Proyecciones Journal of Mathematics
Vol. 40, N^o 6, pp. 1411-1430, December 2021.
Universidad Católica del Norte
Antofagasta - Chile

10.22199/issn.0717-6279-4737

Domination in the entire nilpotent element graph of a module over a commutative ring

Jituparna Goswami
Gauhati University, India
and
Masoumeh Shabani
University of Mohaghegh Ardabili, Iran
Received : February 2020. Accepted : May 2021

Abstract

Let R be a commutative ring with unity and M be a unitary R-module. Let $Nil(M)$ be the set of all nilpotent elements of M. The entire nilpotent element graph of M over R is an undirected graph $E(G(M))$ with vertex set as M and any two distinct vertices x and y are adjacent if and only if $x + y \in Nil(M)$. In this paper we attempt to study the domination in the graph $E(G(M))$ and investigate the domination number as well as bondage number of $E(G(M))$ and its induced subgraphs $N(G(M))$ and $Non(G(M))$. Some domination parameters of $E(G(M))$ are also studied. It has been showed that $E(G(M))$ is excellent, domatically full and well covered under certain conditions.

Keywords: *Entire nilpotent element graph; bondage number; domination number; nilpotent elements; Non-nilpotent elements.*

Mathematics Subject Classification(2010): *16D10; 05E40.*

1. Introduction

The characterization of algebraic structures through association of graphs has become an exciting research topic in the last two decades, leading to many fascinating results and questions. Many fundamental papers assigning graphs to rings and modules have appeared recently, for instance see,[1, 3-5, 7, 19]. In 2008, Anderson and Badawi [4] have introduced the total graph of a commutative ring and later on this notion has been gen- eralised to many algebraic structures, in particular to module over a commutative ring (see [10, 11]).

The concepts of dominating sets and domination numbers play a vital role in graph theory. Dominating sets are the focus of many books of graph theory, for example see [13] and [14]. But not much research has been done on the domination parameters of graphs associated to algebraic structures such as groups, rings, modules in terms of alge- braic properties. However, some works on domination of graphs associated to rings and modules have appeared recently, for instance see,[9, 16, 18, 20].

The study of nilpotent elements is one of the important aspects of module theory. There- fore, as a generalization of the total graph introduced by Anderson and Badawi [4], the second author along with co-researchers in [17] have introduced the entire nilpotent element graph of a module M over a commutative R, denoted by $E(G(M))$, to be an undirected graph with all elements of M as vertices, and for distinct $x + y \in M$, the vertices x and y are adjacent if and only if $x+y \in Nil(M)$. Let $Non(M) = M - Nil(M)$ be the set of all non-nilpotent elements of M. Let $N(G(M))$ be the (induced) subgraph of $E(G(M))$ with vertices $Nil(M)$, and $Non(G(M))$ be the (induced) subgraph of $E(G(M))$ with vertices $Non(M)$. They have studied the characteristics of $E(G(M))$ and its two induced subgraphs $N(G(M))$ and $Non(G(M))$ by considering two cases, $Nil(M)$ is a submodule of M or is not a submodule of M.

The organization of this paper is as follows: In Section 2, we discuss some preliminary definitions and results releted to module theory and graph theory which is required in the next sections. In Section 3, we determine the domination number of the graph $E(G(M))$ and its induced subgraphs $N(G(M))$ and $Non(G(M))$. In Section 4, we determine the bondage number of the graph $E(G(M))$ and its induced subgraph $Non(G(M))$. In Section 5, we study some domination parameters of the graph $E(G(M))$. We have also obtained some conditions under which the graph $E(G(M))$ is excellent, domatically full and well covered. In Section 6, we give conclusions of our paper based on the results we have obtained and mention some of its future aspects.

2. Preliminary Results

Throughout this paper all rings are considered to be commutative with non-zero identity and all modules are unitary unless otherwise stated.

Let R be a ring, M an R-module and N be a submodule of M. We denote by $(N : RM)$ the set of all r in R such that $rM \subset N$. The annihilator of M denoted by $annR(M)$ is $(0 :_R M)$. An R-module M is called faithful if $annR(M) = 0$. An R-module M is called a multiplication module if every submodule N of M has the form IM for some ideal I of R. Note that since $I \subset (N :_R M)$, then $N = IM \subset (N :_R M)M \subset N$. So that $N = (N :_R M)M$. If K is a multiplication submodule of M, then for all submodules N of M, $N \cap K = ((N \cap K) : K)K = (N : K)K$. If M is a finitely generated faithful multiplication R-module, then M is cancellation, from which it follows that $(IN : M) = I(N : M)$. A proper submodule N of M is prime whenever $rm \in N$, for some $r \in R$ and $m \in M$ implies that $m \in N$ or $r \in (N :_R M)$. In this case, $P = (N :_R M)$ is a prime ideal of R and N is called a P-prime submodule of M.

An ideal I of R is nilpotent if $I^k = 0$ for some positive integer k and an element r of R is nilpotent if $r^k = 0$ for some $k \in \mathbf{N}$. Also we denote by $Nil(R)$ the set of all nilpotent elements of R. A submodule N of M is called nilpotent if $(N :_R M)^k N = 0$ for some $k \in \mathbf{N}$. We say that $m \in M$ is nilpotent if Rm is a nilpotent submodule of M [2]. Clearly, the zero submodule of M is nilpotent and hence the zero element of M is nilpotent. We denote by $Nil(M)$ the set of all nilpotent elements of M. $Nil(M)$ is not necessarily a submodule of M, but if M is faithful, then $Nil(M)$ is a submodule of M [2, Theorem 6]. If I is a nilpotent ideal of R or N a nilpotent submodule of M, then IN is a nilpotent submodule of M [2, Proposition 4]. Hence, if $r \in Nil(R)$ or $m \in Nil(M)$, then $rm \in Nil(M)$. Moreover, if M is a faithful multiplication R-module, then $Nil(M) = Nil(R)M = \cap P$, where P runs over all prime submodules of M. For any undefined terminology in rings and modules we refer to [6, 15].

