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(Δ_v^m, f)-lacunary statistical convergence of order α

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Abstract

In this paper, we define the space $S_\theta^\alpha(\Delta_v^m, f)$ of all (Δ_v^m, f)-lacunary statistical convergent sequences of order α with the help of unbounded modulus function f, lacunary sequence (θ), generalized difference operator Δ_v^m and real number $\alpha \in (0, 1]$. We also introduce the space $\omega_\theta^\alpha(\Delta_v^m, f)$ of all strong (Δ_v^m, f)-lacunary summable sequences of order α. Properties related to these spaces are studied. Inclusion relations between spaces $S_\theta^\alpha(\Delta_v^m, f)$ and $\omega_\theta^\alpha(\Delta_v^m, f)$ are established under certain conditions.

Keywords: *Statistical convergence; Modulus function; Lacunary sequence; Difference sequence space.*
2010 Mathematics Subject Classification: *40A05; 40A35; 40F05; 46A45.*

1. Introduction

The idea of statistical convergence was introduced by Steinhaus [19] and Fast [8] in 1951 and later reintroduced by Schoenberg [16] independently in 1959. Further, it was investigated from the point of view of sequence spaces and related with summability theory by Fridy [10]. After work of Fridy, statistical convergence is extensively studied in summability theory by many researchers till now. Statistical convergence is closely related to the natural density of subsets positive integers. The natural density of a subset K of $\mathbf{N}$ is defined by

$$\delta(K) = \lim_{n\to\infty} \frac{1}{n}|\{k \in K : k \leq n\}|, \text{ provided limit exists,}$$

where $|A|$ denotes the cardinality of set A.
A sequence $x = (x_k)$ is said to be statistical convergent to the number L if for every $\varepsilon > 0$,

$$\delta\left(\{k \in \mathbf{N} : |x_k - L| \geq \varepsilon\}\right) = 0, \tag{1.1}$$

$$\text{i.e., } \lim_{n\to\infty} \frac{1}{n}|\{k \leq n : |x_k - L| \geq \varepsilon\}| = 0. \tag{1.2}$$

Lacunary sequence is an increasing integer sequence $\theta = (k_r)$ with $k_0 = 0$ and $h_r = (k_r - k_{r-1}) \to \infty$ as $r \to \infty$. In this paper, we denote I_r and q_r by an interval $(k_{r-1}, k_r]$ and the ratio $\dfrac{k_r}{k_{r-1}}$, respectively.
In 1978, Freedman et al.[9] introduced the space N_θ of lacunary strongly convergent sequences as follows:

$$N_\theta = \left\{ x = (x_k) \in w : \text{there exists } L \text{ such that } h_r^{-1} \sum_{k\in I_r} |x_k - L| \to 0 \right\},$$

where w denotes the space of all sequences of complex numbers.
In 1993, Fridy and Orhan[11] introduced the concept of lacunary statistical convergence as follows:
A sequence $x = (x_k)$ is said to be lacunary statistical convergent to l if for every $\varepsilon > 0$,

$$\lim_{r\to\infty} \frac{1}{h_r}|\{k \in I_r : |x_k - l| \geq \varepsilon\}| = 0.$$

In 2014, Şengül and Et [17] introduced lacunary statistical convergent sequence of order $\alpha (0 < \alpha \leq 1)$. The sequence $x = (x_k)$ is said to be

S_θ^α-statistically convergent to l if for every $\varepsilon > 0$,

$$\lim_{r\to\infty} \frac{1}{h_r^\alpha} |\{k \in I_r : |x_k - l| \geq \varepsilon\}| = 0,$$

where $(h_r^\alpha) = (h_1^\alpha, h_2^\alpha, \cdots, h_r^\alpha, \cdots)$.

Kizmaz [13] introduced difference operator Δ for ℓ_∞, c and c_0. Further, Et and Çolak [4] generalized the notion of difference operator Δ for fixed positive integer m by

$$X(\Delta^m) = \{x = (x_k) \in w : \Delta^m x \in X\} \text{ for } X = \ell_\infty, c \text{ and } c_0,$$

where $\Delta^m x = (\Delta^m x_k) = (\Delta^{m-1} x_k - \Delta^{m-1} x_{k+1})$ for $m \geq 1$ and $\Delta^0 x = (x_k)$. Et and Esi [5] generalized the space $X(\Delta^m)$ by taking the sequence $v = (v_k)$ of non-zero complex numbers. They defined the sequence space $X(\Delta_v^m)$ as follows:

$$X(\Delta_v^m) = \{x = (x_k) \in w : \Delta_v^m x \in X\} \text{ for } X = \ell_\infty, c \text{ and } c_0,$$

where $\Delta_v^0 x_k = (v_k x_k)$ and $\Delta_v^m x_k = \sum_{i=0}^{m} (-1)^i \binom{m}{i} v_{k+i} x_{k+i}$, for $m \geq 1$.

