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Laplacian integral graphs with a given degree sequence constraint *

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Abstract

Let G be a graph on n vertices. The Laplacian matrix of G, denoted by $L(G)$, is defined as $L(G) = D(G) - A(G)$, where $A(G)$ is the adjacency matrix of G and $D(G)$ is the diagonal matrix of the vertex degrees of G. A graph G is said to be L-integral if all eigenvalues of the matrix $L(G)$ are integers. In this paper, we characterize all L-integral non-bipartite graphs among all connected graphs with at most two vertices of degree larger than or equal to three.

Keywords: *Laplacian matrix; Eigenvalues; L-integral graphs.*

Mathematics Subject Classifications: *AMS 15A15, 15F10.*

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1. Introduction

Let $G = (V, E)$ be an undirected graph, without loops or multiple edges. Let $d(G) = (d_1(G), d_2(G), \ldots, d_n(G))$ be the sequence degree of G, such that $\Delta(G) = d_1(G) \geq d_2(G) \geq \cdots \geq \delta(G) = d_n(G)$. The vertex connectivity of G, $k(G)$, is the minimum number of vertices that need to be removed such that the graph G gets disconnected and P_n is a path with n vertices. We define $\mathcal{G}_1$ as the family of connected graphs with sequence degree in a way that $d_1 \geq 3$ and $1 \leq d_i \leq 2$, $i = 2, \ldots, n$. Also, we define $\mathcal{G}_2$ as the family of connected graphs with sequence degree such that $d_1 \geq d_2 \geq 3$ and $1 \leq d_i \leq 2$, $i = 3, \ldots, n$. The *sum* graph, denoted by $G = G_1 + G_2$, is the graph G such that $V = V_1 \times V_2$ and each pair of vertices (u_1, u_2) and (v_1, v_2) are adjacent in G if and only if $u_1 = v_1$ and $(u_2, v_2) \in E_2$ or $u_2 = v_2$ and $(u_1, v_1) \in E_1$. The *Firefly graph*, denoted by $F_{r,s,t}$, is the graph with $2r + s + 2t + 1$ vertices that contains r triangles, s pendant edges and t pendant paths of length 2 sharing a common vertex. We write $A(G)$ for the (0,1)-adjacency matrix of a graph and $D(G)$ for the diagonal matrix of the vertex degrees of G. Also, we write $L(G) = D(G) - A(G)$ and $Q(G) = D(G) + A(G)$ for the Laplacian matrix and signless Laplacian matrix of G. A graph G is L-integral (resp. Q-integral) if all of its L-eigenvalues (resp. Q-eigenvalues) are integers. The spectrum of the Laplacian matrix of G is denoted by $Spec_L(G) = \left\{\mu_1(G)^{[n_1]}, \mu_2(G)^{[n_2]}, \ldots, \mu_s(G)^{[n_s]}\right\}$, where $\mu_i(G)$ is the i-th largest Laplacian eigenvalue and n_i is its algebraic multiplicity. The *algebraic connectivity* of G is denoted by $a(G) = \mu_{n-1}(G)$.

In 1994, Grone and Merris [3] initiated the study of the L-integral graph and Merris in [11] presented an explicit construction of all maximal graphs which are L-integral. After this, some infinite families of L-integral graphs were characterized in the literature as it can be seen in [8, 9, 10, 11, 12]. In particular, Kirkland in [9] determined all *Laplacian integral* graphs such that $\Delta(G) = 3$. Motivated by that, we study all Laplacian integral graphs with at most two vertices of degree greater than or equal to 3. It is worth mentioning that Novanta et al. in [13] found all bipartite Q-integral graphs in $\mathcal{G}_1$ and $\mathcal{G}_2$. Since for bipartite graphs the L- and Q-eigenvalues coincide all L-integral bipartite graphs within those families are determined. In this paper, we determine all non bipartite L-integral graphs in the families $\mathcal{G}_1$ and $\mathcal{G}_2$. Thus, we state our main result:

Theorem 1. *Let G be a graph on $n \geq 9$ vertices with at most two vertices of degree greater than two. Then G is L-integral if and only if G is one of* *the following:* $K_{1,n-1}$, $K_2 + K_{1,n-3}$, $K_{2,n-2}$, $F_{r,s,0}$, *where* $s \geq 1$ *and* $r \geq 1$, $K_1 \vee (rK_1 \cup sK_2 \cup K_{1,t})$, *where* $t \geq 2$ *and* $r + s \geq 2$ *or* $K_2 \vee (n-2)K_1$.

The remaining content of the paper is organized as follows. In Section 2, we will give some important results that will be needed in the sequel. In Section 3, we present all non bipartite L-integral graphs in the family $\mathcal{G}_1$. In Section 4, we present all non bipartite L-integral graphs in the family $\mathcal{G}_2$. In this paper, we use the same demonstration techniques presented in [13].

2. Preliminaries

In this section, we present some results that will be useful to prove the main results of the paper.

Lemma 1. *Let G be a connected graph. Then $a(G) > 0$.*

Lemma 2. *Let G be a non-complete graph. Then $a(G) \leq k(G) \leq \delta(G)$.*

Theorem 3. *Let G be a non-complete and connected graph on n vertices. Then $k(G) = a(G)$ if only if G can be written as $G = G_a \vee G_b$, where G_a is a disconnected graph on $(n - k(G))$ vertices and G_b is a graph on $k(G)$ vertices with $a(G_b) \geq 2k(G) - n$.*

Lemma 4. *Let A be a block diagonal matrix whith diagonal blokcs A_{ii}, for $1 \leq i \leq k$ are blocks of A. Then,* $\det A = \prod_{i=1}^{k} \det A_{ii}$.