By a graph G, we mean a simple undirected graph without loops. For a graph G, we denote by $V(G)$ and $E(G)$ the set of all vertices and edges respectively. We recall that a graph is finite if both $V(G)$ and $E(G)$ are finite sets, and we use the symbol $|G|$ to denote the number of vertices in the graph G. We say that G is a null graph if $E(G) = \phi$. Two vertices x and y of a graph G are connected if there is a path in G connecting them. Also, a graph G is connected if there is a path between any two distinct vertices. A graph G is disconnected if it is not connected. A graph G is complete if any two distinct vertices are adjacent. We denote the complete graph on n vertices by K_n. If the vertex set $V(G)$ of the graph G are partitioned into two non-empty disjoint sets X and Y of cardinality $|X| = m$ and $|Y| = n$, and two vertices are adjacent if and only if they are not in the same partite set, then G is called a bipartite graph. A graph G is called a complete bipartite graph if every vertex in X is connected to every vertex in Y. We denote the complete bipartite graph on m and n vertices by $K_{m,n}$. For vertices $x, y \in G$ one defines the distance $d(x, y)$, as the length of the shortest path between x and y, if the vertices $x, y \in G$ are connected and $d(x, y) = \infty$, if they are not. Then, the diameter of the graph G is

$$diam(G) = sup\{d(x, y)|x, y \in G\}.$$

The cycle is a closed path which begins and ends in the same vertex. The cycle of n vertices is denoted by C_n. The girth of the graph G,denoted by $gr(G)$ is the length of the shortest cycle in G and $gr(G) = \infty$ if G has no cycles.

For a subset $S \subseteq V(G)$, $< S >$ denotes the subgraph of G induced by S. For a vertex $v \in V(G)$, $deg(v)$ is the degree of the vertex v, $N(v) = \{u \in V(G) \mid u \text{ is adjacent to } v\}$ and $N[v] = N(v) \cup \{v\}$. A subset S of $V(G)$ is called a dominating set if every vertex in $V(G) - S$ is adjacent to atleast one vertex in S. A dominating set S is called a strong(or weak) dominating set if for every vertex $u \in V(G) - S$ there is a vertex $v \in S$ with $deg(v) \geq deg(u)$ (or $deg(v) \leq deg(u)$) and u is adjacent to v. The domination number $\gamma(G)$ of G is defined to be minimum cardinality of a dominating set in G and such a dominating set is called γ-set of G. If G is a trivial graph, then $\gamma(G) = 0$. In a similar way, we define the strong domination number γ_s and the weak domination number γ_w. A graph G is called excellent if for every vertex $v \in V(G)$, there exists a γ-set

S containing v. A domatic partition of G is a partition of $V(G)$ into dominating sets in G. The maximum number of classes of a domatic partition of G is called the domatic number of G and is denoted by $d(G)$. A graph G is called domatically full if $d(G) = \delta(G) + 1$, which is the maximum possible order of a domatic partition of $V(G)$ and $\delta(G)$ is the minimum degree of a vertex of G. The disjoint domination number $\gamma\gamma(G)$ defined by $\gamma\gamma(G) = \min\{|S_1| + |S_2| : S_1, S_2$ are disjoint dominating sets of $G\}$. Similarly, we can define $ii(G)$ and $\gamma i(G)$. The double domination parameters are referred to [12]. The bondage number $b(G)$ is the minimum number of edges whose removal increases the domination number. A set of vertices $S \subseteq V(G)$ is said to be independent if no two vertices in S are adjacent in G. The independence number $\beta_0(G)$, is the maximum cardinality of an independent set in G. A graph G is called well-covered if $\beta_0(G) = i(G)$. For basic definitions and results in domination we refer to [8, 13] and for any undefined graph-theoretic terminology we refer to [8].

Now we summarize some results on domination number and bondage number of a graph which will be useful for the later sections.

Lemma 2.1 [8]:

(i) If G is a graph of order n, then $1 \leq \gamma(G) \leq n$. A graph G of order n has domination number 1 if and only if G contains a vertex v of degree $n-1$; while $\gamma(G) = n$ if and only if $G \cong \overline{K_n}$.

(ii) $\gamma(K_n) = 1$ for a complete graph K_n, but the converse is not true, in general and $\gamma(\overline{K_n}) = n$ for a null graph $\overline{K_n}$.

(iii) Let G be a complete r-partite graph ($r \geq 2$) with partite sets $V_1, V_2, ..., V_r$. If $|V_i| \geq 2$ for $1 \leq i \leq r$, then $\gamma(G) = 2$; because one vertex of V_1 and one vertex of V_2 dominate G. If $|V_i| = 1$ for some i, then $\gamma(G) = 1$.

(iv) $\gamma(K_{1,n}) = 1$ for a star graph $K_{1,n}$.

(v) If G is a partition of disjoint subgraphs $G_1, G_2, ..., G_k$, then $\gamma(G) = \gamma(G_1) + \gamma(G_2) + ... + \gamma(G_k)$.

(vi) Domination number of a bistar graph is 2; because the set consisting of two centres of the graph is a minimal dominating set.

(vii) Let C_n and P_n be a n-cycle and a path with n vertices, respectively. Then $\gamma(C_n) = \lceil \frac{n}{3} \rceil = \gamma(P_n)$.

Lemma 2.2 [14]:

(i) If G is a simple graph of order n, then $1 \leq b(G) \leq n-1$.

(ii) $b(K_n) = n-1$ for a complete graph K_n, but the converse is not true, in general and $b(\overline{K_n}) = 0$ for a null graph $\overline{K_n}$.