In 2005, Tripathy and Et [21] used operator Δ^m to introduce idea of Δ^m-lacunary statistical convergent sequences and Δ^m-lacunary strongly summable sequences. The notion of difference operator was investigated from different aspects by Tripathy and Mahanta [22], Tripathy and Dutta [20], Haloi et al.[12], Et and Gidemen[6] and many others over the years.

Following Maddox[14], we recall that $f : [0, \infty) \to [0, \infty)$ is called modulus function if (i) $f(x) = 0$ if and only if $x = 0$, (ii) $f(x+y) \leq f(x) + f(y)$, (iii) f is increasing, (iv) f is continuous from the right at 0.
In 2014, Aizpuru et al.[1] generalized concept of natural density by introducing f-density of a subset K of positive integers with the help of unbounded modulus function f by

$$\delta^f(K) = \lim_{n\to\infty} \frac{f(|\{k \in K : k \leq n\}|)}{f(n)}, \text{ provided limit exists}$$

Bhardwaj and Dhawan[7] defined the concepts of f-density and f-statistical convergence of order α. Further, Şengül and Et [18] introduced f-lacunary statistical convergence and strong f-lacunary summability of order α. In 2019, Et and Gidemen[6] have introduced $\Delta_v^m(f)$-statistical convergence of order α as follows:

A sequence $x = (x_k)$ is said to be $\Delta_v^m(f)$-statistical convergent of order α to L for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{f(n^\alpha)} f(|\{k \leq n : |\Delta_v^m x_k - L| \geq \varepsilon\}|) = 0.$$

The set of all (Δ_v^m, f)-statistical convergent sequences of order α is denoted by $S^\alpha(\Delta_v^m, f)$. Recently, Dowari and Tripathy [2, 3] investigated concepts of complex uncertain sequences with the help of lacunary convergence and derived some interesting results.

Aim of this paper is to generalize and unify some well-known results in the area of lacunary statistical convergence and f-statistical convergence. In this paper, we have defined (Δ_v^m, f)-lacunary statistical convergence of order α by introducing the space $S_\theta^\alpha(\Delta_v^m, f)$ with the help of lacunary sequence θ, generalized difference operator Δ_v^m and unbounded modulus function f. We also introduced space $\omega_\theta^\alpha(\Delta_v^m, f)$ of all strongly (Δ_v^m, f)-lacunary summable sequences of order α. Some inclusion relations between these spaces are obtained under certain conditions.

Throughout this paper, we assume that f is an unbounded modulus function, (v_k) is a fixed sequence of non-zero complex numbers, m is a fixed positive integer, $\theta = (k_r)$ is a lacunary sequence and α, β are real numbers such that $0 < \alpha \leq \beta \leq 1$.

1.1. Introduced definitions

Definition 1.1. Let $\theta = (k_r)$ be a lacunary sequence and $\alpha \in (0, 1]$. A sequence $x = (x_k)$ is said to be (Δ_v^m, f)-lacunary statistical convergent of order α to L if for every $\varepsilon > 0$,

$$\lim_{r\to\infty} \frac{1}{f(h_r^\alpha)} f(|\{k \in I_r : |\Delta_v^m x_k - L| \geq \varepsilon\}|) = 0.$$

In this case, we write $S_\theta^\alpha(\Delta_v^m, f)$-$\lim x_k = L$. The set of all (Δ_v^m, f)-lacunary statistical convergent sequences of order α is denoted by $S_\theta^\alpha(\Delta_v^m, f)$. $S_\theta^\alpha(\Delta_v^m, f)$ is denoted by $S_\theta^\alpha(\Delta_v^m)$ if $f(x) = x$ and by $S_\theta(\Delta_v^m, f)$ if $\alpha = 1$.