Defintion 5. *Given a graph G, and a matrix $M = [m_{ij}]$ associated with G, a partition π of $V(G)$, $V(G) = V_1 \cup \cdots \cup V_k$ is equitable with respect to G and M, if for all $i, j \in \{1, 2, \cdots, k\}$*

$$\sum_{t \in V_j} m_{st} = d_{ij}$$

is a constant d_{ij} for any $s \in V_i$.

The matrix $M_\pi = [d_{ij}]$ of order k is called the divisor matrix of M associated with the partition π.

Theorem 6. *Any eigenvalue of M_π is also an eigenvalue of M.*

Lemma 7. *Let G be a graph on n vertices and f an edge of G. If $H \cong G \backslash f$ then*

$$\mu_1(G) \geq \mu_1(H) \geq \mu_2(G) \geq \mu_2(H) \geq ... \geq \mu_n(G) \geq \mu_n(H).$$

If H is a subgraph of G obtained by removing r edges, then for each $i = 1, \ldots, n-r$.

$$\mu_i(G) \geq \mu_i(H) \geq \mu_{i+r}(G).$$

Let A be a matrix of order n and $1 \leq r \leq n$. The matrix A_r of order r is a principal submatrix of A obtained by deleting $n-r$ rows and the corresponding columns from A.

Proposition 8. *Let A be a Hermitian matrix of order n, let r be an integer with $1 \leq r < n$, and let A_r be a principal submatrix of A of order r with eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$ and $\theta_1 \geq \cdots \geq \theta_r$ respectively. Then, for each $i = 1, \ldots, r$*

$$\lambda_i \geq \theta_i \geq \lambda_{i+n-r}.$$

Remark 9. *Let B_{n-2} be a principal submatrix of $L(G)$ of order $n-2$. From Proposition 8, we have that $\theta_{n-3}(B_{n-2}) \geq \mu_{n-1}(G)$. If G is connected and B_{n-2} has at least two eigenvalues in the interval $(0, 1)$, we conclude that $0 < \mu_{n-1}(G) < 1$.*

Remark 10. *For $n \geq 7$, there are at least 2 eigenvalues of $L(P_n)$ in the interval $(0, 1)$.*

Consider a family of p graphs, $\mathcal{F} = \{G_1, \cdots, G_p\}$, where each graph G_j has order n_j, for $j = 1, \cdots, p$, and a graph H such that $V(H) = \{1, \cdots, p\}$. Each vertex $j \in V(H)$ is assigned to the graph $G_j \in \mathcal{F}$. The $H - join$ (generalized composition) of $G_1, \cdots, G_p$ is the graph $G = H[G_1, \cdots, G_p]$ such that $V(G) = \bigcup_{j=1}^{p} V(G_j)$ and $E(G) = (\bigcup_{j=1}^{p} V(E_j)) \cup (\bigcup_{rs \in E(H)} \{uv : u \in V(G_r), v \in V(G_s)\})$.

Theorem 11. *Let G be a graph on n vertices with at most two vertices of degree greater than or equal to 3. Then G is Q-integral if and only if G is one of the following: $K_{1,n-1}$, $K_2 + K_{1,n-3}$, $K_{2,n-2}$ or $P_4[K_2, K_1, K_1, K_2]$.*

The following results characterize cographs from forbidden P_4 and show that all cographs are L-integral.

Theorem 12. *A graph is cograph if and only if it does not have an induced subgraph isomorphic to P_4.*

Theorem 13. *If G is cograph then G is a L-integral.*

3. L-integral graphs in $\mathcal{G}_1$

In [13], Novanta *et al.* characterized all L-integral bipartite graphs belonging to $\mathcal{G}_1$. In this section, we characterize all L-integral non-bipartite graphs in $\mathcal{G}_1$. The graphs that belong to family $\mathcal{G}_1$ are graphs that contain cycles, paths and pending vertices with one vertex, say u, in common such that $d(u) \geq 3$. Notice that the Firefly graphs, $F_{r,s,t}$, belong to the family $\mathcal{G}_1$. In particular, $F_{0,n-1,0} \cong K_{1,n-1}$, $F_{0,0,1} \cong K_{1,2}$ are L-integral graphs. Below, we present the main result of this section.

Theorem 1. *Let $G \in \mathcal{G}_1$ be a graph on n vertices. Then, G is L-integral if only if either $G \cong K_{1,n-1}$ or $G \cong F_{r,s,0}$, with $s \geq 1$ and $r \geq 1$.*

Proof. Let $G \in \mathcal{G}_1$. If G is bipartite, from Theorem 11, G is L-integral if and only if $G \cong K_{1,n-1}$. Now, suppose that G is non-bipartite and L-integral. From Lemmas 1 and 2, $0 < a(G) \leq k(G) = 1$, and consequently $a(G) = 1$. From Theorem 3, $G \cong G_a \vee G_b$ where $V(G_b) = \{u\}$. As $G \in \mathcal{G}_1$ and for any $x \in V(G)$, $d(x) \leq 2$, we have that $G_a \cong r \cdot K_1 \cup s \cdot K_2$ where $r \geq 1$ and $s \geq 1$. So, $G \cong F_{r,s,0}$ which is a cograph. From Theorem 13, G is $L-$ integral and the result follows. $\square$

4. L-integral graphs in $\mathcal{G}_2$

In [13], Novanta *et al.* characterized all L-integral bipartite graphs belonging to $\mathcal{G}_2$. Let $\mathcal{G}_2'$ be the subfamily of non-bipartite graphs belonging to $\mathcal{G}_2$. From Lemmas 1 and 2, we have that $0 < a(G) \leq k(G) \leq 2$. So, in order to characterize all L-integral graphs in $\mathcal{G}_2'$ we need to consider the cases: $a(G) = k(G)$ and $a(G) < k(G)$.

Case 1: $G \in \mathcal{G}_2'$ and $a(G) = k(G)$.

