Proyecciones Journal of Mathematics
Vol. 41, N^o 5, pp. 1199-1212, October 2022.
Universidad Católica del Norte
Antofagasta - Chile

10.22199/issn.0717-6279-4722

Fractional metric dimension of generalized prism graph

Nosheen Goshi
University of Management and Technology (UMT), Pakistan
Sohail Zafar
University of Management and Technology (UMT), Pakistan
and
Tabasam Rashid
University of Management and Technology (UMT), Pakistan
Received : February 2021. Accepted : August 2022

Abstract

Fractional metric dimension of connected graph G was introduced by Arumugam et al. in [Discrete Math. 312, (2012), 1584-1590] as a natural extension of metric dimension which have many applications in different areas of computer sciences for example optimization, intelligent systems, networking and robot navigation. In this paper fractional metric dimension of generalized prism graph $P_m \times C_n$ is computed using combinatorial criterion devised by Liu et al. in [Mathematics, 7(1), (2019), 100].

Keywords: *Resolving neighborhood; fractional metric dimension; cartesian product; generalized prism graph.*

1. Introduction and Preliminaries

Let $G = (V(G), E(G))$ be a finite simple connected graph and $u, v \in V(G)$ then $d(u, v)$ denotes the length of shortest path between u and v in G. If the ordering $(v_1, v_2, \ldots, v_k)$ is imposed on $W = \{v_1, v_2, \ldots, v_k\} \subseteq V(G)$, then W is called ordered set. For $u, v \in V(G)$, the resolving neighborhood of u and v, denoted by $R\{u, v\}$, is given by the collection of all $w \in V(G)$ which are not equidistant from u and v. The vertex set $W \subseteq V(G)$ is resolving set of G if $W \cap R\{u, v\} \neq \emptyset$ for all distinct pair of vertices u, v in $V(G)$ and the minimum cardinality of such set is called metric dimension, $\dim(G)$, of G. Slater [25, 26] and Harary et al. [15] independently introduced the concept of locating set and resolving sets/ metric dimension respectively. The metric dimension of different classes of graphs have been studied by many authors (see [15], [16], [17] and [18]). Khuller et al. [21] discussed metric dimension as an application to the navigation of robots in a graph space and showed its an NP-hard problem. Garey and Johnson [13] used reduction from 3D matching to show the minimum metric dimension problem is NP-Complete for general graphs. In science, social science and technology metric dimension of graphs possesses diverse applications. Pharmaceutical chemistry [6], combinatorial optimization [23], drug discovery [7], determining routing protocols geographically[19] and telecommunication[3] are some of them. For more applications see [4], [5], [8], [9], [10], [21] and [24].

Recently, fractionalization of various parameters of graphs are rapidly developing. These ideas are being studied under the name of fractional graph theory. For further details on fractional graph theory see [22]. The problem of finding metric dimension $dim(G)$ of a graph has been suggested as an integer programming problem by Chartrand and Lesnaik [8]. Afterwards Currie and Oellermann [10] described fractional metric dimension to be the optimal solution of the linear relaxation of such problems. Fehr et al. [11] assert the equivalent formulation of fractional metric dimension of graph G, as; suppose $V(G) = W = \{v_1, v_2, \ldots, v_n\}$ and $W_p = \{s_1, s_2, \ldots, s_{\binom{n}{2}}\}$ where s_i denotes the distinct pair of vertices in G. Let $A = (a_{ij})_{\binom{n}{2} \times n}$ be a matrix with

$$a_{ij} = \begin{cases} 1, & \text{if } s_i v_i \in E(R(G)); \\ 0, & \text{otherwise} \end{cases}$$

where $1 \leq i \leq \binom{n}{2}$, $1 \leq j \leq n$ and $R(G)$ is the a bipartite graph with bipartition (W, W_p) where $v_i \in W$ is connected to $s_j \in W_p$ if v_i resolves it, called resolving graph of G. The integer programming construction of metric dimension is describe as: Minimize $f(x_1, x_2, \ldots, x_n) = x_1 + x_2 + \ldots + x_n$ subject to $A\overline{X} \geq \overline{1}$ where $\overline{X} = (x_1, x_2, \ldots, x_n)^T, x_i \in \{0, 1\}$ and $\overline{1}$ is the $\binom{n}{2} \times 1$ column vector all of whose entries are 1. The fractional metric dimension of G is given by the optimal solution of the integer programming relaxation of this integer programming problem after replacing $x_i \in \{0, 1\}$ by $0 \leq x_i \leq 1$. Arumugam and Mathew [1] formulated fractional metric dimension in terms of resolving functions.

A least function $f : V(G) \to [0, 1]$ is called minimal resolving function of G if $f(R\{u, v\}) = \sum_{x \in R\{u,v\}} f(x) \geq 1$ for any distinct pair of vertices in $V(G)$. $min\{|f| : f$ is a minimal resolving function of $G\}$ is called fractional metric dimension of G, denoted by $dim_f(G)$, where $|f| = \sum_{v \in V(G)} f(v)$.