Definition 1.2. A sequence $x = (x_k)$ is said to be strongly (Δ_v^m, f)-lacunary summable of order α if there is number L such that

$$\lim_{r\to\infty} \frac{1}{f(h_r^\alpha)} \sum_{k\in I_r} f(|\Delta_v^m x_k - L|) = 0.$$

The set of all strongly (Δ_v^m, f)-lacunary summable sequences of order α is denoted by $\omega_\theta^\alpha(\Delta_v^m, f)$. We write $\omega_\theta^\alpha(\Delta_v^m)$ and $\omega_\theta(\Delta_v^m, f)$ for $f(x) = x$ and $\alpha = 1$, respectively.
For $f(x) = x, \alpha = 1, m = 0$ and $(v_k) = (1, 1, 1, \cdots)$, the space $\omega_\theta^\alpha(\Delta_v^m, f)$ coincides with the space N_θ, studied by Freedman et al.[9].
Also, the space of all strongly $\Delta_v^m(f)$-Cesàro summable of sequences of order α is defined as:

$$\omega^\alpha(\Delta_v^m, f) = \left\{ x \in \omega : \lim_{n\to\infty} \frac{1}{f(n^\alpha)} \sum_{k=1}^{n} f\left(|\Delta_v^m x_k - L|\right) = 0, \text{ for some } L \right\}.$$

2. Main results

Theorem 2.1. *Let $x = (x_k)$ and $y = (y_k)$ be any two sequences. Then*

1. *If $S_\theta^\alpha(\Delta_v^m, f)$-$\lim x_k = x_0$ and $c \in \mathbf{C}$, then $S_\theta^\alpha(\Delta_v^m, f)$-$\lim cx_k = cx_0$.*
2. *If $S_\theta^\alpha(\Delta_v^m, f)$-$\lim x_k = x_0$ and $S_\theta^\alpha(\Delta_v^m, f)$-$\lim y_k = y_0$, then $S_\theta^\alpha(\Delta_v^m, f)$-$\lim(x_k + y_k) = x_0 + y_0$.*

Proof. Omitted. $\square$

2.1. Results on $S_\theta^\alpha(\Delta_v^m, f)$ & $\omega_\theta^\alpha(\Delta_v^m, f)$

Theorem 2.2. *Following inclusions hold for any α, β with $0 < \alpha \leq \beta \leq 1$:*

1. *$S_\theta^\alpha(\Delta_v^m, f) \subseteq S_\theta^\beta(\Delta_v^m, f)$,*
2. *$\omega_\theta^\alpha(\Delta_v^m, f) \subseteq \omega_\theta^\beta(\Delta_v^m, f)$.*

Proof. Proof of both parts follows by noting that $f(h_r^\alpha) \leq f(h_r^\beta)$ for all $r \in \mathbf{N}$ if $0 < \alpha \leq \beta \leq 1$ due to increasing property of modulus function f. $\square$

Theorem 2.3. *Let $\theta = (k_r)$ be lacunary sequence such that $\liminf_r q_r > 1$, f be an unbounded modulus function such that for some positive constant c, $f(xy) \geq cf(x)f(y)$ for all $x \geq 0, y \geq 0$. Then*

1. *$S^\alpha(\Delta_v^m, f) \subseteq S_\theta^\beta(\Delta_v^m, f)$,*
2. *$\omega^\alpha(\Delta_v^m, f) \subseteq \omega_\theta^\beta(\Delta_v^m, f)$.*

Proof. Let $\liminf_r q_r > 1$. Then there exists a $\delta > 0$ such that $q_r \geq 1+\delta$ for sufficiently large r. This implies
$\frac{k_{r-1}}{k_r} \leq \frac{1}{1+\delta} \Rightarrow \frac{h_r}{k_r} \geq \frac{\delta}{1+\delta} \Rightarrow h_r^\alpha \geq \frac{\delta^\alpha}{(1+\delta)^\alpha} k_r^\alpha \Rightarrow f(h_r^\alpha) \geq f\left(\frac{\delta^\alpha}{(1+\delta)^\alpha} k_r^\alpha\right)$,
as f is increasing.
This means $f(h_r^\alpha) \geq cf\left(\frac{\delta^\alpha}{(1+\delta)^\alpha}\right) f\left(k_r^\alpha\right)$, due to assumption $f(xy) \geq cf(x)f(y)$.
As modulus function f is increasing and $\alpha \leq \beta$, so $f(h_r^\beta) \geq f(h_r^\alpha)$. This gives us

$$f(h_r^\beta) \geq cf\left(\frac{\delta^\alpha}{(1+\delta)^\alpha}\right) f\left(k_r^\alpha\right) \tag{2.1}$$