Theorem 1. *Let $G \in \mathcal{G}_2'$ with $n \geq 7$ vertices. Then G is L-integral if and only if $G \cong K_1 \vee (r \cdot K_1 \cup s \cdot K_2 \cup K_{1,t})$, where $t \geq 2$ and $r + s \geq 2$ or $G \cong K_2 \vee (n-2) \cdot K_1$.*

Proof. Let $G \in \mathcal{G}_2'$ with $n \geq 7$ vertices. Suppose that G is L-integral. So, $a(G) = k(G) = 1$ or $a(G) = k(G) = 2$. Firstly, suppose that $a(G) = k(G) = 1$. From Theorem 3, $G \cong G_a \vee G_b$ where $V(G_b) = \{u\}$ and, consequently, $v \in V(G_a)$. Let $x \in V(G_a)$ such that $x \neq v$. As $G \in \mathcal{G}_2$, $d(x) \leq 2$, we conclude that $G_a \cong r \cdot K_1 \cup s \cdot K_2 \cup K_{1,t}$, where $t \geq 2$ and $r + s \geq 2$. Now, suppose that $a(G) = k(G) = 2$. From Theorem 3, $G \cong G_a \vee G_b$ such that G_b is a graph on two vertices. So, $u, v \in V(G_b)$ and $G_b \cong K_2$. Let $x \in V(G_a)$. As $G \in \mathcal{G}_2'$ and $d(x) \leq 2$, we conclude that $G_a \cong (n-2) \cdot K_1$. Then, $G \cong K_2 \vee (n-2) \cdot K_1$ and the result follows. $\square$

Case 2: $G \in \mathcal{G}_2'$ and $a(G) < k(G)$.

In this case, we characterize all $L-$integral graphs in $\mathcal{G}_2'$ such that $a(G) < k(G) \leq 2$. If $k(G) = 1$, G is not L-integral. So, we only need to consider that $k(G) = 2$. Then, G has only cycles that contain two vertices u and v of degree larger than or equal to 3. Consequently, we need to analyze the length of paths with end vertices u and v.

Proposition 2. *Let $G \in \mathcal{G}_2'$ with $n \geq 11$ vertices. If G has a subgraph P_k, for $k \geq 9$, with end vertices u and v, then G is not L-integral.*

Proof. Let $G \in \mathcal{G}_2'$ with $n \geq 11$ vertices. For $k \geq 9$, suppose that G contains a path P_k with sequence of vertices $ux_1 \cdots x_{k-2}v$. Let H be the subgraph of G obtained by removing the edges ux_1 and $x_{k-2}v$. So, $H \cong H_1 \cup P_{k-2}$, where H_1 is a non bipartite graph and $\mu_n(H) = \mu_{n-1}(H) = 0$. From Remark 10, P_{k-2} has at least 2 eigenvalues in the interval $(0, 1)$. Then, we assume that $0 < \mu_{n-2}(H) \leq \mu_{n-3}(H) < 1$. From Lemma 7, we conclude that $0 < \mu_{n-1}(G) \leq \mu_{n-3}(H) < 1$. Therefore, G is not L-integral. $\square$

From Proposition 2 (Section 4), now we need to consider the remaining cases when G has a subgraph P_k for $3 \leq k \leq 8$. First, we consider that $G \in \mathcal{G}_2'$ is a graph that contains r paths $P^p_{n_p}$, for $1 \leq p \leq \Delta(G) = \Delta$ with the sequences of vertices $ux^p_1 \cdots x^p_{n_p-2}v$ such that $n_p \in \{3, 5, 7\}$. As G is non bipartite graph, the vertices u and v should be adjacent. By a convenient labeling for the vertices, $L(G)$ can be seen written in the following way:

$$L(G) = \begin{bmatrix} \mathcal{D}_{2\times 2} & \mathcal{T}_{2\times(n_1-2)} & \mathcal{T}_{2\times(n_2-2)} & \cdots & \mathcal{T}_{2\times(n_\Delta-2)} \\ \mathcal{T}_{(n_1-2)\times 2} & \mathcal{A}_{n_1-2} & 0_{(n_1-2)\times(n_2-2)} & \cdots & 0_{(n_1-2)\times(n_\Delta-2)} \\ \mathcal{T}_{(n_2-2)\times 2} & 0_{(n_2-2)\times(n_2-2)} & \mathcal{A}_{(n_2-2)} & \cdots & 0_{(n_2-2)\times(n_\Delta-2)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathcal{T}_{(n_\Delta-2)\times 2} & 0_{(n_\Delta-2)\times(n_1-2)} & 0_{(n_\Delta-2)\times(n_2-2)} & \cdots & \mathcal{A}_{n_\Delta-2} \end{bmatrix} (I),$$

where $\mathcal{D} = [d_{ij}]_{2\times 2}$ such that $d_{ij} = \begin{cases} \Delta, & if \quad i = j \\ -1, & if \quad i \neq j \end{cases}$ and $\mathcal{T} = [t_{ij}]_{2\times n_{p-2}}$ such that $t_{ij} = \begin{cases} -1, \; i = j = 1 \\ -1, \; i = 2 \text{ and } j = n_p - 2 \\ 0, \; \text{otherwise.} \end{cases}$

Observe that $A_{n_p-2} \in \{A_1, A_3, A_5\}$, where

$$A_1 = \begin{bmatrix} 2 \end{bmatrix}, A_3 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \text{ and } A_5 = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{bmatrix}.$$

Notice that since G is non-bipartite, the vertices u and v should be adjacent and we will use this fact to prove Proposition 3, 4 and 5.