(iii) Let G be a complete r-partite graph with partite sets $V_1, V_2, ..., V_r$. Then $b(G) = min\{|V_1|, |V_2|, ..., |V_r|\}$. In particular, $b(K_{m,n}) = min\{m, n\}$.

(iv) If G is a partition of disjoint subgraphs $G_1, G_2, ..., G_k$, then $b(G) = min\{b(G_1), b(G_2), ..., b(G_k)\}$.

(v) Let C_n and P_n be a n-cycle and a path with n vertices, respectively. Then $b(P_n) = 1$ and $b(C_n) = 2$.

3. Domination number of $E(G(M))$ and induced subgraphs

In this section, an attempt has been made to study the domination in the entire graph $E(G(M))$ and find out the domination number of $E(G(M))$ and its induced subgraphs under different conditions. We begin with the following theorem.

Theorem 3.1:[17] Let R be a ring, M an R-module and N a submodule of M. Then the following hold:

(1) The graph $E(G(M))$ is complete if and only if $Nil(M) = M$.

(2) The graph $E(G(M))$ is null if and only if $Nil(M) = \{0\}$ and $|M| \geq 2$.

(3) If M is a faithful R-module, then $N(G(M))$ is a complete subgraph of $E(G(M))$.

(4) $E(G(N))$ is a subgraph of $E(G(M))$.

Proposition 3.2: Let R be a ring, M an R-module and N a submodule of M. Then

(1) $\gamma(E(G(M))) = 1$ if $Nil(M) = M$.

(2) $\gamma(E(G(M))) = |M|$ if and only if $Nil(M) = \{0\}$ and $|M| \geq 2$.

(3) If M is a faithful R-module, then $\gamma(N(G(M))) = 1$.

Example 1: Let us consider the $\mathbf{Z}_2$-module $\mathbf{Z}_6$. We know that $< \overline{x} >$, the submodule of $\mathbf{Z}_6$ generated by $\overline{x}$ is equal to $\{\overline{0}, \overline{x}\}$ and hence $(< \overline{x} >:_{\mathbf{Z}_2} \mathbf{Z}_6) = \{\overline{0}\}$, for any $\overline{x} \neq \overline{0}$. Therefore, $Nil_{\mathbf{Z}_2}(\mathbf{Z}_6) = \mathbf{Z}_6$ and the graph $E(G(\mathbf{Z}_6))$ is complete. Thus, we have $\gamma(E(G(\mathbf{Z}_6))) = 1$.

Theorem 3.3[17, Theorem 2.6]: Let R be a ring and M an R-module such that $Nil(M)$ is a submodule of M. Then $N(G(M))$ and $Non(G(M))$ are disjoint. In particular, if $Nil(M)$ is a proper submodule of M, then $E(G(M))$ is disconnected.

In some next theorems, let $|Nil(M)| = \alpha$ and $|\frac{M}{Nil(M)}| = \beta$ and we allow α and β to be infinite cardinals. If β is infinite, then of course $\beta - 1 = \frac{(\beta - 1)}{2} = \beta$.

Theorem 3.4 [17, Theorem 2.10]: Let R be a ring and M an R-module such that $Nil(M)$ is a submodule of M. If $2 = 1_R + 1_R \in Nil(R)$, then $Non(G(M))$ is the union of $\beta - 1$ disjoint K_α's.

Proposition 3.5: Let R be a ring and M a faithful R-module. If $2 = 1_R + 1_R \in Nil(R)$, then $\gamma(E(G(M))) = \beta$.

Proof. Suppose that $2 = 1_R + 1_R \in Nil(R)$. Then we have from theorem 3.4 that the graph $Non(G(M))$ is the union of $\beta - 1$ disjoint K_α's and we know that $\gamma(K_\alpha) = 1$. Thus, $\gamma(Non(G(M))) = \beta - 1$.
As M is faithful, so $Nil(M)$ is a submodule of M. By theorem 3.3, we have the subgraphs $N(G(M))$ and $Non(G(M))$ are disjoint.
Again, being M faithful we have by theorem 3.1 that $N(G(M))$ is complete. Therefore, $\gamma(N(G(M))) = 1$.
Consequently, $\gamma(E(G(M))) = \gamma(N(G(M)) \cup Non(G(M))) = \gamma(N(G(M))) + \gamma(Non(G(M))) = 1 + \beta - 1 = \beta$.

Theorem 3.6 [17, Theorem 2.12]: Let R be a ring and M a finitely generated faithful multiplication R-module such that $Nil(M)$ is a prime submodule of M. If $2 = 1_R + 1_R \notin Nil(R)$, then $Non(G(M))$ is the union of

$\dfrac{\beta-1}{2}$ disjoint $K_{\alpha,\alpha}$'s.

Proposition 3.7: Let R be a ring and M a finitely generated faithful multiplication R-module such that $Nil(M)$ is a prime submodule of M. If $2 = 1_R + 1_R \notin Nil(R)$, then $\gamma(E(G(M))) = \beta$.

Proof. Suppose that $2 = 1_R + 1_R \notin Nil(R)$. Then we have from theorem 3.6 that the graph $Non(G(M))$ is the union of $\dfrac{\beta-1}{2}$ disjoint $K_{\alpha,\alpha}$'s and we know that $\gamma(K_{\alpha,\alpha}) = 2$. Thus, $\gamma(Non(G(M))) = \dfrac{\beta-1}{2} \times 2 = \beta - 1$.

Since $Nil(M)$ is a prime submodule of M, we have by theorem 3.3 that the subgraphs $N(G(M))$ and $Non(G(M))$ are disjoint.
Again as M is faithful, we have by theorem 3.3 that $N(G(M))$ is complete. Therefore, $\gamma(N(G(M))) = 1$.