1. Let $x \in S^\alpha(\Delta_v^m, f)$. Now, by using increasing property of modulus function f and inequality (2.1), we have

$$\begin{aligned}
&\tfrac{1}{f(k_r^\alpha)} f(|\{k \leq k_r : |\Delta_v^m x_k - L| \geq \varepsilon\}|) \\
&\qquad \geq \tfrac{1}{f(k_r^\alpha)} f(|\{k \in I_r : |\Delta_v^m x_k - L| \geq \varepsilon\}|) \\
&\qquad \geq cf\left(\tfrac{\delta^\alpha}{(1+\delta)^\alpha}\right) \tfrac{1}{f(h_r^\beta)} f(|\{k \in I_r : |\Delta_v^m x_k - L| \geq \varepsilon\}|).
\end{aligned}$$

   Taking limit $r \to \infty$, we get $x \in S_\theta^\alpha(\Delta_v^m, f)$ and inclusion follows.

2. Let $x \in \omega^\alpha(\Delta_v^m, f)$. Then by inequality (2.1), we have

$$\frac{1}{f(k_r^\alpha)} \sum_{k \leq k_r} f\left(|\Delta_v^m x_k - L|\right) \geq cf\left(\frac{\delta^\alpha}{(1+\delta)^\alpha}\right) \frac{1}{f(h_r^\beta)} \sum_{k \in I_r} f\left(|\Delta_v^m x_k - L|\right).$$

Taking limit $r \to \infty$ on both sides, we can obtain the required inclusion. □

Theorem 2.4. *Let $\theta = (k_r)$ be a lacunary sequence such that $\limsup_r q_r < \infty$ and unbounded modulus function f be such that for some positive constant c, $f(t) \leq ct$ for all $t \geq 0$. Then*

1. $S_\theta^\alpha(\Delta_v^m, f) \subseteq S^\beta(\Delta_v^m, f)$,
2. $\omega_\theta^\alpha(\Delta_v^m, f) \subseteq \omega^\beta(\Delta_v^m, f)$.

Proof. (i) If $\limsup_r q_r < \infty$, then there exist $H > 0$ such that $q_r < H$ for all $r \in \mathbf{N}$. Let $x \in S_\theta^\alpha(\Delta_v^m, f)$. Then for any $\varepsilon > 0$, there exists $r_0 \in \mathbf{N}$ such that

$$\begin{aligned}\frac{1}{f(h_r^\alpha)} f(|\{k \in I_r : |\Delta_v^m x_k - L| \geq \varepsilon\}|) &< \varepsilon \text{ for all } r > r_0\\ \Rightarrow \frac{N_r}{f(h_r)} &< \varepsilon \text{ for all } r > r_0,\end{aligned}$$

where $N_r = f(|\{k \in I_r : |\Delta_v^m x_k - L| \geq \varepsilon\}|)$.
Now, let $M = \max\{N_r : 1 \leq r \leq r_0\}$ and n be any integer satisfying $k_{r-1} < n \leq k_r$. By subadditive property of f, we have

$$\begin{aligned}
&\tfrac{1}{f(n^\beta)} f\left(|\{k \leq n : |\Delta_v^m x_k - L| \geq \varepsilon\}|\right)\\
&\quad\leq \tfrac{1}{f(k_{r-1}^\alpha)} f\left(|\{k \leq k_r : |\Delta_v^m x_k - L| \geq \varepsilon\}|\right)\\
&\quad= \tfrac{1}{f(k_{r-1}^\alpha)} \left(N_1 + N_2 + \cdots + N_{r_0} + N_{r_0+1} + \cdots + N_r\right)\\
&\quad\leq \tfrac{r_0 \cdot M}{f(k_{r-1}^\alpha)} + \tfrac{1}{f(k_{r-1}^\alpha)} \left(f(h_{r_0+1}) \tfrac{N_{r_0+1}}{f(h_{r_0+1})} + \cdots + f(h_r) \tfrac{N_r}{f(h_r)}\right)\\
&\quad\leq \tfrac{r_0 \cdot M}{f(k_{r-1}^\alpha)} + \tfrac{1}{f(k_{r-1}^\alpha)} \left(\sup_{r > r_0} \tfrac{N_r}{f(h_r)}\right) c \cdot (h_{r_0+1} + \cdots + h_r)\\
&\quad\leq \tfrac{r_0 \cdot M}{f(k_{r-1}^\alpha)} + \varepsilon \cdot c \tfrac{k_r - k_{r_0}}{f(k_{r-1}^\alpha)}\\
&\quad\leq \tfrac{r_0 \cdot M}{f(k_{r-1}^\alpha)} + \varepsilon(cH).
\end{aligned}$$

