(3)$. The Programme D only works on split extensions $N{:}G$, where N is elementary abelian. Note that since $N = 3^6$ is an elementary abelian p-group, then all the relations hold. For example, considering the conjugacy class $2A$ of $G = L_4(3)$, and by making use of Theorem 5.2.4 and property (e) in [18], $M(2A)$ has the following form with corresponding weights attached to the rows and columns.

$$\begin{array}{lr}
 & \\
 & \\
|C_{H_1}(2A)| = & 2880 \\
|C_{H_2}(2B)| = & 96 \\
|C_{H_3}(2B)| = & 96 \\
|C_{H_4}(2B)| = & 144 \\
 & m_j
\end{array}
\begin{array}{c}
\begin{array}{cccc}
|C_{\overline{G}}(2A)| & |C_{\overline{G}}(6A)| & |C_{\overline{G}}(6B)| & |C_{\overline{G}}(6C)| \\
233280 & 11664 & 7776 & 7776
\end{array} \\
\begin{pmatrix}
1 & 1 & 1 & 1 \\
30 & f & g & h \\
30 & j & k & l \\
20 & n & o & p
\end{pmatrix} \\
\begin{array}{cccc}
9 & 180 & 270 & 270
\end{array}
\end{array}$$

In order to determine the entries f, g, h, j, k, l, n, o and p of the Fischer-Clifford matrix $M(2A)$, the GAP output for programme D for the matrix $M(2A)$ is first generated as,

$$M(2A') = \begin{pmatrix}
1 & 1 & 1 & 1 \\
20 & 2 & 2 & -7 \\
30 & 3 & -6 & 3 \\
30 & -6 & 3 & 3
\end{pmatrix}.$$

Clearly, row 2 of matrix $M(2A')$ becomes row 4 of Fischer-Clifford matrix $M(2A)$. Using the centralizer orders for the classes $6A$, $6B$ and $6C$ of $\overline{G}$, column 4 of $M(2A')$ becomes column 2 of $M(2A)$ and thus, $f = 3$, $j = 3$ and $n = 7$. Now, columns 3 and 4 of matrix $M(2A)$ are going to be identified using the fact that for any $p-$singular element g of any finite group G and irreducible character χ of G , then $\chi(g) \equiv \chi(g^p) \bmod p$. Noting from Table 3 that $\chi(6B) \bmod 2 \equiv \chi(3A)$, the second power map of the class $6B$ of $\overline{G}$ is applied. If we suppose that column 3 in $M(2A')$ becomes column 3 of $M(2A)$, it turns out that $\chi(6B) \bmod 2 \not\equiv \chi(3A)$ for all irreducible characters of $\overline{G}$ on classes $3A$ and $6B$ of $\overline{G}$ and coming from inertia factor groups H_2, H_3 and H_4. Thus column 2 of matrix $M(2A')$ becomes column 3 of $M(2A)$ while column 3 of $M(2A')$ automatically becomes column 4 of $M(2A)$. The final Fischer-Clifford matrix assumes the following structure with $g = 3$, $h = -6$, $k = -6$, $l = 3$, $o = 2$ and $p = 2$.

$$M(2A) = \begin{array}{c} \\ 2880 \\ 96 \\ 96 \\ 144 \\ \\ \end{array} \begin{array}{c} \begin{array}{cccc} 233280 & 11664 & 7776 & 7776 \end{array} \\ \left(\begin{array}{cccc} 1 & 1 & 1 & 1 \\ 30 & 3 & 3 & -6 \\ 30 & 3 & -6 & 3 \\ 20 & -7 & 2 & 2 \end{array} \right) \\ \begin{array}{cccc} 9 & 180 & 270 & 270 \end{array} \end{array}$$