Arumugam et al. [1, 2] determined fractional metric dimension of some classes of graphs. Following theorem includes some of those classes.

Theorem 1.1. *[1, 2]*

1. *For any n-cycle,* $dim_f(C_n) = \begin{cases} \frac{n}{n-2}, & \text{if } n \text{ is even;} \\ \frac{n}{n-1}, & \text{if } n \text{ is odd} \end{cases}$

2. *For complete graph K_n, $dim_f(K_n) = \frac{n}{2}$*

3. *For Petersen graph $\mathcal{P}$, $dim_f(\mathcal{P}) = \frac{5}{3}$*

Arumugam et al. [1, 2] initiated the problem of finding fractional metric dimension of cartesian product of graphs and proved several results including $dim_f(P_m \times P_n) = 2$ and $dim_f(P_2 \times C_n)$ when n is even. They further proposed an open problem of finding $dim_f(P_2 \times C_n)$ when n is odd. This problem was addressed by Min Feng et al. in [12]. They prove that $P_2 \times C_n$ being vertex transitive graph have fractional metric dimension $\frac{2n}{n+1}$. This motivated us to compute fractional metric dimension of more general case $P_m \times C_n$. This class of graphs is known as generalized prism graph. Moreover, this class of graphs is not vertex transitive since $A \times B$ is vertex transitive if and only if A and B are vertex transitive (see [14]). The path graph P_m is not vertex transitive for $m \geq 3$. Recently Liu et al. [20] calculated fractional metric dimension of generalized Jahangir graph using combinatorial technique. The following theorem states the combinatorial criterion proposed in [20] to compute fractional metric dimension of graphs.

Theorem 1.2. *[20] Let $R = \{R_i, R_j | i \in I$ and $j \in J\}$ be collection of all pairwise resolving sets of G such that $|R_i| = \alpha < |R_j|$ and $|R_j \cap (\cup R_i)| \geq \alpha$, then $dim_f(G) = \sum_{t=1}^{\beta(G)} \frac{1}{\alpha}$ where $\beta(G) = | \cup_{i \in I} R_i|$.*

In this paper, first section is reserved for introduction and preliminaries. In Section 2 fractional metric dimension of generalized prism graph is calculated by determining resolving neighborhoods of all possible pair of distinct vertices in it. Finally, the paper is concluded in Section 3.

2. Fractional metric dimension of generalized prism graph $P_m \times C_n$.

In this section, we will compute the fractional metric dimension of generalized prism graph $P_m \times C_n$. For simplicity, throughout the paper, generalized prism graph $P_m \times C_n$ is denoted by $G_{m,n}$. The vertex set of $G_{m,n}$ is $\{x_{ij} : 1 \leq i \leq m, 1 \leq j \leq n\}$ and edge set consists of all edges between two vertices if they have one common subscript, as shown in Fig. 1. For any set A, $A^c = V(G_{m,n}) \setminus A$. Throughout this paper $m \geq 3$. From now on, fix one vertex on ith circle and label it as x_{i1}, pair it with proceeding $\frac{n}{2}$ vertices on the same circle and on circles at distance k where $1 \leq k \leq m$ and compute resolving neighborhoods of these pair. Remaining vertices of $G_{m,n}$ behave symmetrically.

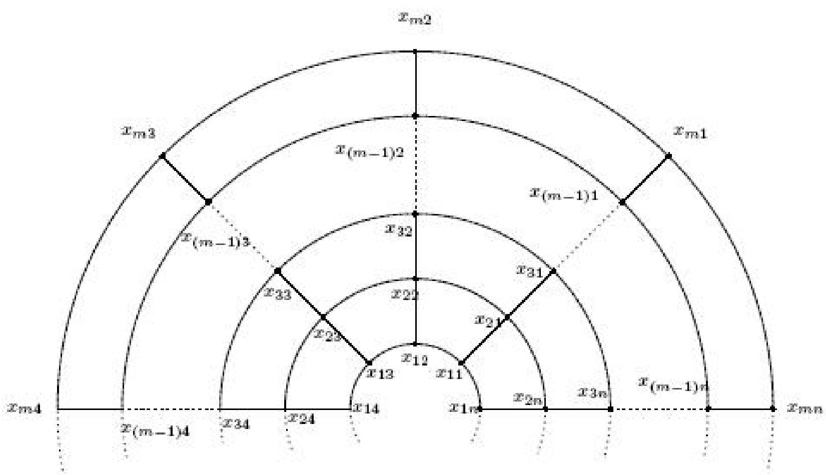

Figure 1: The Family of Graphs $G_{m,n}$

The proof of main result depends upon parity of n (even or odd). Therefore, in following subsections fractional metric dimension of $G_{m,n}$ for n even and odd are calculated separately by considering cardinalities of resolving neighborhoods of distinct pair of vertices.

2.1. Fractional metric dimension of generalized prism graph $G_{m,n}$ graph when n is even.

In order to compute fractional metric dimension of $G_{m,n}$ for n even, all possible pair of vertices are classified depending upon distance between them (even or odd). Following lemma gives the resolving neighborhood of pair of vertices at odd distance.