Taking limit in above inequality, the inclusion follows.
(ii) Proof is similar to part (i), so we omit it. □

Theorem 2.5. *Let $\theta = (k_r)$ be lacunary sequence such that $\lim_{r\to\infty} \dfrac{f(h_r^\beta)}{f(k_r^\alpha)} > 0$ and α, β be real numbers such that $0 < \alpha \leq \beta \leq 1$. Then*

1. $S^\alpha(\Delta_v^m, f) \subseteq S_\theta^\beta(\Delta_v^m, f)$,
2. $\omega^\alpha(\Delta_v^m, f) \subseteq \omega_\theta^\beta(\Delta_v^m, f)$.

Proof. (i) Let $x \in S^\alpha(\Delta_v^m, f)$. Then $\lim_{n\to\infty} \dfrac{1}{f(n^\alpha)} f(|\{k \leq n : |\Delta_v^m x_k - L| \geq \varepsilon\}|) = 0$, which gives us $\lim_{r\to\infty} \dfrac{1}{f(k_r^\alpha)} f(|\{k \leq k_r : |\Delta_v^m x_k - L| \geq \varepsilon\}|) = 0$.
Let $\varepsilon > 0$. Then for each r,
$\{k \in I_r : |\Delta_v^m x_k - L| \geq \varepsilon\} \subseteq \{k \leq k_r : |\Delta_v^m x_k - L| \geq \varepsilon\}$.
By increasing property of modulus function, we get

$$\begin{aligned} f(|\{k\in I_r : |\Delta_v^m x_k - L| \geq \varepsilon\}|) &\leq f(|\{k \leq k_r : |\Delta_v^m x_k - L| \geq \varepsilon\}|) \\ &\Rightarrow \tfrac{f(h_r^\beta)}{f(k_r^\alpha)} \tfrac{1}{f(h_r^\beta)} f(|\{k\in I_r : |\Delta_v^m x_k - L| \geq \varepsilon\}|) \\ &\leq \tfrac{1}{f(k_r^\alpha)} f(|\{k \leq k_r : |\Delta_v^m x_k - L| \geq \varepsilon\}|). \end{aligned}$$

By taking $r \to \infty$ both sides and using given assumption, inclusion follows.
(ii) Inclusion follows same as in part (i) by noting that

$$\sum_{k\in I_r} f\left(|\Delta_v^m x_k - L|\right) \leq \sum_{k\leq k_r} f\left(|\Delta_v^m x_k - L|\right), \text{ holds for any } r \in \mathbf{N}.$$

□

Theorem 2.6. *Let α and β be real numbers such that $0 < \alpha \leq \beta \leq 1$. Then*
$S_\theta^\alpha(\Delta_v^m, f) \subseteq S_\theta^\beta(\Delta_v^m)$.

Proof. Let $x \in S_\theta^\alpha(\Delta_v^m, f)$. Then for every $i \in \mathbf{N}$, there exists $n_0 \in \mathbf{N}$ such that whenever $n \geq n_0$, we have

$$\begin{aligned} \frac{1}{f(h_r^\alpha)} f(|\{k\in I_r : |\Delta_v^m x_k - L| \geq \varepsilon\}|) &\leq \frac{1}{i} \\ \Rightarrow f(|\{k\in I_r : |\Delta_v^m x_k - L| \geq \varepsilon\}|) &\leq \frac{1}{i} f\left(\frac{i\cdot h_r^\alpha}{i}\right). \end{aligned}$$

By using subadditive property of modulus function f, we get

$$\begin{aligned} f(|\{k\in I_r : |\Delta_v^m x_k - L| \geq \varepsilon\}|) &\leq \frac{1}{i} i f\left(\frac{h_r^\alpha}{i}\right) \\ \Rightarrow \quad |\{k\in I_r : |\Delta_v^m x_k - L| \geq \varepsilon\}| &\leq \frac{h_r^\alpha}{i}, \text{ as } f \text{ is increasing} \\ \Rightarrow \quad \frac{1}{h_r^\beta} |\{k\in I_r : |\Delta_v^m x_k - L| \geq \varepsilon\}| &\leq \frac{1}{i}. \end{aligned}$$