Proposition 3. *Let $G \in \mathcal{G}_2'$ with $n \geq 9$ vertices. If G has at least two subgraphs P_5, or at least two subgraphs P_7 or one subgraph P_5 together with a subgraph P_7 with end vertices u and v, then G is not L-integral.*

Proof. Let $G \in \mathcal{G}_2'$ with $n \geq 9$ vertices. Suppose that G contains at least two paths P_5, or at least two paths P_7 or one path P_5 together with a path P_7 with the sequence of vertices $ux_1^i \cdots x_{n_p-2}^i v$ such that $n_p \in \{5, 7\}$ and $i \geq 2$. In all cases, let B_{n-2} be the principal submatrix of $L(G)$ obtained by removing both rows and columns that correspond to the vertices u and v. It is easy to see that B_{n-2} is a block diagonal matrix and its blocks are the matrices A_3 and/or A_5 which have eigenvalues in the interval $(0, 1)$. From Proposition 8, we have $0 < \mu_{n-1}(G) \leq \theta_{n-3}(B_{n-2}) < 1$. Then, G is not L-integral. □

As G is a non-bipartite graph, the following remaining cases are described as: (i) G has at least one subgraph P_3 and one subgraph P_5, and (ii) G has at least one subgraph P_3 and one subgraph P_7. Next, Proposition 4 proves case (i), and Proposition 5 proves case (ii).

Proposition 4. *Let $G \in \mathcal{G}_2'$ with $n \geq 6$ vertices. If G has $s \geq 1$ subgraphs P_3 and one subgraph P_5 with end vertices u and v, then G is not L-integral.*

Proof. Let $G \in \mathcal{G}_2'$ with $n \geq 6$ vertices. Recall that $uv \in G$. It is easy to see that G is not L-integral graph for $s = 1$. Suppose that $s \geq 2$. From matrix (I), $L(G)$ can be seen written in the following way:

$$L(G) = \begin{bmatrix} \mathcal{D}_{2\times2} & \mathcal{T}_{2\times1} & \mathcal{T}_{2\times1} & \cdots & \mathcal{T}_{2\times1} & \mathcal{T}_{2\times3} \\ \mathcal{T}_{1\times2} & \mathcal{A}_1 & 0_{1\times1} & \cdots & 0_{1\times1} & 0_{1\times3} \\ \mathcal{T}_{1\times2} & 0_{1\times1} & \mathcal{A}_1 & \cdots & 0_{1\times1} & 0_{1\times3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathcal{T}_{1\times2} & 0_{1\times1} & 0_{1\times1} & \cdots & \mathcal{A}_1 & 0_{1\times3} \\ \mathcal{T}_{3\times2} & 0_{3\times1} & 0_{3\times1} & \cdots & 0_{3\times1} & \mathcal{A}_3 \end{bmatrix},$$

where $\Delta = s+2$. According to Theorem 6, the eigenvalues of the matrix

$$R_{L(G)} = \begin{bmatrix} s+2 & -1 & -s & -1 & 0 & 0 \\ -1 & s+2 & -s & 0 & 0 & -1 \\ -1 & -1 & 2 & 0 & 0 & 0 \\ -1 & 0 & 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & -1 & 0 & 0 & -1 & 2 \end{bmatrix}$$

are eigenvalues of $L(G)$, whose characteristic polynomial is $p(\lambda) = \lambda^6 + (-2s-12)\lambda^5 + (s^2+18s+55)\lambda^4 + (-6s^2-56s-120)\lambda^3 + (10s^2+70s+125)\lambda^2 + (-4s^2-30s-50)\lambda$. As $p(1) = s^2-1 > 0$, for $s \geq 2$, and $p(2) = -4s < 0$, we conclude that there is a root in the interval $(1,2)$, and consequently G is not L-integral. $\square$

Proposition 5. *Let $G \in \mathcal{G}_2'$ with $n \geq 8$ vertices. If G has at least $s \geq 1$ subgraphs P_3 and one subgraph P_7 with end vertices u and v, then G is not L-integral.*

Proof. Let $G \in \mathcal{G}_2'$ with $n \geq 8$ vertices. It is easy to see that G is not L-integral for $s=1$. Suppose that $s \geq 2$. From matrix (I), $L(G)$ has the following form:

$$L(G) = \begin{bmatrix} \mathcal{D}_{2\times2} & \mathcal{T}_{2\times1} & \mathcal{T}_{2\times1} & \cdots & \mathcal{T}_{2\times1} & \mathcal{T}_{2\times5} \\ \mathcal{T}_{1\times2} & \mathcal{A}_1 & 0_{1\times1} & \cdots & 0_{1\times1} & 0_{1\times5} \\ \mathcal{T}_{1\times2} & 0_{1\times1} & \mathcal{A}_1 & \cdots & 0_{1\times1} & 0_{1\times5} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathcal{T}_{1\times2} & 0_{1\times1} & 0_{1\times1} & \cdots & \mathcal{A}_1 & 0_{1\times5} \\ \mathcal{T}_{5\times2} & 0_{5\times1} & 0_{5\times1} & \cdots & 0_{5\times1} & \mathcal{A}_5 \end{bmatrix},$$

where $\Delta = s+2$. Applying the Intermediate Value Theorem to the characteristic polynomial associated to matrix obtained by Theorem 6 for the matrix $L(G)$, we conclude that there is a root in the interval $(3,4)$, and consequently G is not L-integral. $\square$

Remark 6. *If $G \in \mathcal{G}_2'$ and G has only subgraph P_3 with end vertices u and v, $a(G) = k(G)$ which was analyzed in Case 4.1.*