Hence, $\gamma(E(G(M))) = \gamma(N(G(M)) \cup Non(G(M))) = \gamma(N(G(M))) + \gamma(Non(G(M))) = 1 + \beta - 1 = \beta$.

Proposition 3.8: Let R be a ring and M a non-zero finitely generated faithful multiplication R-module such that $Nil(M) = 0$, then $\gamma(E(G(M))) = \dfrac{\beta+1}{2}$.

Proof. According hypothesis $Nil(M) = 0$. Therefore, $|\dfrac{M}{Nil(M)}| = |M| = \beta$. As M is a faithful multiplication R-module, so $Nil(M) = Nil(R)M$, by theorem 6 of [2] yielding $Nil(R) = 0$. Therefore, $2 = 1_R + 1_R \notin Nil(R)$ and from theorem 3.8 we have the graph $Non(G(M))$ is the union of $\dfrac{\beta-1}{2}$ disjoint $K_{1,1}$'s and we know that $\gamma(K_{1,1}) = 1$.

Moreover, M is faithful, so we have by theorem 3.1 that $N(G(M))$ is complete. Therefore, $\gamma(N(G(M))) = 1$.

Hence,

$$\gamma(E(G(M))) = \gamma(N(G(M)) \cup Non(G(M)))$$

$$= \gamma(N(G(M))) + \gamma(Non(G(M))) = 1 + (\frac{\beta - 1}{2}) \times 1 = \frac{\beta + 1}{2}.$$

The following example explains the facts discussed above.

Example 2: Let us now consider the $\mathbf{Z}$-module $\mathbf{Z}_5$. In this module $< \overline{x} >$, the submodule of $\mathbf{Z}_5$ generated by $\overline{x}$ is equal to $\mathbf{Z}_5$ and hence $(< \overline{x} >:_{\mathbf{Z}} \mathbf{Z}_5) = \mathbf{Z}$, for any $\overline{x} \neq \overline{0}$. Also, $(< \overline{x} >:_{\mathbf{Z}} \mathbf{Z}_5) < \overline{x} > \neq \overline{0}$, for any $\overline{x} \neq \overline{0}$. Thus, $Nil_{\mathbf{Z}}(\mathbf{Z}_5) = \{\overline{0}\}$.
Moreover, $ann_{\mathbf{Z}}(\mathbf{Z}_5) = 5\mathbf{Z}$. So this module is not faithful, but $Nil_{\mathbf{Z}}(\mathbf{Z}_5)$ is a prime submodule of $\mathbf{Z}_5$. Also, $\mathbf{Z}_5 =< \overline{0} >$ is finitely generated.
These facts imply that $Non(G(\mathbf{Z}_5))$ and $E(G(\mathbf{Z}_5))$ are disconnected, $|Nil_{\mathbf{Z}}(\mathbf{Z}_5)| = 1$ and $Non(G(\mathbf{Z}_5))$ is the union of two disjoint complete bipartite graphs $K_{1,1}$ Therefore, we have $\gamma(Non(G(\mathbf{Z}_5))) = \gamma(K_{1,1} \cup K_{1,1}) = \gamma(K_{1,1}) + \gamma(K_{1,1}) = 1 + 1 = 2$.

Theorem 3.9 [17, Theorem 2.16]: Let R be a ring and M a finitely generated faithful multiplication R-module such that $Nil(M)$ is a prime submodule of M. Then the following hold:

(1) $Non(G(M))$ is complete if and only if either $|\frac{M}{Nil(M)}| = 2$ or $|\frac{M}{Nil(M)}| = |M| = 3$.

(2) $Non(G(M))$ is connected if and only if either $|\frac{M}{Nil(M)}| = 2$ or $|\frac{M}{Nil(M)}| = 3$.

(3) $Non(G(M))$ (and hence $N(G(M))$) and $E(G(M))$ are null if and only if $Nil(M) = 0$ and $2 \in Nil(R)$.

Theorem 3.10 [17, Theorem 2.17]: Let R be a ring and M a finitely generated faithful multiplication R-module such that $Nil(M)$ is a prime submodule of M. Then the following hold:

(1) $diam(Non(G(M))) = 0$ if and only if $Nil(M) = 0$ or $|M| = 2$.

(2) $diam(Non(G(M))) = 1$ if and only if $Nil(M) \neq 0$ and $|\dfrac{M}{Nil(M)}| = 2$ or $Nil(M) = 0$ and $|M| = 3$.

(3) $diam(Non(G(M))) = 2$ if and only if $Nil(M) \neq 0$ and $|\dfrac{M}{Nil(M)}| = 3$.

(4) Otherwise $diam(Non(G(M))) = \infty$.

Note that $m + 0 \in Nil(M)$ for each $m \in Nil(M)\backslash\{0\}$. So 0 is adjacent to any vertex of $Nil(M)\backslash\{0\}$ in $N(G(M))$. Thus, $S = \{0\}$ is a containing set for $N(G(M))$ and hence $\gamma(N(G(M))) = 1$.

Proposition 3.11: Let R be a ring and M a finitely generated faithful multiplication R-module such that $Nil(M)$ is a prime submodule of M. Then the following are equivalent:

(1) $\gamma(E(G(M))) = 2$.

(2) $\gamma(Non(G(M))) = 1$.

(3) $|\dfrac{M}{Nil(M)}| = 2$ or $|\dfrac{M}{Nil(M)}| = |M| = 3$.

Proof. (1) $\Leftrightarrow$ (2): Since $Nil(M)$ is a prime submodule of M and M is faithful, $N(G(M))$ and $Non(G(M))$ are disjoint and $N(G(M))$ is complete. So, $\gamma(N(G(M))) = 1$ and hence $\gamma(E(G(M))) = \gamma(N(G(M)))+\gamma(Non(G(M)))$ which yields $\gamma(E(G(M))) = 1 + \gamma(Non(G(M)))$.