With a quite a number of the Fischer-Clifford matrices of $\overline{G}$ containing entries which involve the complex number $\alpha = -\frac{1}{2} + \frac{\sqrt{-3}}{2}$ such that $\alpha^3 = 1$, the Fischer-Clifford matrix corresponding to the conjugacy class $6D$ of $G = L_4(3)$ is briefly discussed. Using theorem 5.2.4 and property (e) in [18], $M(6D)$ has the following form with corresponding weights attached to the rows and columns:

$$\begin{array}{rc} & \begin{array}{cccccc} |C_{\overline{G}}(6T)| & |C_{\overline{G}}(6U)| & |C_{\overline{G}}(6V)| & |C_{\overline{G}}(6W)| & |C_{\overline{G}}(6X)| & |C_{\overline{G}}(6Y)| \\ 324 & 324 & 324 & 162 & 162 & 162 \end{array} \\ \begin{array}{r} |C_{H_1}(6D)| = 36 \\ |C_{H_2}(6D)| = 36 \\ |C_{H_3}(6E)| = 36 \\ |C_{H_4}(6A)| = 18 \\ |C_{H_4}(6F)| = 18 \\ |C_{H_4}(6J)| = 18 \end{array} & \left(\begin{array}{cccccc} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & a & f & k & p & u \\ 1 & b & g & l & q & v \\ 2 & c & h & m & r & w \\ 2 & d & i & n & s & x \\ 2 & e & j & o & t & y \end{array} \right) \\ m_j & \begin{array}{cccccc} 81 & 81 & 81 & 162 & 162 & 162 \end{array} \end{array}$$

To determine the remaining entries of the Fischer-Clifford matrix $M(6D)$, the GAP output for programme D for $M(6D)$ is computed,

$$M\left(6D'\right) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & -1 & -1 & -1 \\ 2 & 2\overline{\alpha} & 2\alpha & -\alpha & -1 & -\overline{\alpha} \\ 1 & \overline{\alpha} & \alpha & \alpha & 1 & -\overline{\alpha} \\ 2 & 2\alpha & 2\overline{\alpha} & -\overline{\alpha} & -1 & -\alpha \\ 1 & \alpha & \overline{\alpha} & \overline{\alpha} & 1 & \alpha \end{pmatrix}$$

where $\overline{\alpha} = -\frac{1}{2} - \frac{\sqrt{-3}}{2}$ and $\alpha = -\frac{1}{2} + \frac{\sqrt{-3}}{2}$. The rows of Fischer-Clifford matrix $M(6D')$ are then rearranged to match the structure above. Thus we have

$$M\left(6D'\right) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \overline{\alpha} & \alpha & \alpha & 1 & -\overline{\alpha} \\ 1 & \alpha & \overline{\alpha} & \overline{\alpha} & 1 & \alpha \\ 2 & 2 & 2 & -1 & -1 & -1 \\ 2 & 2\overline{\alpha} & 2\alpha & -\alpha & -1 & -\overline{\alpha} \\ 2 & 2\alpha & 2\overline{\alpha} & -\overline{\alpha} & -1 & -\alpha \end{pmatrix}$$

Taking into consideration properties of Fischer-Clifford matrices discussed in [6], [13], [14] and [23], the values of the irreducible characters of $\overline{G}$ on the classes $3R$, $3P$, $3Q$ and $3S$ that have already been obtained by Fischer-Clifford matrix $M(3C)$, and using the fact that $(6U)^2 = 3R$, $(6V)^2 = 3R$, $(6W)^2 = 3P$, $(6X)^2 = 3Q$ and $(6Y)^2 = 3S$ (see Table 3), the final Fischer-Clifford matrix $M(6D)$ assumes the form below.

$$M(6D) = \begin{array}{cc} & \begin{array}{cccccc} 324 & 324 & 324 & 162 & 162 & 162 \end{array} \\ \begin{array}{c} 36 \\ 36 \\ 36 \\ 18 \\ 18 \\ 18 \end{array} & \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \alpha & \overline{\alpha} & \alpha & \overline{\alpha} & 1 \\ 1 & \overline{\alpha} & \alpha & \overline{\alpha} & \alpha & 1 \\ 2 & 2 & 2 & -1 & -1 & -1 \\ 2 & 2\overline{\alpha} & 2\alpha & -\overline{\alpha} & -\alpha & -1 \\ 2 & 2\alpha & 2\overline{\alpha} & -\alpha & -\overline{\alpha} & -1 \end{pmatrix} \\ & \begin{array}{cccccc} 81 & 81 & 81 & 162 & 162 & 162 \end{array} \end{array}$$

For each class representative $g \in L_4(3)$, a Fischer-Clifford matrix $M(g)$ is constructed and listed in Table 7.

