Proyecciones Journal of Mathematics
Vol. 41, N^o 4, pp. 835-854, August 2022.
Universidad Católica del Norte
Antofagasta - Chile

10.22199/issn.0717-6279-4694

Energy and basic reproduction number of n-Corona graphs prior to order 1

Veninstine Vivik J.
Karunya Institute of Technology and Sciences, India
Received : January 2021. Accepted : January 2022

Abstract

This paper advances the corona product to n times corona in the aspect of increasing and decreasing product of graphs and calibrates its energy and basic reproduction number. The proposed model emanates as a graph with successive generations of complexity, whose structure is constructed as a matrix based on its adjacency. The energy is measured from the sum of the absolute values of the eigenvalues of the adjacency matrix of graph G and the largest eigenvalue is known to be R_0. The energy upper bound for increasing and decreasing n-corona product with order 1 of complete graphs are attained.

2010 Subjclass: *05C50, 05C76, 00A71, 92E10*

Keywords: *Corona product, Eigenvalues, Energy of a graph, Basic reproduction number.*

1. Introduction

The airborne infectious diseases like SARS, MERS, SARS-CoV-2, etc mainly spread through direct, indirect or close contact between infected people and people. It is difficult to measure the outbreak in a population. The mathematical modeling plays a vital role in measuring, analyzing and controlling epidemic outbreaks. The disease propagation of epidemiological models like SIS, SIR, SIER, SEQIR and various other models are discussed in [1, 2].

In this context the people is considered as vertices V and their contact between them as edges E which is represented as a graph $G(V, E)$. More precisely, the first infected person host transmits to a pair of two conected persons and all the infected persons each in turn transmits to three connected persons and so on with increase of order 1 in every subsequent generation. This type of mediation can be modelled as an increasing n corona product of graphs with order 1. Analogously a set of all n infected individuals transmits to $n-1$ connected persons, each in turn transmits to $n-2$ associated individuals and so on with decrease of order 1 in successive generations. This sort of transmission is modelled as decreasing n corona product of graphs. Here the adjaceny matrix of these graphs are considered as the next generation matrix. But the outbreaks in real environments will not be uniform and has substantial change in random order of increase or decrease. In this paper the proposed model is confined to the study of increasing and decreasing n corona product of graphs with order 1. Both these provides the possibility of generating graphs with different random order. The inception of corona product of two graphs was made by Frucht and Harary [3] in 1970. Tavakoli *et al* [4] studied the corona product of graphs. Kaliraj *et al.* [5] investigated the equitable coloring on corona product of graphs. Further Furmaǹcyzk and Kubale analyzed corona product of cubic graphs [6] and multiproduct graph $G \circ^l H$ [7]. The model introduced in this paper is almost the extended idea of l-corona product graph.

The energy of a graph emerges from chemistry owing to importance of total π-electron energy of carbon compounds. It is defined as the sum of the absolute values of the eigenvalues of the adjacency matrix of G and was introduced by Gutman [8]. The study of various graph energy can be found in many papers, see e.g. [9, 10]. Liu *et al.* [11] performed the study on upper bounds for energy of graphs. In another work, Das *et al* [12] and Sridhara *et al* [13] investigated the improved bounds for graph energy.

The notion of finding the energy for corona product of graphs is a recent approach. The dominant eigenvalue of the next generation matrix is called as the basic reproduction number R_0 and was introduced by Diekmann et al *et al.* [14] in 1990. The estimation of R_0 was performed by Diekz [15]. If $R_0 < 1$ the disease is under control and if $R_0 > 1$ the disease can invade. Hence R_0 trace the intensity of an infectious disease outbreak. The largest eigenvalue of the adjacency matrix of a graph G is called the spectral radius of G, which is analogous to R_0. Barik *et al.* [16] intiated the spectrum of corona of two graphs and also studied by Cam McLeman and Erin McNicholas [17]. Recently Vivik and Xavier [20] worked on 2-corona product among different graphs and calculated its energy.

In this paper the concept of increasing and decreasing n-Corona product of graphs with order 1 are established. The energies of both such complete graphs are determined and its nearest bounds are calibrated. The reason behind the choice of complete graphs is that, it has higher degree of contacts in all nodes than other graphs. Also the basic reproduction number is obtained from the dominant eigenvalue of the next generation matrix of these graphs. The magnitude of R_0 helps in analyzing the preventive measures to be adopted to control the epidemic. Not only in epidemics this type can be applied in compter networks where different kinds of malicious objects attack computer systems, robots and mobiles etc.

2. Increasing and Diminishing Corona Product of Graphs

Definition 2.1. *[4] The corona product of G and H is the graph $G \circ H$ obtained by taking one copy of G, called the center graph, $|V(G)|$ copies of H, called the outer graph, and making the i^th vertex of G adjacent to every vertex of the i^th copy of H, where $1 \le i \le |V(G)|$.*

Definition 2.2. *(***Energy***) [9]*
Let G be a graph having n vertices and if $A(G)$ is the adjacency square matrix of order n whose $(i,j)^{th}$-entry is defined as

$$a_{ij} = \begin{cases} 1, & \textit{if vertices } v_i \textit{ and } v_j \textit{ are adjacent} \\ 0, & \textit{otherwise.} \end{cases}$$

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of $A(G)$. The energy of a graph $E(G)$ is defined as the sum of the absolute values of its eigenvalues. Hence $E(G) = \sum_{i=1}^{n} |\lambda_i|$. It is also known as simply energy or ordinary energy of a graph.