(2) $\Rightarrow$ (3): Suppose $\gamma(Non(G(M))) = 1$. Then clearly $Non(G(M))$ is connected. If $2 \in Nil(R)$, then $\beta - 1 = 1$ and hence $\beta = 2$, where $\beta = |\dfrac{M}{Nil(M)}|$, by theorem 3.4. Thus $|\dfrac{M}{Nil(M)}| = 2$.
If $2 \notin Nil(R)$, then $\dfrac{\beta - 1}{2} = 1$ and so $\beta = |\dfrac{M}{Nil(M)}| = 3$, by theorem 3.6. Also, by assumption, $\alpha = |Nil(M)| = 1$ and hence $Nil(M) = \{0\}$. Thus $|\dfrac{M}{Nil(M)}| = |M| = 3$.

(3) $\Rightarrow$ (2): Assume $|\dfrac{M}{Nil(M)}| = 2$ or $|\dfrac{M}{Nil(M)}| = |M| = 3$. Then by theorem 3.9, $Non(G(M))$ is complete and hence $\gamma(Non(G(M))) = 1$.

Corollary 3.12: Let R be a ring and M a finitely generated faithful multiplication R-module such that $Nil(M)$ is a prime submodule of M. Then

(1) $diam(Non(G(M))) = 1$ if and only if $\gamma(Non(G(M))) = 1$.

(2) $diam(Non(G(M))) = 2$ if and only if $\gamma(Non(G(M))) = 2$.

Proof. (1) It is clear by theorem 3.10 and proposition 3.11.
(2) If $diam(Non(G(M))) = 2$, then $Nil(M) \neq 0$ and $|\frac{M}{Nil(M)}| = 3$, by theorem 3.10. Hence $Non(G(M))$ is connected, by theorem 3.9. Therefore $Non(G(M))$ is a complete bipartite graph $K_{\alpha,\alpha}$ with $\alpha \geq 2$. So $\gamma(Non(G(M))) = 2$.
Conversely, if $\gamma(Non(G(M))) = 2$, then $Non(G(M))$ is the union of two K_α's or is a complete bipartite graph $K_{\alpha,\alpha}$ with $\alpha \geq 2$, by theorem 3.4 and theorem 3.6. So $\beta - 1 = 2$ or $\frac{\beta - 1}{2} = 1$. In either case, $|\frac{M}{Nil(M)}| = 3$ and $|Nil(M)| \geq 2$. Thus $|Nil(M)| \neq 0$ and $diam(Non(G(M))) = 2$, by theorem 3.10.

4. Bondage number of $E(G(M))$

In this section, we find certain domination parameters of $E(G(M))$. We begin with the following proposition.

Proposition 4.1: Let R be a ring and M a faithful R-module such that $Nil(M)$ is a submodule of M, $|Nil(M)| = \alpha$ and $|\frac{M}{Nil(M)}| = \beta$. If $2 = 1_R + 1_R \in Nil(R)$, then $b(E(G(M))) = \alpha - 1$.

Proof. Suppose that $2 = 1_R + 1_R \in Nil(R)$. Then, by theorem 3.4, the graph $Non(G(M))$ is the union of $\beta - 1$ disjoint K_α's and we know that $b(K_\alpha) = \alpha - 1$. Hence $b(Non(G(M))) = \alpha - 1$. Also M is faithful, so $N(G(M))$ is complete, by theorem 3.1 (3). Thus, $b(N(G(M))) = \alpha - 1$. On the other hand, $N(G(M))$ and $Non(G(M))$ are disjoint, by theorem 3.3. Therefore, $b(E(G(M))) = \alpha - 1$.

Proposition 4.2: Let R be a ring and M a finitely generated faithful multiplication R-module such that $Nil(M)$ is a prime submodule of M, $|Nil(M)| = \alpha$ and $|\frac{M}{Nil(M)}| = \beta$. Then $b(E(G(M))) = \alpha - 1$.

Proof. If $2 = 1_R + 1_R \in Nil(R)$, then $b(E(G(M))) = \alpha - 1$, by proposition 4.1.
Now, suppose that $2 = 1_R + 1_R \notin Nil(R)$. Then, by theorem 3.6, $Non(G(M))$ is the union of $\frac{\beta - 1}{2}$ disjoint $K_{\alpha,\alpha}$'s and we know that $b(K_{\alpha,\alpha}) = \alpha$. Thus $b(Non(G(M))) = \alpha$. But $N(G(M))$ is complete, by theorem 3.1 (3) and disjoint from $Non(G(M))$, by theorem 3.3. So, $b(N(G(M)))$ and hence $b(E(G(M)))$ is equal to $\alpha - 1$.

Example 3.

(1) If $E(G(M))$ is complete, then $b(E(G(M))) = n - 1$. But $Nil(M) = M$, by theorem 3.1(1). So, $b(E(G(M))) = |Nil(M)| - 1$.

(2) If $\gamma(G) = |V(G)$, then $b(G) = 0$. So, by proposition 3.2(2), if $Nil(M) = 0$ and $|M| \geq 2$, $then b(E(G(M))) = 0$.

(3) If M is a faithful R-module, then $b(N(G(M))) = |Nil(M)| - 1$.

Theorem 4.3 [17, Theorem 2.15]: Let R be a ring and M a finitely generated faithful multiplication R-module such that $Nil(M)$ is a prime submodule of M. Then the following hold:

(1) $gr(Non(G(M))) = 3$ if and only if $2 \in Nil(R)$ and $|Nil(M)| \geq 3$.

(2) $gr(Non(G(M))) = 4$ if and only if $2 \notin Nil(R)$ and $|Nil(M)| \geq 2$.

(3) $gr(E(G(M))) = 3$ if and only if $|Nil(M)| \geq 3$.

(4) $gr(E(G(M))) = 4$ if and only if $2 \notin Nil(R)$ and $|Nil(M)| = 2$.