TABLE 7. The Fischer-Clifford Matrices of $\overline{G} = 3^6{:}L_4(3)$

$M(g)$	$M(g)$
$M(1A) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 234 & -9 & 18 & -9 \\ 234 & 18 & -9 & -9 \\ 260 & -10 & -10 & 17 \end{pmatrix}$	$M(2A) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 30 & 3 & 3 & -6 \\ 30 & 3 & -6 & 3 \\ 20 & -7 & 2 & 2 \end{pmatrix}$
$M(2B) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & -1 & -1 \\ 2 & -1 & 2 & -1 \\ 4 & -2 & -2 & 1 \end{pmatrix}$	$M(3A) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 9 & 9 & 9 & 9\overline{\alpha} & 9\overline{\alpha} & 9\overline{\alpha} & 9\alpha & 9\alpha & 9\alpha & 0 \\ 9 & 9 & 9 & 9\alpha & 9\alpha & 9\alpha & 9\overline{\alpha} & 9\overline{\alpha} & 9\overline{\alpha} & 0 \\ 9 & 9\overline{\alpha} & 9\alpha & 9 & 9\overline{\alpha} & 9\alpha & 9 & 9\overline{\alpha} & 9\alpha & 0 \\ 9 & 9\alpha & 9\overline{\alpha} & 9 & 9\alpha & 9\overline{\alpha} & 9 & 9\alpha & 9\overline{\alpha} & 0 \\ 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & -1 \\ 9 & 9\overline{\alpha} & 9\alpha & 9\overline{\alpha} & 9\alpha & 9 & 9\alpha & 9 & 9\overline{\alpha} & 0 \\ 9 & 9\alpha & 9\overline{\alpha} & 9\alpha & 9\overline{\alpha} & 9 & 9\overline{\alpha} & 9 & 9\alpha & 0 \\ 9 & 9\overline{\alpha} & 9\alpha & 9\alpha & 9 & 9\overline{\alpha} & 9\overline{\alpha} & 9\alpha & 9 & 0 \\ 9 & 9\alpha & 9\overline{\alpha} & 9\overline{\alpha} & 9 & 9\alpha & 9\alpha & 9\overline{\alpha} & 9 & 0 \end{pmatrix}$
$M(3B) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & \overline{\alpha} & \overline{\alpha} & \overline{\alpha} & \alpha & \alpha & \alpha \\ 1 & 1 & 1 & \alpha & \alpha & \alpha & \overline{\alpha} & \overline{\alpha} & \overline{\alpha} \\ 1 & \overline{\alpha} & \alpha & 1 & \overline{\alpha} & \alpha & 1 & \overline{\alpha} & \alpha \\ 1 & \overline{\alpha} & \alpha & \overline{\alpha} & \alpha & 1 & \alpha & 1 & \overline{\alpha} \\ 1 & \overline{\alpha} & \alpha & \alpha & 1 & \overline{\alpha} & \overline{\alpha} & \alpha & 1 \\ 1 & \alpha & \overline{\alpha} & 1 & \alpha & \overline{\alpha} & 1 & \alpha & \overline{\alpha} \\ 1 & \alpha & \overline{\alpha} & \overline{\alpha} & 1 & \alpha & \alpha & \overline{\alpha} & 1 \\ 1 & \alpha & \overline{\alpha} & \alpha & \overline{\alpha} & 1 & \overline{\alpha} & 1 & \alpha \end{pmatrix}$	$M(3C) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 18 & -9 & -9 & 9 & 0 & 0 \\ 36 & 9 & 9 & 0 & -9 & 0 \\ 2 & 2 & 2 & 2 & 2 & -1 \\ 12 & 6\alpha - 3\overline{\alpha} & -3\alpha + 6\overline{\alpha} & -6 & 3 & 0 \\ 12 & -3\alpha + 6\overline{\alpha} & 6\alpha - 3\overline{\alpha} & -6 & 3 & 0 \end{pmatrix}$
$M(3D) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 18 & -9 & -9 & 9 & 0 & 0 \\ 36 & 9 & 9 & 0 & -9 & 0 \\ 2 & 2 & 2 & 2 & 2 & -1 \\ 12 & 6\alpha - 3\overline{\alpha} & -3\alpha + 6\overline{\alpha} & -6 & 3 & 0 \\ 12 & -3\alpha + 6\overline{\alpha} & 6\alpha - 3\overline{\alpha} & -6 & 3 & 0 \end{pmatrix}$	$M(4A) = (1)$
$M(4B) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & -1 & -1 & -1 \\ 2 & -1 & -1 & 2 & -1 \\ 2 & -1 & 2 & -1 & -1 \\ 2 & -1 & -1 & -1 & 2 \end{pmatrix}$	$M(4C) = \begin{pmatrix} 1 & 1 & 1 \\ 4 & -2 & 1 \\ 4 & 1 & -2 \end{pmatrix}$
$M(5A) = \begin{pmatrix} 1 & 1 & 1 \\ 4 & 1 & -2 \\ 4 & -2 & 1 \end{pmatrix}$	$M(6A) = \begin{pmatrix} 1 & 1 & 1 \\ 6 & -3 & 0 \\ 2 & 2 & -1 \end{pmatrix}$