Now let us analyze the cases in which $G \in \mathcal{G}_2'$ is a graph that contains r paths $P_{n_p}^p$ with the sequence of vertices $ux_1^p \cdots x_{n_p-2}^p v$, for $1 \leq p \leq r$, and $n_p \in \{4,6,8\}$. By a convenient labeling to the vertices of G we obtain

$$L(G) = \begin{bmatrix} \mathcal{D}_{2\times 2} & \mathcal{T}_{2\times(n_1-2)} & \mathcal{T}_{2\times(n_2-2)} & \cdots & \mathcal{T}_{2\times(n_r-2)} \\ \mathcal{T}_{(n_1-2)\times 2} & \mathcal{A}_{n_1-2} & 0_{(n_1-2)\times(n_2-2)} & \cdots & 0_{(n_1-2)\times(n_r-2)} \\ \mathcal{T}_{(n_2-2)\times 2} & 0_{(n_2-2)\times(n_2-2)} & \mathcal{A}_{(n_2-2)} & \cdots & 0_{(n_2-2)\times(n_r-2)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathcal{T}_{(n_r-2)\times 2} & 0_{(n_r-2)\times(n_1-2)} & 0_{(n_r-2)\times(n_2-2)} & \cdots & \mathcal{A}_{n_r-2} \end{bmatrix} (II),$$

where

$\mathcal{D} = [d_{ij}]_{2\times 2}$ such that $d_{ij} = \begin{cases} \Delta, & if \quad i = j \\ -1, & if \quad i \neq j \quad and \quad u \sim v \\ 0, & if \quad i \neq j \quad and \quad u \not\sim v \end{cases}$ and $\mathcal{T} = [t_{ij}]_{2\times n_p-2}$ such that $t_{ij} = \begin{cases} 1, & \text{if } i = j = 1, \\ -1, & \text{if } i = 2 \text{ and } j = n_p - 2, \\ 0, & \text{otherwise.} \end{cases}$

Observe that $A_{n_p-2} \in \{A_2, A_4, A_6\}$, where

$A_2 = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$,

$A_4 = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$ and $A_6 = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix}$.

By using the matrix $L(G)$ presented above, we obtain the following propositions.

Proposition 7. *Let $G \in \mathcal{G}_2'$ with $n \geq 9$ vertices. If G has a subgraph P_8 with end vertices u and v, then G is not L-integral.*

Proof. Let $G \in \mathcal{G}_2'$ with $n \geq 9$ vertices. Suppose that G contains a path P_8 with the sequence of vertices $ux_1 \cdots x_6 v$. Let B_{n-2} be the principal submatrix of $L(G)$ obtained by removing both rows and columns that correspond to vertices u and v. Note that B_{n-2} is a block diagonal matrix and one of its blocks is the matrix A_6, which has two eigenvalues in the interval $(0, 1)$. From Lemma 4, we have $0 < \theta_{n-2}(B_{n-2}) < \theta_{n-3}(B_{n-2}) < 1$ and consequently from Proposition 8, we conclude that $0 < \mu_{n-1}(G) = a(G) \leq \theta_{n-3}(B_{n-2}) < 1$. Therefore, G is not L-integral. □

Proposition 8. *Let $G \in \mathcal{G}_2'$ with $n \geq 7$ vertices. If G has a subgraph P_6 with end vertices u and v, then G is not L-integral.*

Proof. Let $G \in \mathcal{G}_2'$ with $n \geq 7$ vertices. Suppose that G contains $a \geq 2$ paths P_6 with the sequence of vertices $u{x_1}^i \cdots {x_4}^i v$ for $2 \leq i \leq a$. Let B_{n-2} be the principal submatrix of $L(G)$ obtained by removing both rows and columns that correspond to vertices u and v. Note that B_{n-2} is a block diagonal matrix and $a \geq 2$ of its blocks is the matrix A_4. It is easy to see that A_4 has one eigenvalue in the interval $(0, 1)$. Then, B_{n-2} contains $a \geq 2$ eigenvalues in the interval $(0, 1)$. From Lemma 4, we have $0 < \theta_{n-2}(B_{n-2}) < \theta_{n-3}(B_{n-2}) < 1$, and consequently, from Proposition 8, we conclude that $0 < \mu_{n-1}(G) \leq \theta_{n-3}(B_{n-2}) < 1$. Therefore, G is not L-integral. Now, suppose that G contains one path P_6 with the sequence of vertices $ux_1 \cdots x_4 v$ along with one path P_5 or one path P_7. Therefore the principal submatrix of $L(G)$ obtained by removing both rows and columns that correspond to vertices u and v, B_{n-2}, is a block diagonal matrix wich has at least two blocks A_6 and A_5 or A_6 and A_7. In both cases, B_{n-2} contains $a \geq 2$ eigenvalues in the interval $(0, 1)$ and, consequently, from Proposition 8, we conclude that $0 < \mu_{n-1}(G) \leq \theta_{n-3}(B_{n-2}) < 1$. Then, G is not L-integral. Finally, suppose that G contains one path P_6, with the sequence of vertices $ux_1 \cdots x_4 v$, with $s \geq 1$ paths P_3 and $t \geq 0$ path P_4. So, we need to consider the following cases:

Case 1: G contains $t \geq 1$ paths P_4, $s \geq 1$ paths P_3 and one path P_6.

Case 1.1: u and v are adjacent.

By a convenient labeling for the vertices, $L(G)$ can be seen represented in the following way:

$$L(G)=\left[\begin{array}{c|c|ccc|ccc|ccc|cccc}
s+t+2 & -1 & -1 & \cdots & -1 & -1 & \cdots & -1 & 0 & \cdots & 0 & -1 & 0 & 0 & 0 \\ \hline
-1 & s+t+2 & -1 & \cdots & -1 & 0 & \cdots & 0 & -1 & \cdots & -1 & 0 & 0 & 0 & -1 \\ \hline
-1 & -1 & 2 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
-1 & -1 & 0 & \cdots & 2 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ \hline
-1 & 0 & 0 & \cdots & 0 & 2 & \cdots & 0 & -1 & \cdots & 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
-1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 2 & 0 & \cdots & -1 & 0 & 0 & 0 & 0 \\ \hline
0 & -1 & 0 & \cdots & 0 & -1 & \cdots & 0 & 2 & \cdots & 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & -1 & 0 & \cdots & 0 & 0 & \cdots & -1 & 0 & \cdots & 2 & 0 & 0 & 0 & 0 \\ \hline
-1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 2 & -1 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & -1 & 2 & -1 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & -1 & 2 & -1 \\
0 & -1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & -1 & 2
\end{array}\right].$$

Applying the Intermediate Value Theorem to the characteristic polynomial associated to matrix obtained by Theorem 6 for the matrix $L(G)$, we conclude that there is a root in the interval $(3,4)$, and consequently G is not L-integral.