Lemma 2.1. *Let $u, v \in V(G_{m,n})$ for n even. If $d(u,v) \equiv 1(\mod 2)$, then $R\{u,v\} = V(G_{m,n})$.*

Proof. Let n be even, then $G_{m,n}$ is bipartite. Assume $u, v \in V(G_{m,n})$ such that $d(u,v) \equiv 1(\mod 2)$. Then, length of every $u-v$ path is odd. On contrary, suppose $R\{u,v\} \neq V(G_{m,n})$, then there exists $x \in V(G_{m,n})$ such that $x \notin R\{u,v\}$ and $d(u,x) = d(v,x) = k$. Choose k to be minimal. Let $P_1 : u \to \ldots \to x_i$ and $P_2 : x_i \to \ldots \to v$ be their respective path of length k. Since $P_1 \cap P_2 = \emptyset$ because of minimality of k. Therefore, $P_1 \cup P_2$ is a $u - v$ path of length $2k$. A contradiction. Hence, there is no x which does not resolves the vertices u, v. $\square$

Now consider vertices u, v for which $d(u,v) \not\equiv 1(\mod 2)$. Such possible pairs may be lie on same or different circles in $G_{m,n}$. In the following lemma resolving neighborhoods of pair of vertices lying on adjacent circles is computed.

Lemma 2.2. *If $G_{m,n}$ be generalized prism graph where n is even, then $|R\{x_{i1}, x_{(i+1)2}\}| = \frac{mn}{2}$ where $1 \leq i \leq m-1$. Moreover $|\bigcup_{j=1}^{n-1}\bigcup_{i=1}^{m-1} R\{x_{ij}, x_{(i+1)(j+1)}\}| = mn$.*

Proof. Since $R\{x_{i1}, x_{(i+1)2}\} = \left\{\{x_{ts} : 1 \leq t \leq i, 2 \leq s \leq 1 + \frac{n}{2}\} \cup \{x_{t1}, x_{ts} : i+1 \leq t \leq m, \frac{n}{2} + 2 \leq s \leq n\}\right\}^c$. Therefore, $|R\{x_{i1}, x_{(i+1)2}\}| = \frac{mn}{2}$. Further, $\bigcup_{i=1}^{m-1} R\{x_{ij}, x_{(i+1)(j+1)}\} = \{x_{1(1+j)}, x_{1(2+j)}, \ldots, x_{1(\frac{n}{2}+j)}\}^c$.

Implies that
$\bigcup_{j=1}^{n-1}\bigcup_{i=1}^{m-1} R\{x_{ij}, x_{(i+1)(j+1)}\} = V(G_{m,n})$. □

In the following lemma relation between the cardinalities of resolving neighborhoods, computed in Lemma 2.1 and 2.2, has established.

Lemma 2.3. *Let $G_{m,n}$ be generalized prism graph where n is even. If $d(u,v) \equiv 1(\bmod 2)$, then $|R\{x_{i1}, x_{(i+1)2}\}| \leq |R\{u,v\}|$ and $|R\{u,v\} \cap (\bigcup_{j=1}^{n-1}\bigcup_{i=1}^{m-1} R\{x_{ij}, x_{(i+1)(j+1)}\})| \geq |R\{x_{i1}, x_{(i+1)2}\}|$ where $1 \leq i \leq m-1$.*

Proof. Result follows from Lemma 2.1 and 2.2 □

In the following lemma resolving neighborhoods of pair of vertices lying on same circle are computed and their relation with resolving neighborhoods computed in Lemma 2.2 has established.

Lemma 2.4. *If $G_{m,n}$ be generalized prism graph where n is even, then $|R\{x_{i1}, x_{(i+1)2}\}| \leq |R\{x_{il}, x_{ij}\}|$ and*
$|R\{x_{il}, x_{ij}\} \cap (\bigcup_{j=1}^{n-1}\bigcup_{i=1}^{m-1} R\{x_{ij}, x_{(i+1)(j+1)}\})| \geq |R\{x_{i1}, x_{(i+1)2}\}|$.

Proof. Proof is divided into the following two cases:

Case 1 When l, j are both either even or odd. Then, $R\{x_{il}, x_{ij}\} = \{x_{t(\frac{l+j}{2})}, x_{t(\frac{l+j+n}{2})} : 1 \leq t \leq m\}^c$ where $1 \leq i \leq m$. Implies that $|R\{x_{il}, x_{ij}\}| = mn - 2m \geq |R\{x_{i1}, x_{(i+1)2}\}|$. Clearly $R\{x_{il}, x_{ij}\} \cap (\bigcup_{j=1}^{n-1}\bigcup_{i=1}^{m-1} R\{x_{ij}, x_{(i+1)(j+1)}\}) = R\{x_{il}, x_{ij}\}$.

Case 2 When l, j are not both even or odd. Then, $d(x_{il}, x_{ij}) \equiv 1(\bmod 2)$ where $1 \leq i \leq m$. Thus, result follows from Lemma 2.3.