(5) If $gr(E(G(M))) \neq 3$ or 4, then $gr(E(G(M))) = \infty$.

Proposition 4.4: Let R be a ring and M a finitely generated faithful multiplication R-module such that $Nil(M)$ is a prime submodule of M, $|Nil(M)| = \alpha$ and $|\frac{M}{Nil(M)}| = \beta$. Then

(1) $gr(Non(G(M))) = 3$ if and only if $b(Non(G(M))) = \alpha - 1$ and $|Nil(M)| \geq 3$.

(2) $gr(Non(G(M))) = 4$ if and only if $b(Non(G(M))) = \alpha$ and $|Nil(M)| \geq 2$.

Proof.

(1) If $gr(Non(G(M))) = 3$, then $2 \in Nil(R)$ and $|Nil(M)| \geq 3$, by theorem 4.3. So $Non(G(M))$ is the union of $\beta - 1$ disjoint K_α's, by theorem 3.4. Therefore, $b(Non(G(M))) = \alpha - 1$.
Now assume that $b(Non(G(M))) = \alpha - 1$ and $|Nil(M)| \geq 3$. If $2 \notin Nil(R)$, then $Non(G(M))$ is the union of $\dfrac{\beta - 1}{2}$ disjoint $K_{\alpha,\alpha}$'s, by theorem 3.6 and hence $b(Non(G(M))) = \alpha$, a contradiction by assumption. Therefore $2 \in Nil(R)$, and then $gr(Non(G(M))) = 3$, by theorem 4.3.

(2) If $gr(Non(G(M))) = 4$, then $2 \notin Nil(R)$ and $|Nil(M)| \geq 2$, by theorem 4.3. So $b(Non(G(M))) = \alpha$, by the same argument to above. Now, let $b(Non(G(M))) = \alpha$ and $|Nil(M)| \geq 2$. If $2 \in Nil(R)$, then $b(Non(G(M))) = \alpha - 1$, by theorem 3.4, a contradiction. So $2 \notin Nil(R)$. Therefore, $Non(G(M))$ is the union of $K_{\alpha,\alpha}$'s, where $\alpha \geq 2$. Thus $gr(K_{\alpha,\alpha})$ and hence $gr(Non(G(M)))$ is equal to 4.

5. When $E(G(M))$ is excellent, domatically full and well covered

In this section, some domination parameters of $E(G(M))$ has been studied. It has been proved that $E(G(M))$ is excellent, domatically full and well covered under some conditions. We begin with the following proposition.

Proposition 5.1: Let R be a ring and M a finitely generated faithful multiplication R-module such that $Nil(M)$ is a prime submodule of M, $|Nil(M)| = \alpha$ and $|\dfrac{M}{Nil(M)}| = \beta$. A set $S = \{x_1, x_2, ..., x_\beta\} \subset V(E(G(M)))$ is a γ-set of $E(G(M))$ if and only if $x_j \notin x_i + Nil(M)$ for all $1 \leq i, j \leq \beta$ and $i \neq j$.

Proof. If part follows directly from proposition 3.4 and 3.6 as $\gamma(E(G(M))) = \beta$.
Conversely, let S be a γ-set of $E(G(M))$. Let us assume that there exist $j, k \in \{1, 2, ..., \beta\}$ such that $x_j \in x_k + Nil(M)$. Since $|S| = \beta$, so there exist a coset $x + Nil(M)$ such that $x_i \notin x + Nil(M)$ for all $x_i \in S$. Now, the vertices in $-x + Nil(M)$ cannot be dominated by S, a contradiction.

Proposition 5.2: Let R be a ring and M an R-module such that $Nil(M)$ is a submodule of M. Let x be a vertex of the graph $E(G(M))$. Then

$$deg(x) = \begin{cases} |Nil(M)| - 1, & \text{if } 2 \in Nil(R) \text{ or } x \in Nil(M) \\ |Nil(M)|, & \text{otherwise.} \end{cases}$$

.

Proof. If $x_i \in Nil(M)$, the vertex $x \in M$ is adjacent to vertices $x_i - x$. Then $deg(x) = |Nil(M)| - 1$ if and only if $x = x_i - x$ for some $x_i \in Nil(M)$, that is, if and only if $2x \in Nil(M)$. If $2x \notin Nil(M)$, then $deg(x) = |Nil(M)|$.
If $2 \in Nil(R)$, then $2x \in Nil(M)$ for all $x \in M$. Therefore, $deg(x) = |Nil(M)| - 1$.
Again, if $2 \notin Nil(R)$, then the following two cases arise.
Case-1: If $x \in Nil(M)$, then $deg(x) = |Nil(M)| - 1$.
Case-2: If $x \notin Nil(M)$, then $deg(x) = |Nil(M)|$.
It follows that

$$deg(x) = \begin{cases} |Nil(M)| - 1, & \text{if } 2 \in Nil(R) \text{ or } x \in Nil(M) \\ |Nil(M)|, & \text{otherwise.} \end{cases}$$

.

Proposition 5.3: Let R be a ring and M an R-module such that $Nil(M)$ is a submodule of M, $|Nil(M)| = \alpha \neq 0$ and $|\frac{M}{Nil(M)}| = \beta$, then

(1) $E(G(M))$ is excellent.

(2) the domatic number $d(E(G(M))) = \alpha$.

(3) $E(G(M))$ is domatically full.

Proof. The proof for (1) and (2) are trivial.
(3) By (2) we have $d(E(G(M))) = \alpha = |Nil(M)|$. Also, we have by proposition 4.2 that $\delta(E(G(M))) = |Nil(M)| - 1 = \alpha - 1$. Therefore, we have $d(E(G(M))) = \delta(E(G(M))) + 1$. Hence, $E(G(M))$ is domatically full.