Case 1.2: u and v are non-adjacent.

By a convenient labeling for the vertices, $L(G)$ can be seen written in the following way:

$$L(G)=\left[\begin{array}{c|c|ccc|ccc|ccc|cccc}
s+t+1 & 0 & -1 & \cdots & -1 & -1 & \cdots & -1 & 0 & \cdots & 0 & -1 & 0 & 0 & 0 \\ \hline
0 & s+t+1 & -1 & \cdots & -1 & 0 & \cdots & 0 & -1 & \cdots & -1 & 0 & 0 & 0 & -1 \\ \hline
-1 & -1 & 2 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
-1 & -1 & 0 & \cdots & 2 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ \hline
-1 & 0 & 0 & \cdots & 0 & 2 & \cdots & 0 & -1 & \cdots & 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
-1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 2 & 0 & \cdots & -1 & 0 & 0 & 0 & 0 \\ \hline
0 & -1 & 0 & \cdots & 0 & -1 & \cdots & 0 & 2 & \cdots & 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & -1 & 0 & \cdots & 0 & 0 & \cdots & -1 & 0 & \cdots & 2 & 0 & 0 & 0 & 0 \\ \hline
-1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 2 & -1 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & -1 & 2 & -1 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & -1 & 2 & -1 \\
0 & -1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & -1 & 2
\end{array}\right].$$

Applying the same technique as in Case 1.1, we conclude that there is a root in the interval $(3,4)$, and consequently G is not L-integral.

Case 2: G contains $s \geq 1$ paths P_3 and one path P_6.

Case 2.1: u and v are adjacent.

From the matrix (II), $L(G)$ can be seen written in the following way:

$$L(G)=\begin{bmatrix} \mathcal{D}_{2\times 2} & \mathcal{T}_{2\times 1} & \mathcal{T}_{2\times 1} & \cdots & \mathcal{T}_{2\times 1} & \mathcal{T}_{2\times 4} \\ \mathcal{T}_{1\times 2} & \mathcal{A}_1 & 0_{1\times 1} & \cdots & 0_{1\times 1} & 0_{1\times 4} \\ \mathcal{T}_{1\times 2} & 0_{1\times 1} & \mathcal{A}_1 & \cdots & 0_{1\times 1} & 0_{1\times 4} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathcal{T}_{1\times 2} & 0_{1\times 1} & 0_{1\times 1} & \cdots & \mathcal{A}_1 & 0_{1\times 4} \\ \mathcal{T}_{4\times 2} & 0_{4\times 1} & 0_{4\times 1} & \cdots & 0_{4\times 1} & \mathcal{A}_4 \end{bmatrix},$$

where $\Delta = s+2$. Applying the same technique as in Case 1.1, we conclude that there is a root in the interval $(1,2)$, and consequently G is not L-integral.

Case 2.2: u and v are non-adjacent.

It is easy to see that for $s = 1$ or $s = 2$, G is not L-integral. Suppose that $s \geq 3$. From the matrix (II), $L(G)$ can be seen written in the following way:

$$L(G)=\begin{bmatrix} \mathcal{D}_{2\times 2} & \mathcal{T}_{2\times 1} & \mathcal{T}_{2\times 1} & \cdots & \mathcal{T}_{2\times 1} & \mathcal{T}_{2\times 4} \\ \mathcal{T}_{1\times 2} & \mathcal{A}_1 & 0_{1\times 1} & \cdots & 0_{1\times 1} & 0_{1\times 4} \\ \mathcal{T}_{1\times 2} & 0_{1\times 1} & \mathcal{A}_1 & \cdots & 0_{1\times 1} & 0_{1\times 4} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathcal{T}_{1\times 2} & 0_{1\times 1} & 0_{1\times 1} & \cdots & \mathcal{A}_1 & 0_{1\times 4} \\ \mathcal{T}_{4\times 2} & 0_{4\times 1} & 0_{4\times 1} & \cdots & 0_{4\times 1} & \mathcal{A}_4 \end{bmatrix},$$

where $\Delta = s+1$. Applying the same technique as in Case 1.1, we conclude that there is a root in the interval $(2,3)$, and consequently G is not L-integral. □

Proposition 9. *Let $G \in \mathcal{G}_2'$ with $n \geq 6$ vertices. If G has a subgraph P_4 with end vertices u and v, then G is not L-integral.*

Proof. Let $G \in \mathcal{G}_2'$ with $n \geq 6$ vertices. Suppose that G contains one path P_4 with the sequence of vertices ux_1x_2v and with at least two paths in the set $\{P_5, P_7\}$. Let B_{n-2} be the submatrix principal of $L(G)$ obtained by removing boths rows and columns corresponding to vertices u and v. As B_{n-2} is a block diagonal matrix and its blocks belong to the set $\{A_3, A_5\}$ which have one eigenvalue in the interval $(0,1)$, from Proposition 8 (Section 2), we conclude that $0 < \mu_{n-1}(G) \leq \theta_{n-3}(B_{n-2}) < 1$. Then, G is not L-integral. Now, suppose that G contains $t \geq 1$ paths P_4, with the sequence of vertices $ux_1^q x_2^q v$, such that $1 \leq q \leq t$, $s \geq 1$ paths P_3 or/and one path of the set $\{P_5, P_7\}$. So we need to analyse the following cases:

Case 1: G contains $t \geq 1$ paths P_4 and $s \geq 1$ paths P_3.

Case 1.1: u and v are adjacent.