□

In the following two lemmas resolving neighborhoods of pair of vertices lying on circles at distance k are computed.

Lemma 2.5. *If $G_{m,n}$ be generalized prism graph where n is even and $m \geq 3$, then $|R\{x_{i1}, x_{(i+1)2}\}| \leq |R\{x_{ij}, x_{(i+k)j}\}|$ where $1 \leq i+k \leq m$. Further $|R\{x_{ij}, x_{(i+k)j}\} \cap (\bigcup_{j=1}^{n-1}\bigcup_{i=1}^{m-1} R\{x_{ij}, x_{(i+1)(j+1)}\})| \geq |R\{x_{i1}, x_{(i+1)2}\}|$.*

Proof. Proof comprises of two cases:

Case 1 When k is odd. Then, $R\{x_{ij}, x_{(i+k)j}\} = V(G_{m,n})$

Case 2 When k is even. Then, $R\{x_{ij}, x_{(i+k)j}\} = \{x_{(i+\frac{k}{2})s} : 1 \leq s \leq n\}^c$.

Both the cases implies $|R\{x_{ij}, x_{(i+2)j}\}| \in \{mn, (m-1)n\}$. Also, $R\{x_{ij}, x_{(i+k)j}\} \cap (\bigcup_{j=1}^{n-1} \bigcup_{i=1}^{m-1} R\{x_{ij}, x_{(i+1)(j+1)}\}) = R\{x_{ij}, x_{(i+k)j}\}$. □

Lemma 2.6. *If $G_{m,n}$ be the generalized prism graph where n is even, then $|R\{x_{i1}, x_{(i+k)(1+j)}\}| \geq |R\{x_{i1}, x_{(i+1)2}\}|$ where $1 \leq i+k \leq m$. Further, $|R\{x_{i1}, x_{(i+k)(1+j)}\} \cap (\bigcup_{j=1}^{n-1} \bigcup_{i=1}^{m-1} R\{x_{ij}, x_{(i+1)(j+1)}\})| \geq |R\{x_{i1}, x_{(i+1)2}\}|$.*

Proof. Consider the following two cases to prove the claim:

Case 1 When k, j are not both even or odd. Then $d(x_{i1}, x_{(i+k)(1+j)}) \equiv 1(\bmod 2)$. Thus result follows from Lemma 2.3.

Case 2 When k, j are both either even or odd. Then the following subcases arise:

Case 2.1 When $k = j$. Then,

$$R\{x_{i1}, x_{(i+k)(1+j)}\} = \left\{ \{x_{ts} : 1 \leq t \leq i, j+1 \leq s \leq 1+\frac{n}{2}\} \cup \{x_{(i+1)j}, x_{(i+2)(j-1)}, \ldots, x_{(i+j-1)2}\} \cup \{x_{t1}, x_{ts} : i+j \leq t \leq m, \frac{n}{2}+j+1 \leq s \leq n\} \cup \{x_{(i+1)(\frac{n}{2}+2)}, x_{(i+2)(\frac{n}{2}+3)}, \ldots, x_{(i+j-1)(\frac{n}{2}+j)}\} \right\}^c.$$

This shows $|R\{x_{i1}, x_{(i+k)(1+j)}\}| = mn-(m+\frac{n}{2}-1+(m-j)(\frac{n}{2}-j))$. Which is least for $j=1$, i.e., $|R\{x_{i1}, x_{(i+k)(1+j)}\}| = \frac{mn}{2}$ for $j=1$.

Case 2.2 When $k \geq j+2$. Then,

$$R\{x_{i1}, x_{(i+k)(1+j)}\} = \left\{ \{x_{(i+\frac{k-j}{2})s} : 1+j \leq s \leq \frac{n}{2}+1\} \cup \{x_{(i+\frac{k-j}{2}+1)j}, x_{(i+\frac{k-j}{2}+2)(j-1)}, \ldots, x_{(i+\frac{k+j}{2})1}\} \cup \{x_{(i+\frac{k+j}{2})s} : 1+j+\frac{n}{2} \leq s \leq n\} \cup \{x_{(i+\frac{k-j}{2}+1)(2+\frac{n}{2})}, x_{(i+\frac{k-j}{2}+2)(3+\frac{n}{2})}, \ldots, x_{(i+\frac{k+j}{2})(1+j+\frac{n}{2})}\} \right\}^c.$$

This shows $|R\{x_{i1}, x_{(i+k)(1+j)}\}| = mn-n$.

Case 2.3 When $j \geq k+2$. Then,
$R\{x_{i1}, x_{(i+k)(1+j)}\} = \Big\{\{x_{t(\frac{j+k}{2}+1)}, x_{t(\frac{n+j-k}{2}+1)} : 1 \leq t \leq i\} \cup \{x_{(i+1)(\frac{j+k}{2})}, x_{(i+2)(\frac{j+k}{2}-1)}, \dots, x_{(i+k-1)(\frac{j-k}{2}+2)}\} \cup \{x_{(i+1)(\frac{n+j-k}{2}+2)}, x_{(i+2)(\frac{n+j-k}{2}+3)}, \dots, x_{(i+k-1)(\frac{n+j+k}{2})}\} \cup \{x_{t(\frac{j-k}{2}+1)}, x_{t(\frac{n+j+k}{2}+1)} : i+k \leq t \leq m\}\Big\}^c$. Hence, $|R\{x_{i1}, x_{(i+k)(1+j)}\}| = mn - 2m$.