Theorem 5.4 [17, Theorem 3.2]: Let R be a ring and M an R-module such that $Nil(M)$ is not a submodule of M. Then $E(G(M))$ is connected if and only if $M =< Nil(M) >$.

Lemma 5.5: Let M be a module over a ring R and N be a maximum annihilator submodule in M such that $|N| = \alpha \neq 0$ and $|\frac{M}{N}| = \beta$. If $\gamma(E(G(M))) = \mu$, then the set $S = \{x_1, x_2, ..., x_\mu\} \subset V(E(G(M)))$ is a γ-set of $E(G(M))$ where $x_j \notin x_i + N$ for all $1 \leq i, j \leq \beta$ and $i \neq j$.

Proposition 5.6: Let R be a ring and M be an R-module. If $Nil(M)$ is not a submodule of M, $M =< Nil(M) >$ and $\gamma(E(G(M))) = \mu$,
then $\gamma_t(E(G(M))) = \gamma_c(E(G(M))) = \mu$.

Proof. If $Nil(M)$ is not a submodule of M and $M =< Nil(M) >$, then by theorem 4.4, $E(G(M))$ is connected. Let N be a maximum annihilator submodule in M and $x_1 \in N$. Since $E(G(M))$ is connected, there exists a vertex $x_2 \in a_1 + N$ for some $a_1 \in M - N$ such that x_2 is adjacent to x_1. Again by connectedness of $E(G(M))$, there exists a coset $a_2 + N$ for some $a_2 \notin N$ as well as $a_2 \notin a_1 + N$ such that atleast one element of $a_2 + N$ is adjacent to either a vertex in N or in $a_1 + N$, say N.

If there exists an element $a \in a_i + N$ which is adjacent to some $b \in a_j + N$ with $a \notin a_j + N$, then each vertex in $a_i + N$ is adjacent to atleast one vertex in $a_j + N$. For, if $a + b = c$ for some $c \in Nil(M)$, then $c \in a_i + a_j + N$. Let $d_1 \in a_i + N$ and take $d_2 \in M$ such that $d_1 + d_2 = c$. From this $d_2 \in a_j + N$ and d_1 is adjacent to d_2. Therefore, each vertex in $a_i + N$ is adjacent to atleast one vertex in $a_j + N$.

Thus x_1 is adjacent to some vertex $x_3 \in a_2 + N$. Similarly, we can choose coset representatives x_i, for $4 \leq i \leq \mu$, in distinct cosets of N in M other than N, $a_1 + N$ and $a_2 + N$ such that $< x_1, x_2, ..., x_\mu > \subseteq E(G(M))$ is connected. Then by lemma 4.5, $\{x_1, x_2, ..., x_\mu\}$ is a γ_c-set of $E(G(M))$ and so $\gamma_c(E(G(M))) = \mu$. Since, for any graph G, we have $\gamma(G) \leq \gamma_t(G) \leq \gamma_c(G)$, so $\gamma_t(E(G(M))) = \mu$.

We now find the bondage number of the graph $E(G(M))$. We begin with the following lemma.

Lemma 5.7: Let R be a ring and M a finitely generated faithful multiplication R-module such that $Nil(M)$ is a prime submodule of M, $|Nil(M)| = \alpha$ and $|\frac{M}{Nil(M)}| = \beta$. Then

$$E(G(M)) = \begin{cases} K_\alpha \cup \underbrace{K_\alpha \cup K_\alpha \cup ... \cup K_\alpha}_{(\beta-1) copies}, & \text{if } 2 \in Nil(R) \\ K_\alpha \cup \underbrace{K_{\alpha,\alpha} \cup K_{\alpha,\alpha} \cup ... \cup K_{\alpha,\alpha}}_{(\frac{\beta-1}{2}) copies}, & \text{if } 2 \notin Nil(R). \end{cases}$$

Proof. It follows from theorem 3.4 and theorem 3.6 directly.

Proposition 5.8: Let R be a ring and M a finitely generated faithful multiplication R-module such that $Nil(M)$ is a prime submodule of M, $|Nil(M)| = \alpha$ and $|\frac{M}{Nil(M)}| = \beta$. Then $E(G(M))$ is well covered.

Proof. If $2 \in Nil(R)$, then by lemma 5.7 we have $i(E(G(M))) = \beta$.
If $2 \notin Nil(R)$, then all the vertices in one partition of $K_{\alpha,\alpha}$ together with a vertex of $Nil(M)$, form an i-set of $E(G(M))$ and so $i(E(G(M))) = (\frac{\beta-1}{2})\alpha + 1$. Similarly $\beta_0(E(G(M)))$ is same as $i(E(G(M)))$. Thus

$$i(E(G(M))) = \beta_0(E(G(M))) = \begin{cases} , \\ \text{if } 2 \in Nil(R) \\ (\frac{\beta-1}{2})\alpha + 1, \\ \text{otherwise.} \end{cases}$$

Hence, $E(G(M))$ is well covered.

Corollary 5.9: Let R be a ring and M a finitely generated faithful multiplication R-module such that $Nil(M)$ is a prime submodule of M and $|Nil(M)| = \alpha$, then $\omega(E(G(M))) = \alpha$.

As proved above, we can prove the following.

Proposition 5.10: Let R be a ring and M a finitely generated faithful multiplication R-module such that $Nil(M)$ is a prime submodule of M, $|Nil(M)| = \alpha$ and $|\frac{M}{Nil(M)}| = \beta$. Then

(1)
$$\gamma_t(E(G(M))) = \begin{cases} 2\beta, & \text{if } 2 \in Nil(R) \\ \beta + 1, & \text{otherwise.} \end{cases}$$
.

(2) $\gamma_s(E(G(M))) = \gamma_w(T(\Gamma(M))) = \beta$.

(3) $\gamma_p(E(G(M))) = \beta$.