By a convenient labeling for the vertices, $L(G)$ can be seen written in the following way:

$$\left[\begin{array}{c|c|ccc|ccc|ccc} s+t+1 & -1 & -1 & \cdots & -1 & -1 & \cdots & -1 & 0 & \cdots & 0 \\ \hline -1 & s+t+1 & -1 & \cdots & -1 & 0 & \cdots & 0 & -1 & \cdots & -1 \\ \hline -1 & -1 & 2 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & 0 & \cdots & 2 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \hline -1 & 0 & 0 & \cdots & 0 & 2 & \cdots & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 2 & 0 & \cdots & -1 \\ \hline 0 & -1 & 0 & \cdots & 0 & -1 & \cdots & 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & -1 & 0 & \cdots & 0 & 0 & \cdots & -1 & 0 & \cdots & 2 \end{array}\right].$$

Applying the Intermediate Value Theorem to the characteristic polynomial associated to matrix obtained by Theorem 6 for the matrix $L(G)$, we conclude that there is a root in the interval $(1, 2)$, and, consequently, G is not L-integral.

Case 1.2: u and v are non-adjacent.

By a convenient labeling for the vertices, $L(G)$ can be seen written in the following way:

$$L(G) = \left[\begin{array}{c|c|ccc|ccc|ccc} s+t & 0 & -1 & \cdots & -1 & -1 & \cdots & -1 & 0 & \cdots & 0 \\ \hline 0 & s+t & -1 & \cdots & -1 & 0 & \cdots & 0 & -1 & \cdots & -1 \\ \hline -1 & -1 & 2 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & 0 & \cdots & 2 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \hline -1 & 0 & 0 & \cdots & 0 & 2 & \cdots & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 2 & 0 & \cdots & -1 \\ \hline 0 & -1 & 0 & \cdots & 0 & -1 & \cdots & 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & -1 & 0 & \cdots & 0 & 0 & \cdots & -1 & 0 & \cdots & 2 \end{array}\right].$$

According to Theorem 6, the eigenvalues of the matrix

$$R_{L(G)} = \begin{bmatrix} s+t & 0 & -s & -t & 0 \\ 0 & s+t & -s & 0 & -t \\ -1 & -1 & 2 & 0 & 0 \\ -1 & 0 & 0 & 2 & -1 \\ 0 & -1 & 0 & -1 & 2 \end{bmatrix}$$

are eigenvalues of $L(G)$, whose characteristic polynomial is $p(\lambda) = \lambda^5 + (-2s - 2t - 6)\lambda^4 + (s^2 + 2st + t^2 + 10s + 10t + 11)\lambda^3 + (-4s^2 - 8st - 4t^2 - 14s - 14t - 6)\lambda^2 + (3s^2 + 8st + 4t^2 + 6s + 4t)\lambda$. Then, we have:

(i) for $s = 1$ and $t \geq 2$, $p(\lambda) = \lambda(\lambda^2 - \lambda(4+t) + 3 + 2t)^2$, whose roots are 0, $\frac{-\sqrt{t^2+4}+t+4}{2}$ with multiplicity 2, and $\frac{\sqrt{t^2+4}+t+4}{2}$ with multiplicity 2 as well. As $t < \sqrt{t^2+4} < t+1$, $p(\lambda)$ has non-integer roots ;

(ii) for $s = 2$ and $t = 1$, $Spec_L(G) = \{4.73^{[1]}, 4^{[1]}, 2^{[2]}, 1.27^{[1]}, 0^{[1]}\}$;

(iii) for $s = 2$ and $t = 2$, $Spec_L(G) = \{5.56^{[1]}, 5^{[1]}, 3^{[1]}, 2^{[2]}, 1.44^{[1]}, 1^{[1]}, 0^{[1]}\}$;

(iv) for $s = 2$ and $t \geq 3$, $p(\lambda) = (-2t - 10)\lambda^4 + \lambda^5 + (t^2 + 14t + 35)\lambda^3 + (-4t^2 - 30t - 50)\lambda^2 + (4t^2 + 20t + 24)\lambda$, whose roots are $0, 2, t+3, \frac{-\sqrt{t^2+2t+9}+t+5}{2}$, $\frac{\sqrt{t^2+2t+9}+t+5}{2}$. As $t + 1 < \sqrt{t^2 + 2t + 9} < t + 2$, $p(\lambda)$ has non-integer roots ;

(v) for $s \geq 3$ and $t \geq 1$, $p(2) = -2s^2 + 4s < 0$ and $p(3) = 6st + 3t^2 - 6t > 0$. So, we conclude that there is a root in the interval $(2, 3)$.

Therefore, in all previous cases we obtain that G is not L-integral.

Case 2: G contains $t \geq 1$ paths P_4, $s \geq 0$ paths P_3, and one path P_5.

Case 2.1: u and v are adjacent. By a convenient labeling for the vertices, $L(G)$ can be seen written in the following way:

$$L(G) = \left[\begin{array}{c|c|ccc|ccc|ccc|ccc}
s+t+2 & -1 & -1 & \cdots & -1 & -1 & \cdots & -1 & 0 & \cdots & 0 & -1 & 0 & 0 \\ \hline
-1 & s+t+2 & -1 & \cdots & -1 & 0 & \cdots & 0 & -1 & \cdots & -1 & 0 & 0 & -1 \\ \hline
-1 & -1 & 2 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
-1 & -1 & 0 & \cdots & 2 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \hline
-1 & 0 & 0 & \cdots & 0 & 2 & \cdots & 0 & -1 & \cdots & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
-1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 2 & 0 & \cdots & -1 & 0 & 0 & 0 \\ \hline
0 & -1 & 0 & \cdots & 0 & -1 & \cdots & 0 & 2 & \cdots & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & -1 & 0 & \cdots & 0 & 0 & \cdots & -1 & 0 & \cdots & 2 & 0 & 0 & 0 \\ \hline
-1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 2 & -1 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & -1 & 2 & -1 \\
0 & -1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & -1 & 2
\end{array}\right]$$

.