Combining all cases, $|R\{x_{i1}, x_{(i+k)(1+j)}\}| \geq |R\{x_{i1}, x_{(i+1)2}\}|$. Moreover, $|R\{x_{i1}, x_{(i+k)(1+j)}\} \cap (\bigcup_{j=1}^{n-1}\bigcup_{i=1}^{m-1} R\{x_{ij}, x_{(i+1)(j+1)}\})| = |R\{x_{i1}, x_{(i+k)(1+j)}\}| \geq |R\{x_{i1}, x_{(i+1)2}\}|$. □

In the following theorem fractional metric dimension of generalized prism graph is calculated using Theorem 1.2.

Theorem 2.7. *The fractional metric dimension of generalized prism graph* $G_{m,n}; m \geq 3$ *is* $\frac{n}{\lceil \frac{n}{2} \rceil}$, *when* n *is even.*

Proof. From Lemmas 2.1 to 2.6,

$$|R\{x_{ij}, x_{lm}\} \cap (\bigcup_{j=1}^{n-1}\bigcup_{i=1}^{m-1} R\{x_{ij}, x_{(i+1)(j+1)}\})| \geq |R\{x_{i1}, x_{(i+1)2}\}|.$$

Thus, using Theorem 1.2 define the function $f : V(G_{m,n}) \rightarrow [0,1]$, defined by $f(v) = \frac{2}{mn}$ for all $v \in V(G_{m,n})$, is a minimal resolving function. Hence, $dim_f(G_{m,n}) = |f| = \sum f(v) = (mn)(\frac{2}{mn}) = 2 = \frac{n}{\lceil \frac{n}{2} \rceil}$. □

2.2. Fractional metric dimension of generalized prism graph $G_{m,n}$ graph when n is odd.

In this subsection distinct vertices are classified on the bases of distance between cycles on which they are lying in $G_{m,n}$ and cardinalities of resolving neighborhoods of these pairs are calculated. Using these cardinalities fractional metric dimension of $G_{m,n}$ when n is odd is computed. In the following lemma resolving neighborhoods of pair of vertices lying on adjacent cycles is determined.

Lemma 2.8. *If* $G_{m,n}$ *be generalized prism graph where* n *is odd, then* $|R\{x_{i1}, x_{(i+1)2}\}| = \frac{m(n+1)}{2}$ *where* $1 \leq i \leq m-1$. *Moreover* $|\bigcup_{j=1}^{n-1}\bigcup_{i=1}^{m-1} R\{x_{ij}, x_{(i+1)(j+1)}\}| = mn$.

Proof. Since $R\{x_{i1}, x_{(i+1)2}\} = \Bigg\{\{x_{ts} : 1 \leq t \leq i, 2 \leq s \leq \lceil \frac{n}{2} \rceil\} \cup \{x_{ts} : i+1 \leq t \leq m, \lceil \frac{n}{2} \rceil + 2 \leq s \leq n\} \cup \{x_{t1} : i+1 \leq t \leq m\}\Bigg\}^c$. Therefore, $|R\{x_{i1}, x_{(i+1)2}\}| = \frac{m(n+1)}{2}$. Similarly, $R\{x_{i2}, x_{(i+1)3}\} = \Bigg\{\{x_{ts} : 1 \leq t \leq i, 3 \leq s \leq \lceil \frac{n}{2} \rceil + 1\} \cup \{x_{ts} : i+1 \leq t \leq m, \lceil \frac{n}{2} \rceil + 3 \leq s \leq n\} \cup \{x_{t1}, x_{t2} : i+1 \leq t \leq m\}\Bigg\}^c$. Continuing same procedure and taking their union give $\bigcup_{j=1}^{n-1} \bigcup_{i=1}^{m-1} R\{x_{ij}, x_{(i+1)(j+1)}\} = V(G_{m,n})$. Hence, $|\bigcup_{j=1}^{n-1} \bigcup_{i=1}^{m-1} R\{x_{ij}, x_{(i+1)(j+1)}\}| = mn$. □

Now calculate the cardinality of resolving neighborhood of pair of vertices lying on same cycle and compare it with cardinality given in Lemma 2.8.