Proposition 5.11: Let R be a ring and M a finitely generated faithful multiplication R-module such that $Nil(M)$ is a prime submodule of M, $|Nil(M)| = \alpha$ and $|\frac{M}{Nil(M)}| = \beta$. Then

(1) $\gamma\gamma(E(G(M))) = 2\beta$.

(2)
$$\gamma i(E(G(M))) = \begin{cases} 2\beta, & \text{if } 2 \in Nil(R) \\ \beta + (\frac{\beta - 1}{2})\alpha + 1, & \text{otherwise.} \end{cases}$$
.

(3)
$$ii(E(G(M))) = \begin{cases} 2\beta, & \text{if } 2 \in Nil(R) \\ (\beta - 1)\alpha + 2, & \text{otherwise.} \end{cases}$$
.

(4)
$$tt(E(G(M))) = \begin{cases} 4\beta, & \text{if } 2 \in Nil(R) \text{ and } \alpha \geq 4 \\ 2(\beta + 1), & \text{if } 2 \notin Nil(R) \\ \text{does not exist,} & \text{otherwise.} \end{cases}$$
.

6. Conclusion

In this paper we study the domination properties of the entire nilpotent element graph $EG(M)$ of an R-module M. We determine the domination number of $E(G(M))$ and its two induced subgraphs $N(G(M))$ and $Non(G(M))$. We obtain an equivalent condition describing the reletionship between the domination number and the diameter of $Non(G(M))$. Again we determine the bondage number of $E(G(M))$. In addition to this, we establish a relationship between the bondage number and the girth of $Non(G(M))$. Finally we study some domination parameters of $E(G(M))$. We have shown that $E(G(M))$ is excellent, domatically full and well covered under certain conditions.

In future, some more graph theoretic properties such as planarity, traversability, colorability etc. can be studied in the graph $E(G(M))$ and its induced subgraphs. Also, the domination properties of some other graphs defined on rings and modules can be studied in near future.

Acknowledgement:

The author would like to express his deep gratitude to the referee for a very careful reading of the article, and many valuable suggestions to improve the article.

References

[1] S. Akbari, D. Kiani, F. Mohammadi, and S. Moradi, "The total graph and regular graph of a commutative ring", *Journal of pure and applied algebra*, vol. 213, no. 12, pp. 2224–2228, 2009.

[2] M. M. Ali, "Idempotent and nilpotent submodules of multiplication modules", *Communications in algebra*, vol. 36, no. 12, pp. 4620–4642, 2008.

[3] D. D. Anderson and M. Naseer, "Beck's coloring of a commutative ring", *Journal of algebra*, vol. 159, no. 2, pp. 500–514, 1993.

[4] D. F. Anderson and A. Badawi, "The total graph of a commutative ring", *Journal of algebra*, vol. 320, no. 7, pp. 2706–2719, 2008.

[5] D. F. Anderson and P. S. Livingston, "The zero-divisor graph of a commutative ring", *Journal of algebra*, vol. 217, no. 2, pp. 434–447, 1999.

[6] F. W. Anderson and K. R. Fuller, *Rings and categories of modules,* 2nd ed. New York. NY: Springer, 1992.

[7] I. Beck, "Coloring of commutative rings", *Journal of algebra*, vol. 116, no. 1, pp. 208–226, 1988.

[8] G. Chartrand and P. Zhang, *Introduction to graph theory*. New Delhi (India): Tata McGraw-Hill, 2006.

[9] J. Goswami, "Domination in the total graph of a module with respect to singular submodule", *submitted.*

[10] J. Goswami, K. K. Rajkhowa, and H. K. Saikia, "Total graph of a module with respect to singular submodule", *Arab journal of mathematical sciences*, vol. 22, no. 2, pp. 242–249, 2016.

[11] J. Goswami and H. K. Saikia, "On the Line graph associated to the total graph of a module", *Matematika*, vol. 31, no. 1, pp. 7-13, 2015.

[12] F. Harary, and T. W. Haynes, "Double domination in graphs", A*rs combinatoria*, vol. 55, pp. 201-213, 2000.

[13] T. W. Haynes, S. T. Hedetniemi, and P. J. Slatar, F*undamental of domination in graphs.* New York, NY: Marcel Dekker, 1998.

[14] T. W. Haynes, S. T. Hedetniemi, and P. J. Slatar, *Domination in graphs: advanced topics*. Marcel Dekker, 2000.

[15] J. Lambek, *Lectures on rings and modules*. Waltham (MA): Blaisdell, 1966.

[16] D. A. Mojdeh and A. M. Rahimi, "Dominating sets of some graphs associated to commutative rings", *Communications in algebra*, vol. 40, no. 9, pp. 3389–3396, 2012.

[17] M. Shabani, A. Y. Darani, and S. E. Atani, "The entire nilpotent element graph of a module over a commutative ring", *submitted.*

[18] A. Shariatinia and A. Tehranian, "Domination number of total graph of module", *Algebraic structures and their applications,* vol. 2, no. 1, pp. 1-8, 2015.

[19] M. H. Shekarriz, M. H. Shirdareh Haghighi, and H. Sharif, "On the total graph of a finite commutative ring", *Communications in algebra*, vol. 40, no. 8, pp. 2798–2807, 2012.

[20] T. Tamizh Chelvam and T. Asir, "Domination in the total graph of a commutative ring", *Journal of combinatorial mathematics and combinatorial computing*, vol. 87, pp. 147-158, 2013.

Jituparna Goswami
Department of Mathematics,
Gauhati University
Guwahati-14, Assam,
India
e-mail: jituparnagoswami18@gmail.com
Corresponding author

and

Masoumeh Shabani
Department of Mathematics and Applications,
University of Mohaghegh Ardabili
P. O. Box 179, Ardabil,
Iran
e-mail: m.shabani@uma.ac.ir