$\overline{\alpha} = -\frac{1}{2} - \frac{\sqrt{-3}}{2}, \alpha = -\frac{1}{2} + \frac{\sqrt{-3}}{2}$

TABLE 7. (continued)

$M(g)$	$M(g)$
$M(6B) = \begin{pmatrix} 1 & 1 & 1 \\ 6 & -3 & 0 \\ 2 & 2 & -1 \end{pmatrix}$	$M(6C) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & \overline{\alpha} & \overline{\alpha} & \overline{\alpha} & \alpha & \alpha & \alpha \\ 1 & 1 & 1 & \alpha & \alpha & \alpha & \overline{\alpha} & \overline{\alpha} & \overline{\alpha} \\ 1 & \overline{\alpha} & \alpha & \alpha & 1 & \overline{\alpha} & \overline{\alpha} & \alpha & 1 \\ 1 & \alpha & \overline{\alpha} & \overline{\alpha} & 1 & \alpha & \alpha & \overline{\alpha} & 1 \\ 1 & \alpha & \overline{\alpha} & 1 & \alpha & \overline{\alpha} & 1 & \alpha & \overline{\alpha} \\ 1 & \overline{\alpha} & \alpha & 1 & \overline{\alpha} & \alpha & 1 & \overline{\alpha} & \alpha \\ 1 & \alpha & \overline{\alpha} & \alpha & \overline{\alpha} & 1 & \overline{\alpha} & 1 & \alpha \\ 1 & \overline{\alpha} & \alpha & \overline{\alpha} & \alpha & 1 & \alpha & 1 & \overline{\alpha} \end{pmatrix}$
$M(6D) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \alpha & \overline{\alpha} & \alpha & \overline{\alpha} & 1 \\ 1 & \overline{\alpha} & \alpha & \overline{\alpha} & \alpha & 1 \\ 2 & 2 & 2 & -1 & -1 & -1 \\ 2 & 2\overline{\alpha} & 2\alpha & -\overline{\alpha} & -\alpha & -1 \\ 2 & 2\alpha & 2\overline{\alpha} & -\alpha & -\overline{\alpha} & -1 \end{pmatrix}$	$M(6E) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & -1 & -1 & -1 \\ 1 & \alpha & \overline{\alpha} & \alpha & \overline{\alpha} & 1 \\ 1 & \overline{\alpha} & \alpha & \overline{\alpha} & \alpha & 1 \\ 2 & 2\overline{\alpha} & 2\alpha & -\overline{\alpha} & -\alpha & -1 \\ 2 & 2\alpha & 2\overline{\alpha} & -\alpha & -\overline{\alpha} & -1 \end{pmatrix}$
$M(8A) = \begin{pmatrix} 1 \end{pmatrix}$	$M(9A) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 3 & 3\overline{\alpha} & 3\alpha & 0 & 0 \\ 3 & 3\alpha & 3\overline{\alpha} & 0 & 0 \\ 1 & 1 & 1 & \overline{\alpha} & \alpha \\ 1 & 1 & 1 & \alpha & \overline{\alpha} \end{pmatrix}$
$M(9B) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 3 & 3\overline{\alpha} & 3\alpha & 0 & 0 \\ 3 & 3\alpha & 3\overline{\alpha} & 0 & 0 \\ 1 & 1 & 1 & \overline{\alpha} & \alpha \\ 1 & 1 & 1 & \alpha & \overline{\alpha} \end{pmatrix}$	$M(10A) = \begin{pmatrix} 1 \end{pmatrix}$
$M(12A) = \begin{pmatrix} 1 \end{pmatrix}$	$M(12B) = \begin{pmatrix} 1 \end{pmatrix}$
$M(12C) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & \overline{\alpha} & \overline{\alpha} & \overline{\alpha} & \alpha & \alpha & \alpha \\ 1 & 1 & 1 & \alpha & \alpha & \alpha & \overline{\alpha} & \overline{\alpha} & \overline{\alpha} \\ 1 & \alpha & \overline{\alpha} & \overline{\alpha} & 1 & \alpha & \alpha & \overline{\alpha} & 1 \\ 1 & \overline{\alpha} & \alpha & \alpha & 1 & \overline{\alpha} & \overline{\alpha} & \alpha & 1 \\ 1 & \overline{\alpha} & \alpha & 1 & \overline{\alpha} & \alpha & 1 & \overline{\alpha} & \alpha \\ 1 & \alpha & \overline{\alpha} & 1 & \alpha & \overline{\alpha} & 1 & \alpha & \overline{\alpha} \\ 1 & \overline{\alpha} & \alpha & \overline{\alpha} & \alpha & 1 & \alpha & 1 & \overline{\alpha} \\ 1 & \alpha & \overline{\alpha} & \alpha & \overline{\alpha} & 1 & \overline{\alpha} & 1 & \alpha \end{pmatrix}$	$M(13A) = \begin{pmatrix} 1 \end{pmatrix}$
$M(13B) = \begin{pmatrix} 1 \end{pmatrix}$	$M(13C) = \begin{pmatrix} 1 \end{pmatrix}$
$M(13D) = \begin{pmatrix} 1 \end{pmatrix}$	$M(20A) = \begin{pmatrix} 1 \end{pmatrix}$
$M(20B) = \begin{pmatrix} 1 \end{pmatrix}$	