Lemma 2.9. *If $G_{m,n}$ be generalized prism graph when n is odd, then $|R\{x_{i1}, x_{(i+1)2}\}| \leq |R\{x_{i1}, x_{i(1+j)}\}|$, where $1 \leq j \leq n-1$, $1 \leq i \leq m$. Further $|R\{x_{i1}, x_{i(1+j)}\} \cap (\bigcup_{j=1}^{n-1} \bigcup_{i=1}^{m-1} R\{x_{ij}, x_{(i+1)(j+1)}\})| \geq |R\{x_{i1}, x_{(i+1)2}\}|$.*

Proof. Since

$$R\{x_{i1}, x_{(i)(1+j)}\} = \begin{cases} \{x_{t(1+\frac{j}{2})}\}^c, & \text{if } j \text{ is even;} \\ \{x_{t(1+\frac{n+j}{2})}\}^c, & \text{if } j \text{ is odd} \end{cases}$$

where $1 \leq i \leq m$. Therefore, $|R\{x_{i1}, x_{i(1+j)}\}| = mn - m \geq |R\{x_{i1}, x_{(i+1)2}\}|$. Clearly, $R\{x_{i1}, x_{i(1+j)}\} \cap (\bigcup_{j=1}^{n-1} \bigcup_{i=1}^{m-1} R\{x_{ij}, x_{(i+1)(j+1)}\}) = R\{x_{i1}, x_{i(1+j)}\}$. □

Now calculate the cardinality of resolving neighborhood of pair of vertices lying on cycles at distance k and compare it with cardinality given in Lemma 2.8. In the following lemma resolving neighborhood of vertices lying on cycles at distance k and vertically aligned are determined.

Lemma 2.10. *If $G_{m,n}$ be generalized prism graph where n is odd and k be a positive integer, then $|R\{x_{i1}, x_{(i+1)2}\}| \leq |R\{x_{ij}, x_{(i+k)j}\}|$. Where $1 \leq j \leq n$, $1 \leq i+k \leq m$. Further $|R\{x_{ij}, x_{(i+k)j}\} \cap (\bigcup_{j=1}^{n-1} R\{x_{ij}, x_{(i+1)(j+1)}\})| \geq |R\{x_{i1}, x_{(i+1)2}\}|$.*

Proof. Since

$$R\{x_{ij}, x_{(i+k)j}\} = \begin{cases} V(G), & \text{if } k \text{ is odd;} \\ \{x_{(i+\frac{k}{2})s} : 1 \leq s \leq n\}^c, & \text{if } k \text{ is even.} \end{cases}$$

Therefore, $|R\{x_{ij}, x_{(i+k)j}\}| \in \{mn, (m-1)n\} \geq |R\{x_{i1}, x_{(i+1)2}\}|$. Also $R\{x_{ij}, x_{(i+k)j}\} \cap (\bigcup_{j=1}^{n-1} R\{x_{ij}, x_{(i+1)(j+1)}\}) = R\{x_{ij}, x_{(i+k)j}\}$. □

In the following lemma resolving neighborhood of vertices lying on cycles at distance k but are not vertically aligned, are computed.

Lemma 2.11. *If $G_{m,n}$ be generalized prism graph where n is odd and k be a positive integer, then $|R\{x_{i1}, x_{(i+1)2}\}| \leq |R\{x_{i1}, x_{(i+k)(1+j)}\}|$, where $1 \leq j \leq n-1$, $1 \leq i+k \leq m$. Further $|R\{x_{i1}, x_{(i+k)(1+j)}\} \cap (\bigcup_{j=1}^{n-1} R\{x_{ij}, x_{(i+1)(j+1)}\})| \geq |R\{x_{i1}, x_{(i+1)2}\}|$.*

Proof. The claim can be proved using the following cases:

Case 1 When $k = j$. Then, $R\{x_{i1}, x_{(i+k)(1+j)}\} = \left\{ \{x_{ts} : 1 \leq t \leq i, j+1 \leq s \leq \lceil \frac{n}{2} \rceil\} \cup \{x_{(i+1)(j)}, x_{(i+2)(j-1)}, \ldots, x_{(i+j-1)2}\} \cup \{x_{t1}, x_{ts} : i+j \leq t \leq m, \lceil \frac{n}{2} \rceil + j + 1 \leq s \leq n\} \right\}^c$. This shows $|R\{x_{i1}, x_{(i+k)(1+j)}\}| = mn - (m + (m-j+1)(\frac{n-1}{2} - j))$. Which is least for $j = 1$, i.e., $|R\{x_{i1}, x_{(i+k)(1+j)}\}| = \frac{m(n+1)}{2}$ for $j = 1$.

Case 2 When $k < j$. Then we have the following subcases:

Case 2.1 When both k and j are either even or odd. Then,

$$R\{x_{i1}, x_{(i+k)(1+j)}\} = \left\{ \{x_{t(\frac{j+k}{2}+1)} : 1 \leq t \leq i\} \cup \{x_{(i+1)(\frac{j+k}{2})}, x_{(i+2)(\frac{j+k}{2}-1)}, \ldots, x_{(i+k-1)(\frac{j-k}{2}+2)}\} \cup \{x_{t(\frac{j-k}{2}+1)} : i+k \leq t \leq m\} \right\}^c.$$

Case 2.2 When k and j are not both even or odd. Then,

$$R\{x_{i1}, x_{(i+k)(1+j)}\} = \left\{ \{x_{t(\frac{n+j-k}{2}+1)} : 1 \leq t \leq i\} \cup \{x_{(i+1)(\frac{n+j-k}{2}+2)}, x_{(i+2)(\frac{n+j-k}{2}+3)}, \ldots, x_{(i+k-1)(\frac{n+j+k}{2})}\} \cup \{x_{t(\frac{n+j+k}{2}+1)} : i+k \leq t \leq m\} \right\}^c.$$

In both the cases $|R\{x_{i1}, x_{(i+k)(1+j)}\}| = m(n-1)$.