Applying the Intermediate Value Theorem to the characteristic polynomial associated to matrix obtained by Theorem 6 for the matrix $L(G)$, we conclude that there is a root in the interval $(0.5, 1)$, and consequently G is not L-integral.

Case 2.2: u and v are non-adjacent.

For $s = 0$ and $t = 2$, it is easy to see that G is not L-integral. By a convenient labeling for the vertices, $L(G)$ can be seen written in the following way:

$$L(G) = \left[\begin{array}{c|c|ccc|ccc|ccc|ccc}
s+t+1 & 0 & -1 & \cdots & -1 & -1 & \cdots & -1 & 0 & \cdots & 0 & -1 & 0 & 0 \\ \hline
0 & s+t+1 & -1 & \cdots & -1 & 0 & \cdots & 0 & -1 & \cdots & -1 & 0 & 0 & -1 \\ \hline
-1 & -1 & 2 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
-1 & -1 & 0 & \cdots & 2 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \hline
-1 & 0 & 0 & \cdots & 0 & 2 & \cdots & 0 & -1 & \cdots & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
-1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 2 & 0 & \cdots & -1 & 0 & 0 & 0 \\ \hline
0 & -1 & 0 & \cdots & 0 & -1 & \cdots & 0 & 2 & \cdots & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & -1 & 0 & \cdots & 0 & 0 & \cdots & -1 & 0 & \cdots & 2 & 0 & 0 & 0 \\ \hline
-1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 2 & -1 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & -1 & 2 & -1 \\
0 & -1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & -1 & 2
\end{array}\right].$$

Applying the same technique as in Case 2.1, we conclude that there is a root in the interval $(0.5, 1)$, and consequently G is not L-integral.

Case 3: G contains $t \geq 1$ paths P_4, $s \geq 1$ paths P_3 and one path P_7.

Case 3.1: u and v are adjacent.

By a convenient labeling for the vertices, $L(G)$ can be seen written in the following way:

$$\left[\begin{array}{c|c|ccc|ccc|ccc|ccccc}
s+t+2 & -1 & -1 & \cdots & -1 & -1 & \cdots & -1 & 0 & \cdots & 0 & -1 & 0 & 0 & 0 & 0 \\ \hline
-1 & s+t+2 & -1 & \cdots & -1 & 0 & \cdots & 0 & -1 & \cdots & -1 & 0 & 0 & 0 & 0 & -1 \\ \hline
-1 & -1 & 2 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
-1 & -1 & 0 & \cdots & 2 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
-1 & 0 & 0 & \cdots & 0 & 2 & \cdots & 0 & -1 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
-1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 2 & 0 & \cdots & -1 & 0 & 0 & 0 & 0 & 0 \\ \hline
0 & -1 & 0 & \cdots & 0 & -1 & \cdots & 0 & 2 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & -1 & 0 & \cdots & 0 & 0 & \cdots & -1 & 0 & \cdots & 2 & 0 & 0 & 0 & 0 & 0 \\ \hline
-1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 2 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & -1 & 2 & -1 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & -1 & 2 & -1 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & -1 & 2 & -1 \\
0 & -1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & -1 & 2
\end{array}\right].$$

Applying the same technique as in Case 2.1, we conclude that there is a root in the interval $(0.5, 1)$, and consequently G is not L-integral.

Case 3.2: u and v are not adjacent.

By a convenient labeling for the vertices, $L(G)$ can be seen written in the following way:

$$\left[\begin{array}{c|c|ccc|ccc|ccc|ccccc}
s+t+1 & 0 & -1 & \cdots & -1 & -1 & \cdots & -1 & 0 & \cdots & 0 & -1 & 0 & 0 & 0 & 0 \\
\hline
0 & s+t+1 & -1 & \cdots & -1 & 0 & \cdots & 0 & -1 & \cdots & -1 & 0 & 0 & 0 & 0 & -1 \\
\hline
-1 & -1 & 2 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
-1 & -1 & 0 & \cdots & 2 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
-1 & 0 & 0 & \cdots & 0 & 2 & \cdots & 0 & -1 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
-1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 2 & 0 & \cdots & -1 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & -1 & 0 & \cdots & 0 & -1 & \cdots & 0 & 2 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & -1 & 0 & \cdots & 0 & 0 & \cdots & -1 & 0 & \cdots & 2 & 0 & 0 & 0 & 0 & 0 \\
\hline
-1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 2 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & -1 & 2 & -1 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & -1 & 2 & -1 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & -1 & 2 & -1 \\
0 & -1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & -1 & 2
\end{array}\right].$$

Applying the same technique as in Case 2.1, we conclude that there is a root in the interval $(0.5, 1)$, and consequently G is not L-integral. $\square$

Theorem 10. *Let $G \in \mathcal{G}_2'$ with $n \geq 9$ vertices. Then G is L-integral if and only if $G \cong K_1 \vee (r \cdot K_1 \cup s \cdot K_2 \cup K_{1,t})$, where $t \geq 2$ and $r + s \geq 2$ or $G \cong K_2 \vee (n-2) \cdot K_1$.*

Proof. Let $G \in \mathcal{G}_2'$ with $n \geq 9$ vertices. Suppose that G is L-integral. From Theorem 1 (Section 4) and Propositions 2, 3, 4, 5, 7, 8 and 9 (Section 4) we conclude that $G \cong K_1 \vee (r \cdot K_1 \cup s \cdot K_2 \cup K_{1,t})$, where $t \geq 2$ and $r + s \geq 2$ or $G \cong K_2 \vee (n-2) \cdot K_1$ and the result follows. $\square$

By Theorems 11 (Section 2), 1 (Section 3) and 10 (Section 4), the proof of Theorem 1 (Section 1) is complete.

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