$\overline{\alpha} = -\frac{1}{2} - \frac{\sqrt{-3}}{2}, \alpha = -\frac{1}{2} + \frac{\sqrt{-3}}{2}$

7. The Character Table of $\bar{G} = 3^6{:}L_4(3)$

Having obtained the conjugacy classes of $\overline{G} = 3^6{:}L_4(3)$, the ordinary character tables of all the inertia factor groups available in GAP [11], the fusions of conjugacy classes of the inertia factors into classes of $L_4(3)$ and the Fischer-Clifford matrices of $\overline{G} = 3^6{:}L_4(3)$, the full character table of $\overline{G}$ can now be constructed by following the theoretical outline discussed in Section 2. This character table is a 111×111 complex-valued matrix partitioned row-wise into four blocks $\triangle_1, \triangle_2, \triangle_3$ and $\triangle_4$, where each block corresponds to an inertia group $\overline{H}_i = 3^6{:}H_i$. In fact, $\triangle_1 = \{\chi_i | 1 \le i \le 29\}$, $\triangle_2 = \{\chi_i | 30 \le i \le 49\}$, $\triangle_3 = \{\chi_i | 50 \le i \le 69\}$ and $\triangle_4 = \{\chi_i | 70 \le i \le 111\}$, where $Irr(3^6{:}L_4(3)) = \cup_{i=1}^{4} \triangle_i$. The consistency and accuracy of this table has been tested using Programme C in [22] and the complete character table of the group $\overline{G}$ can be accessed via the link below.

https://drive.google.com/file/d/16oDwruWA0qWOFJNy8JmgOoESw7UoumVR/view?usp=sharing