Case 3 When $j < k$. Then the following subcases arise:

Case 3.1 When both k and j are either even or odd. Then, $R\{x_{i1}, x_{(i+k)(1+j)}\} = \Big\{\{x_{(i+\frac{k-j}{2})s} : 1+j \leq s \leq \lceil \frac{n}{2} \rceil\}$ $\cup \{x_{(i+\frac{k-j}{2}+1)j}, x_{(i+\frac{k-j}{2}+2)(j-1)}, \ldots, x_{(i+\frac{k+j}{2})1}\} \cup \{x_{(i+\frac{k+j}{2})s} : 1 + j + \lceil \frac{n}{2} \rceil \leq s \leq n\}\Big\}^c$. This shows $|R\{x_{i1}, x_{(i+k)(1+j)}\}| = mn - n + j$. Which is minimum if $j = 1$ and is maximum if $j = n-1$.

Case 3.2 When k and j are not both even or odd. Then, $R\{x_{i1}, x_{(i+k)(1+j)}\} = \{x_{(i+\lceil \frac{k-j}{2} \rceil)(1+\lceil \frac{n}{2} \rceil)}, x_{(i+\lceil \frac{k-j}{2} \rceil+1)(2+\lceil \frac{n}{2} \rceil)}, \ldots, x_{(i+\frac{k+j+1}{2})(j+\lceil \frac{n}{2} \rceil)}\}^c$. This shows $|R\{x_{i1}, x_{(i+k)(1+j)}\}| = mn - j$. Which is minimum if $j = n-1$ and is maximum if $j = 1$.

Combining all above cases $|R\{x_{i1}, x_{(i+1)2}\}| \leq |R\{x_{i1}, x_{(i+k)(1+j)}\}|$, Where $1 \leq j \leq n-1$, $1 \leq i+k \leq m$. Furthermore $R\{x_{i1}, x_{(i+k)(1+j)}\} \cap (\bigcup_{j=1}^{n-1} R\{x_{ij}, x_{(i+1)(j+1)}\}) = R\{x_{i1}, x_{(i+k)(1+j)}\}$. □

In the following theorem fractional metric dimension of generalized prism graph is calculated using Theorem 1.2.

Theorem 2.12. *The fractional metric dimension of generalized prism graph $G_{m,n}$ is $\frac{n}{\lceil \frac{n}{2} \rceil}$, when n is odd.*

Proof. From Lemmas 2.8 to 2.11,

$$|R\{x_{ij}, x_{lm}\} \cap (\bigcup_{j=1}^{n-1} \bigcup_{i=1}^{m-1} R\{x_{ij}, x_{(i+1)(j+1)}\})| \geq |R\{x_{i1}, x_{(i+1)2}\}|$$

. Thus, using Theorem 1.2, define a function $f : V(G_{m,n}) \to [0,1]$ defined by $f(v) = \frac{2}{m(n+1)}$ for all $v \in V(G_{m,n})$, is a minimal resolving function. Hence, $dim_f(G_{m,n}) = |f| = \sum f(v) = (mn)(\frac{2}{m(n+1)}) = \frac{n}{\lceil \frac{n}{2} \rceil}$. □

3. Conclusion

In this paper a non vertex transitive class of graphs, generalized prism graph $G_{m,n}$ where $m \geq 3$, have been discussed. The resolving neighborhoods of all possible distinct pair of vertices in $G_{m,n}$ are computed. Their cardinalities are used to calculate its fractional metric dimension. It is worth noting that results presented in this paper not only verify results in [2] and [12] for $m = 2$ but are their natural extension.

Acknowledgement: The authors are grateful to the editor and reviewers for the careful reading and several suggestions to improve the manuscript.

References

[1] S. Arumugam and V. Mathew, "The fractional metric dimension of graphs", *Discrete Mathematics,* vol. 312, no. 9, pp. 1584-1590, 2012. doi: 10.1016/j.disc.2011.05.039

[2] S. Arumugam, V. Mathew and J. Shen, "On the fractional metric dimension of graphs", *Discrete Mathematics, Algorithms and Applications,* vol. 5, no. 4, pp. 135-137, 2013. doi: 10.1142/s1793830913500377

[3] Z. Beerliova, F. Eberhard, T. Erlebach, A. Hall, M. Hoffman, M. Mihalak and L. Ram, "Network discovery and verification", *IEEE Journal on Selected Areas in Communications,* vol. 24, no. 12, pp. 2168-2181, 2006. doi: 10.1109/jsac.2006.884015