Taking limit $r \to \infty$, we have $x \in S_\theta^\beta(\Delta_v^m)$. □

Lemma 2.1. *[15] For any modulus function f, $\lim_{t\to\infty} \frac{f(t)}{t} = \inf\left\{\frac{f(t)}{t} : t > 0\right\}$.*

Theorem 2.7. *Let modulus function f be such that $\lim_{t\to\infty} \frac{f(t)}{t} > 0$. Then*
$\omega_\theta^\alpha(\Delta_v^m, f) \subseteq \omega_\theta^\beta(\Delta_v^m)$.

Proof. Let $x \in \omega_\theta^\alpha(\Delta_v^m, f)$ and $\lim_{t\to\infty} \frac{f(t)}{t} = \gamma$. Then $\gamma = \inf\left\{\frac{f(t)}{t} : t > 0\right\}$ by Lemma 2.1. This means $t \le \gamma^{-1} f(t)$ for all $t \ge 0$. Now,

$$\begin{aligned} \frac{1}{f(h_r^\beta)} \sum_{k\in I_r} |\Delta_v^m x_k - L| &\le \frac{1}{f(h_r^\beta)} \sum_{k\in I_r} \gamma^{-1} f(|\Delta_v^m x_k - L|) \\ &\le \frac{\gamma^{-1}}{f(h_r^\alpha)} \sum_{k\in I_r} f(|\Delta_v^m x_k - L|). \end{aligned}$$

By taking limit $r \to \infty$, inclusion follows. □

2.2. Inclusion relations between spaces $S_\theta^\alpha(\Delta_v^m, f)$ & $\omega_\theta^\alpha(\Delta_v^m, f)$

Theorem 2.8. *Let f be an unbounded modulus function satisfying $f(xy) \ge cf(x)f(y)$ for some positive constant c. Then $\omega_\theta^\alpha(\Delta_v^m, f) \subseteq S_\theta^\beta(\Delta_v^m, f)$.*

Proof. Let $x \in \omega_\theta^\alpha(\Delta_v^m, f)$ and $\varepsilon > 0$. Then by using subadditive and increasing property of modulus function, we have

$$\begin{aligned} \tfrac{1}{f(h_r^\alpha)} \textstyle\sum_{k\in I_r} f\left(|\Delta_v^m x_k - L|\right) &\ge \tfrac{1}{f(h_r^\alpha)} f\left(\textstyle\sum_{k\in I_r} |\Delta_v^m x_k - L|\right) \\ &\ge \tfrac{1}{f\left(h_r^\beta\right)} f\left(\sum_{\substack{k\in I_r \\ |\triangle_v^m x_k - L| \ge \varepsilon}} |\triangle_v^m x_k - L| \right) \\ &\ge \tfrac{1}{f(h_r^\beta)} f\left(|\{k \in I_r : |\Delta_v^m x_k - L| \ge \varepsilon\}|\,\varepsilon\right) \\ &\ge \tfrac{c}{f(h_r^\beta)} f\left(|\{k \in I_r : |\Delta_v^m x_k - L| \ge \varepsilon\}|\right) f(\varepsilon), \end{aligned}$$

using $f(xy) \ge cf(x)f(y)$.

Taking limit $r \to \infty$, we get $x \in S_\theta^\beta(\Delta_v^m, f)$. □

Theorem 2.9. *Suppose modulus function f and lacunary sequence $\theta = (k_r)$ are such that $\lim_{t\to\infty} \frac{f(t)}{t} > 0$ and $\lim_{r\to\infty} \frac{f(h_r)}{f(h_r^\alpha)} = 1$. Then $S_\theta^\alpha(\Delta_v^m, f) \cap \ell_\infty(\Delta_v^m) \subseteq \omega_\theta^\beta(\Delta_v^m, f) \cap \ell_\infty(\Delta_v^m)$.*