8. The Fusion of $3^6{:}L_4(3)$ of into $O_8^+(3)$

Since $\overline{G} = 3^6{:}L_4(3)$ is a maximal subgroup of $O_8^+(3)$ of index 1120, then the action of $O_8^+(3)$ on the cosets of $\overline{G}$ gives rise to a permutation character $\chi(O_8^+(3)|\bar{G})$ of degree 1120. From the ATLAS of finite groups [8], $\chi(O_8^+(3)|\bar{G}) = 1a + 300a + 819a$, where $1a$, $300a$ and $819a$ are irreducible characters of $O_8^+(3)$ of degrees $1, 300$ and 819 respectively. Using the information provided by the conjugacy classes of $3^6{:}L_4(3)$ and $O_8^+(3)$, the power maps and the permutation character of $O_8^+(3)$ of degree 1120, a partial fusion of $\overline{G}$ into $O_8^+(3)$ is obtained. To complete the fusion maps, irreducible characters of $O_8^+(3)$ of small degrees are restricted to $3^6{:}L_4(3)$. To determine the restrictions of irreducible characters of $O_8^+(3)$ to $3^6{:}L_4(3)$, the technique of set intersections for characters which has been discussed in detail in [1], [17] and [18] is used.

Let ρ be the character of $L_4(3)$ afforded by the regular representation of $L_4(3)$. Then $\rho = \sum_{i=1}^{29} \varepsilon_i \phi_i$, where $\phi_i \in Irr(L_4(3))$ and $\varepsilon_i = deg(\phi_i)$. Thus ρ can be regarded as a character of $3^6{:}L_4(3)$ which contains 3^6 in its kernel such that,

$$\rho(g) = \begin{cases} |L_4(3)| & \text{if } g \in 3^6 \\ 0 & \text{otherwise} \end{cases}$$

If ψ is a character of $O_8^+(3)$, then

$$\begin{aligned}\langle \rho,\psi\rangle_{\overline{G}} &= \tfrac{1}{|3^6:L_4(3)|}\{\rho(1A)\psi(1A)+234\rho(3A)\psi(3A)+234\rho(3B)\psi(3B)+260\rho(3C)\psi(3C)\}\\ &= \tfrac{1}{|3^6:L_4(3)|}\{|L_4(3)|\psi(1A)+234|L_4(3)|\psi(3A)+234|L_4(3)|\psi(3B)+260|L_4(3)|\psi(3C)\}\\ &= \tfrac{1}{729}\{\psi(1A)+234\psi(3A)+234\psi(3B)+260\psi(3C)\}\\ &= \langle \psi_{3^6},\tau_1\rangle,\end{aligned}$$

where τ_1 is the identity character of 3^6 and ψ_{3^6} is the restriction of ψ to 3^6. Also for ψ it is obtained that;

$$\psi_{3^6} = a_1\theta_1 + a_2\theta_2 + a_3\theta_3 + a_4\theta_4$$

where $a_1,\ a_2,\ a_3,\ a_4 \in \mathbf{N}\cup\{0\}$ and $\theta_i,\ i\in\{1,2,3,4\}$ are sums of the irreducible characters of 3^6 which are in one orbit under action of $L_4(3)$ on $Irr(3^6)$. Letting $\tau_j \in Irr(3^6)$, where $j\in\{1,2,...,729\}$, then,

$$\begin{aligned}\theta_1 &= \tau_1, deg(\theta_1)=1\\ \theta_2 &= \textstyle\sum_{j=2}^{235}\tau_j, deg(\theta_2)=234\\ \theta_3 &= \textstyle\sum_{j=236}^{469}\tau_j, deg(\theta_3)=234\\ \theta_4 &= \textstyle\sum_{j=470}^{729}\tau_j, deg(\theta_4)=260\end{aligned}$$

Hence

$$\psi_{3^6} = a_1\tau_1 + a_2\sum_{j=2}^{235}\tau_j + a_3\sum_{j=236}^{469}\tau_j + a_4\sum_{j=470}^{729}\tau_j$$

and,

$$\langle\psi_{3^6},\psi_{3^6}\rangle = a_1^2+234a_2^2+234a_3^2+260a_4^2$$

$$= \tfrac{1}{729}\{\psi(1A)\psi(1A)+234\psi(3A)\psi(3A)+234\psi(3B)\psi(3B)+260\psi(3C)\psi(3C)\},$$