[4] J. C. Bermound, F. Comellas and D. F. Hsu, "Distributed loop computer networks: Survey", *Journal of Parallel and Distributed Computing,* vol. 24, no. 1, pp. 2-10, 1995. doi: 10.1006/jpdc.1995.1002

[5] A. Borchert and S. Gosselin, "The metric dimension of circulant graphs and Cayley hyper-graphs", *Utilitas Mathematica,* 2015. [On line]. Available: https://bit.ly/3pUqCVs

[6] P. J. Cameron and J. H. V. *Lint, Designs, Graphs, Codes and their Links,* vol. 22. Cambridge: Cambridge University Press, 1991. doi: 10.1017/CBO9780511623714

[7] G. Chartrand, L. Eroh, M. Johnson and O. R. Oellermann, "Resolvability in graphs and the metric dimension of a graph", *Discrete Applied Mathematics,* vol. 105, no. 99-113, pp. 99-113, 2000. doi: 10.1016/s0166-218x(00)00198-0

[8] G. Chartrand and L. Lesniak, *Graphs & digraphs,* 4th ed., Chapman & Hall/CRC, 2005.

[9] V. Chvátal, "Mastermind", *Combinatorica*, vol. 3, pp. 325-329, 1983.

[10] J. Currie and O.R. Oellermann, "The metric dimension and metric independence of graphs", *Journal of Combinatorial Mathematics and Combinatorial Computing,* vol. 39, pp. 157-167, 2001.

[11] M. Fehr, S. Gosselin and O.R. Oellermann, "The metric dimension of Cayley digraphs", *Discrete Mathematics,* vol. 306, no. 1, pp. 31-41, 2006. doi: 10.1016/j.disc.2005.09.015

[12] M. Feng, Lv Benjian and K. Wang, "On the fractional metric dimensión of graphs", *Discrete Applied Mathematics,* vol. 170, pp. 55-63, 2014. doi: 10.1016/j.dam.2014.01.006

[13] M. R. Garey and D.S. Johnson, *Computers and Intractability: A Guide to the theory of NP. Completeness.* New York: Freeman, 1979.

[14] Hammack, W. Imrich, Klavžar, *The handbook of product graphs,* 2nd ed. Boca Raton: CRC, 2011.

[15] F. Harary and R. A. Melter, "On the metric dimension of a graph", *Ars Combinatoria,* vol. 2, pp. 191-195, 1976.

[16] M. Imran, A. Q. Baig, S.A.U.H. Bokhary and I. Javaid, "On the metric dimension of circulant graphs", *Applied Mathematics Letters,* vol. 25, pp. 320-325, 2012. doi: 10.1016/j.aml.2011.09.008

[17] M. Imran, M. K. Siddiqui and R. Naeem, "On the metric dimension of generalized Petersen multigraphs", *IEEE Access,* vol. 6, pp. 74328-74338, 2018. doi: 10.1109/access.2018.2883556

[18] I. Javaid, M.T. Rahim and K. Ali, "Families of regular graphs with constant metric dimensión", *Utilitas Mathematica,* vol. 75, pp. 21-33, 2008.

[19] K. Liu and N. Abu-Ghazaleh, *Virtual coordinate back tracking for void traversal in geographic routing,* vol. 4104, pp. 46-59. Springer, 2006.

[20] J. B. Liu, A. Kashif, T. Rashid and M. Javaid, "Fractional metric dimension of generalized Jahangir graph", *Mathematics*, vol. 7, no. 1, 2019. doi: 10.3390/math7010100

[21] S. Khuller, B. Raghavachari and A. Rosenfield, "Landmarks in graphs", *Discrete Applied Mathematics*, vol. 70, no. 3, pp. 217-229, 1996. doi: 10.1016/0166-218x(95)00106-2

[22] E. R. Scheinerman and D. H. Ulleman, *Fractional graph theory: A Rational approach to the theory of graphs.* New York: John Wiley & Sons, 1997.

[23] A. Sebő and E. Tannier, "On metric generators of graphs", *Mathematics of Operations Research,* vol. 29, no. 2, pp. 383-393, 2004. doi: 10.1287/moor.1030.0070

[24] H. Shapiro and S. Soderberg, "A combinatory detection problema", *The American Mathematical Monthly,* vol. 70, no. 10, pp. 1066-1070, 1963. doi: 10.2307/2312835

[25] P. S. Slater, "Leaves of Trees", *Congressus Numerantium,* vol. 14, pp. 549-559, 1975.

[26] P. S. Slater, "Domination and location in acyclic graphs", *Networks,* vol. 17, no. 1, pp. 55-64, 1987. doi: 10.1002/net.3230170105

Nosheen Goshi
University of Management and Technology (UMT),
Lahore Pakistan
email: gohernayyab@gmail.com

Sohail Zafar
University of Management and Technology (UMT),
Lahore Pakistan
email: sohailahmad04@gmail.com
Corresponding author

and

Tabasam Rashid
University of Management and Technology (UMT),
Lahore Pakistan
email: tabasam.rashid@gmail.com