Proof. Suppose $\lim_{t\to\infty}\dfrac{f(t)}{t}=\gamma$. Then $\gamma=\inf\left\{\dfrac{f(t)}{t}:t>0\right\}$ by Lemma 2.1. This means $t\le\gamma^{-1}f(t)$ for all $t\ge 0$. Let $x\in S_\theta^\alpha(\Delta_v^m,f)\cap\ell_\infty(\Delta_v^m)$. Then there exists $M>0$ such that $|\Delta_v^m x_k-L|\le M$ for all $k\in\mathbf{N}$. For each $\varepsilon>0$, suppose Σ_1 and Σ_2 denote the sums over $k\in I_r,|\Delta_v^m x_k-L|\ge\varepsilon$ and $k\in I_r,|\Delta_v^m x_k-L|<\varepsilon$, respectively. Now,

$$\begin{aligned}\tfrac{1}{f(h_r^\beta)}\textstyle\sum_{k\in I_r} f\left(|\Delta_v^m x_k-L|\right) &\le \tfrac{1}{f(h_r^\alpha)}\left(\Sigma_1 f\left(|\Delta_v^m x_k-L|\right)+\Sigma_2 f\left(|\Delta_v^m x_k-L|\right)\right)\\ &\le \tfrac{1}{f(h_r^\alpha)}\Sigma_1 f(M)+\tfrac{1}{f(h_r^\alpha)}\Sigma_2 f(\varepsilon)\\ &\le \tfrac{1}{f(h_r^\alpha)}\left|\{k\in I_r:|\Delta_v^m x_k-L|\ge\varepsilon\}\right| f(M)+\tfrac{h_r}{f(h_r^\alpha)}f(\varepsilon)\\ &\le \tfrac{\gamma^{-1}}{f(h_r^\alpha)}f\left(\left|\{k\in I_r:|\Delta_v^m x_k-L|\ge\varepsilon\}\right|\right)f(M)+\tfrac{\gamma^{-1}f(h_r)}{f(h_r^\alpha)}f(\varepsilon).\end{aligned}$$

Taking limit $r\to\infty$ and using $\lim_{r\to\infty}\dfrac{f(h_r)}{f(h_r^\alpha)}=1$, inclusion follows. □

2.3. Results on lacunary refinements

The lacunary sequence $\theta'=\{k_r'\}$ is called lacunary refinement of lacunary sequence $\theta=\{k_r\}$ if $\{k_r\}\subseteq\{k_r'\}$. In this case $I_r\subseteq I_r'$ where I_r' denotes interval $(k_{r-1}',k_r']$. We denote interval length $(k_r'-k_{r-1}')$ by l_r.

Theorem 2.10. *Let $\theta'=\{k_r'\}$ be lacunary refinement of $\theta=\{k_r\}$ and $0<\alpha\le\beta\le 1$. Further, suppose lacunary sequences $\theta=\{k_r\}$, $\theta'=\{k_r'\}$ and modulus function f satisfy $\liminf\dfrac{f(h_r^\beta)}{f(l_r^\alpha)}>0$. Then*

1. *$S_{\theta'}^\alpha(\Delta_v^m,f)\subseteq S_\theta^\beta(\Delta_v^m,f)$,*
2. *$\omega_{\theta'}^\alpha(\Delta_v^m,f)\subseteq\omega_\theta^\beta(\Delta_v^m,f)$.*

Proof. (i) Let $x\in S_{\theta'}^\alpha(\Delta_v^m,f)$. Since $I_r\subseteq I_r'$ for all $r\in\mathbf{N}$, so for any $\varepsilon>0$, we have

$$\begin{aligned}&\{k\in I_r:|\Delta_v^m x_k-L|\ge\varepsilon\}\subseteq\{k\in I_r':|\Delta_v^m x_k-L|\ge\varepsilon\}\\ &\Rightarrow f(|\{k\in I_r:|\Delta_v^m x_k-L|\ge\varepsilon\}|)\le f(|\{k\in I_r':|\Delta_v^m x_k-L|\ge\varepsilon\}|),\\ &\qquad\text{as } f \text{ is increasing}\\ &\Rightarrow \tfrac{f(h_r^\beta)}{f(l_r^\alpha)}\tfrac{1}{f(h_r^\beta)}f(|\{k\in I_r:|\Delta_v^m x_k-L|\\ &\ge\varepsilon\}|)\le\tfrac{1}{f(l_r^\alpha)}f(|\{k\in I_r':|\Delta_v^m x_k-L|\ge\varepsilon\}|),\\ &\qquad\text{for all } r\in\mathbf{N}.\end{aligned}$$