where

$$a_1 = \langle\psi_{3^6},\tau_1\rangle = \langle\rho,\psi\rangle_{\bar{G}}$$

Now, the above results are applied to some irreducible characters of $O_8^+(3)$ of small degrees, which in this case are $\psi_1 = 260a$, $\psi_2 = 260b$, $\psi_3 = 260c$, $\psi_4 = 819b$, $\psi_5 = 819c$, $\psi_6 = 2275a$, $\psi_7 = 2275b$,$\psi_8 = 2275c$, and $\psi_9 = 2275d$ of degrees 1, 260, 260, 260, 819, 819, 2275, 2275, 2275, and 2275, respectively (see ATLAS for character table of $O_8^+(3)$). From the partial fusion that has already been determined, the class $3C$ of $\overline{G}$ must fuse into the class $3A$ of $O_8^+(3)$. Also, the classes $3A$ and $3B$ of $\overline{G}$ must each fuse into one of the following classes, $3B$, $3C$, $3D$, $3E$, $3F$ and $3G$ of $O_8^+(3)$ such that the condition $a_1 = \langle\psi_{3^6},\tau_1\rangle \in \mathbf{N}\cup\{0\}$ is satisfied. The values of $\psi_1 = 260a$ on the classes $3A$, $3B$ and $3C$ of $O_8^+(3)$ violate this condition. The only combination that satisfies this condition is $3A$, $3B$ and $3E$ and thus for ψ_1 the following result is obtained,

$$a_1 = \langle \rho, \psi \rangle_{\overline{G}} = \tfrac{1}{729}\{260 + 234(17) + 234(44) + 260(17)\} = 26$$

Since the degree of ψ_1 is 260 then,

$$a_1 + 234a_2 + 234a_3 + 260a_4 = 260$$

and so we must have that $a_1 = 26$, $a_2 = 1$, $a_3 = 0$ and $a_4 = 0$ or $a_1 = 26$, $a_2 = 0$, $a_3 = 1$ and $a_4 = 0$. It turns out that $(\psi_1)_{3^6:L_4(3)}$ is a sum of two irreducible characters of $3^6{:}L_4(3)$ of degrees 26 and 234, respectively. Based on the partial fusion of $3^6{:}L_4(3)$ into $O_8^+(3)$ that has already been done, the result below is obtained,

$$(\psi_1)_{3^6:L_4(3)} = \chi_3 + \chi_{50}$$

Similarly, for $\psi_2 = 260b$, a_1 is obtained as follows,

$$a_1 = \langle \rho, \psi \rangle_{\overline{G}} = \tfrac{1}{729}\{260 + 234(-10) + 234(-10) + 260(17)\} = 0$$

Since the degree of ψ_2 is 260 then

$$a_1 + 234a_2 + 234a_3 + 260a_4 = 260$$

so that $a_1 = 0$, $a_2 = 0$, $a_0 = 0$ and $a_4 = 1$ and thus $(\psi_2)_{3^6:L_4(3)}$ is an irreducible character of $3^6{:}L_4(3)$ of degree 260. Based on the partial fusion of $3^6{:}L_4(3)$ into $O_8^+(3)$ that has already been done, it turns out that

$$(\psi_2)_{3^6:L_4(3)} = \chi_{71}.$$

Applying the same procedure to ψ_3, ψ_4, ψ_5, ψ_6, ψ_7, ψ_8 and ψ_9, we obtained the restricted characters below.

$$\begin{aligned}
(\psi_3)_{3^6:L_4(3)} &= \chi_{76}\\
(\psi_4)_{3^6:L_4(3)} &= \chi_4 + \chi_{79}\\
(\psi_5)_{3^6:L_4(3)} &= \chi_4 + \chi_{80}\\
(\psi_6)_{3^6:L_4(3)} &= \chi_6 + \chi_{32} + \chi_{90}\\
(\psi_7)_{3^6:L_4(3)} &= \chi_6 + \chi_{31} + \chi_{91}\\
(\psi_8)_{3^6:L_4(3)} &= \chi_7 + \chi_{51} + \chi_{87}\\
(\psi_9)_{3^6:L_4(3)} &= \chi_7 + \chi_{52} + \chi_{86}
\end{aligned}$$

By making use of the partial fusion which has already been determined, the values of $\psi_1, ..., \psi_9$ on the classes of $O_8^+(3)$ and the values of $(\psi_1)_{3^6:L_4(3)}, ..., (\psi_9)_{3^6:L_4(3)}$ on the classes of $3^6{:}L_4(3)$, the fusion of $3^6{:}L_4(3)$ into $O_8^+(3)$ is completed and is found in Table 3.

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