Now, by taking limit $r \to \infty$ and using $\liminf \dfrac{f(h_r^\beta)}{f(l_r^\alpha)} > 0$, required inclusion follows.
(ii) Let $x \in \omega_{\theta'}^\alpha(\Delta_v^m, f)$. Inclusion follows by taking limit and using given assumption in following inequality

$$\frac{1}{f(l_r^\alpha)} \sum_{k \in I'_r} f\left(|\Delta_v^m x_k - L|\right) \geq \frac{f(h_r^\beta)}{f(l_r^\alpha)} \frac{1}{f(h_r^\beta)} \sum_{k \in I_r} f\left(|\Delta_v^m x_k - L|\right).$$

□

Theorem 2.11. *Let $\theta' = \{k'_r\}$ be lacunary refinement of $\theta = \{k_r\}$ such that the set $\{n : n \in I'_r \setminus I_r\}$ is finite for each r and $0 < \alpha \leq \beta \leq 1$. Then*

1. $S_\theta^\alpha(\Delta_v^m, f) \subseteq S_{\theta'}^\beta(\Delta_v^m, f)$,
2. $\omega_\theta^\alpha(\Delta_v^m, f) \cap \ell_\infty(\Delta_v^m) \subseteq \omega_{\theta'}^\beta(\Delta_v^m, f) \cap \ell_\infty(\Delta_v^m)$.

Proof. (i) Let $x \in S_\theta^\alpha(\Delta_v^m, f)$. $I_r \subseteq I'_r$ implies that $k'_{r-1} \leq k_{r-1} < k_r \leq k'_r$ for all $r \in \mathbf{N}$. Let $\varepsilon > 0$. Then by subadditive property of f, we can write

$$\begin{aligned}
\tfrac{1}{f(l_r^\beta)} f(|\{k \in I'_r : |\Delta_v^m x_k - L| \geq \varepsilon\}|) &\leq \tfrac{1}{f(l_r^\beta)} f(|\{k'_{r-1} < k \leq k_{r-1} : |\Delta_v^m x_k - L| \geq \varepsilon\}|) \\
&+ \tfrac{1}{f(l_r^\beta)} f(|\{k_{r-1} < k \leq k_r : |\Delta_v^m x_k - L| \geq \varepsilon\}|) \\
&+ \tfrac{1}{f(l_r^\beta)} f(|\{k_r < k \leq k'_r : |\Delta_v^m x_k - L| \geq \varepsilon\}|) \\
&\leq \tfrac{f(k_{r-1} - k'_{r-1})}{f(l_r^\beta)} + \tfrac{1}{f(h_r^\alpha)} f(|\{k \in I_r : |\Delta_v^m x_k - L| \geq \varepsilon\}|) \\
&+ \tfrac{f(k'_r - k_r)}{f(l_r^\beta)}
\end{aligned}$$

Now, taking limit $r \to \infty$ on both sides and using finiteness of set $\{n : n \in I'_r \setminus I_r\}$ and continuity of f, we get $x \in S_{\theta'}^\beta(\Delta_v^m, f)$.
(ii) Let $x \in \omega_\theta^\alpha(\Delta_v^m, f) \cap \ell_\infty(\Delta_v^m)$. Then there exists $M > 0$ such that $|\Delta_v^m x_k - L| \leq M$ for all k. Now, for any $r \in \mathbf{N}$, we can write

$$\begin{aligned}
\tfrac{1}{f(l_r^\beta)} \textstyle\sum_{k \in I'_r} f(\Delta_v^m x_k - L) &\leq \tfrac{1}{f(h_r^\alpha)} \left(\textstyle\sum_{k \in I'_r \setminus I_r} f(|\Delta_v^m x_k - L|) + \sum_{k \in I_r} f(|\Delta_v^m x_k - L|) \right) \\
&\leq \tfrac{1}{f(h_r^\alpha)} \textstyle\sum_{k \in I'_r \setminus I_r} f(M) + \tfrac{1}{f(h_r^\alpha)} \sum_{k \in I_r} f(|\Delta_v^m x_k - L|) \\
&\leq \tfrac{l_r - h_r}{f(h_r^\alpha)} f(M) + \tfrac{1}{f(h_r^\alpha)} \textstyle\sum_{k \in I_r} f(|\Delta_v^m x_k - L|).
\end{aligned}$$

By taking limit $r \to \infty$ and using finiteness of set $\{n : n \in I'_r \setminus I_r\}$, inclusion follows. □

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