Definition 2.3. *[2] [18] The basic reproduction number R_0 is defined as the average number of secondary infections resulting from the index case in a wholly susceptible population. Also it is defined mathematically as the dominant eigenvalue of the next generation matrix.*

Definition 2.4. *Let G_1 be a graph of order 1 and G_2 be the next generation of any graph with order 2. Consider one copy of G_1 as the center graph and join 2 copies of G_2 such that the one vertex of G_1 is adjacent to every vertex of the two copies of G_2, is called the first corona and similarly the corona product of upcoming $r, (2 \leq r \leq n-1)$ generations of any graphs G_{r+1} whose orders are increased by 1 is obtained by joining successively p_r copies of G_{r+1} where $p_r = 2\prod_{k=1}^{n-2}(1+k)$. Accordingly in r^{th} generation the i^{th} vertex of G_r is adjacent to every vertex of i^{th} copy of G_{r+1}, where $1 \leq i \leq p_r$, $2 \leq r \leq n-1$ is known as the Increasing n-corona product of order 1 and is denoted by $G_{r-1} \circ_{+1}^{n} G_r, 2 \leq r \leq n$.*

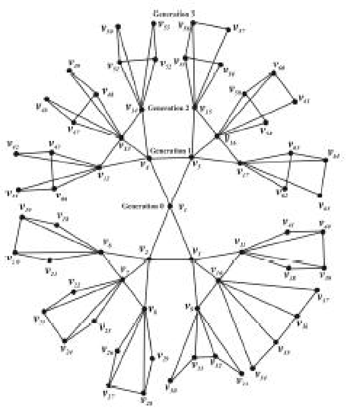

Figure 1: 4-Corona product graph $G_{r-1} \circ_{+1}^{4} G_r, 2 \leq r \leq 4$.

Definition 2.5. *Let G_1 be any graph of order n and G_2 be the first generation of any graph with order $n-1$, take one copy of G_1 as the center graph and join n copies of G_2 such that the i^{th} vertex of G_1 is adjacent to every vertex of the i^{th} copy of G_2, where $1 \leq i \leq n$ is called the first corona and similarly the corona product of following $r, (2 \leq r \leq n-1)$ generations of any graphs G_{r+1} whose orders are reduced by 1 is obtained by joining successively p_r copies of G_{r+1} where*

$$p_r = \begin{cases} n, \; for \; r = 2 \; and \; n = 3 \\ \prod_{k=1}^{r-1} n(n-k), \; for \; 2 \leq r \leq n-2 \; and \; n > 3 \\ \prod_{k=1}^{r-2} n(n-k), \; for \; r = n-1, \; and \; n > 3. \end{cases}$$

Consequently in r^{th} *generation the* j^{th} *vertex on each copy of* G_r *is adjacent to every vertex of the* j^{th} *copy of* G_{r+1}*, where* $1 \leq j \leq p_r$, $2 \leq r \leq n-1$ *is known as the Decreasing* n*-corona product of order 1 and is denoted by* $G_{r-1} \circ_{-1}^{n} G_r, 2 \leq r \leq n$.

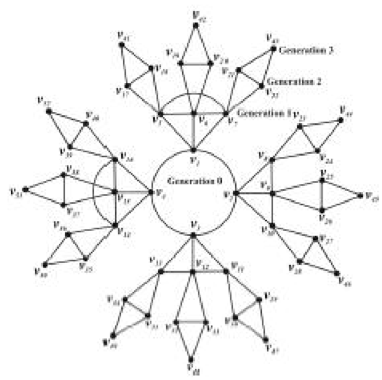

Figure 2: 4-Corona product graph $G_{r-1} \circ_{-1}^{4} G_r, 2 \leq r \leq 4$.

Lemma 2.6. *[19] Let* $\lambda = (\lambda_i)$ *be a real non null* $n \times 1$ *vectors,* $m = \frac{\lambda' e}{n}$, $s^2 = \frac{\lambda' C \lambda}{n}$ *where* e *is the* $n \times 1$ *vector of ones, the centering matrix* $C = I - \frac{ee'}{n}$, e' *is the transpose of* e *and* $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ *then* $\lambda_n \leq m - \frac{s}{(n-1)^{\frac{1}{2}}} \leq m + \frac{s}{(n-1)^{\frac{1}{2}}} \leq \lambda_1$.

Theorem 2.7. *[8] For a graph* G *on* n *vertices and having* m *edges, it is shown that*

$$E(G) \leq \frac{2m}{n} + \sqrt{(n-1)\left[2m - \left(\frac{2m}{n}\right)^2\right]}$$

while if G *is* k*-regular,* $E(G) \leq k + \sqrt{k(n-1)(n-k)}$.

Theorem 2.8. *[9] For each $\epsilon > 0$, there exist infinitely many n for each of which there exists a k-regular graph G of order n with $k < n-1$ and* $\frac{E(G)}{k+\sqrt{k(n-1)(n-k)}} < \epsilon$.

Theorem 2.9. *[10] If $m \leq n$ and A is an $m \times n$ nonnegative matrix with maximum entry α, then* $\varepsilon(A) \leq \alpha \frac{(m+\sqrt{m})\sqrt{n}}{2}$.

Theorem 2.10. *[13] Let G be a graph with $n \geq 3$ vertices and m edges. If $n^2 \geq 4m$, then* $\varepsilon(G) \leq \frac{2m}{n} + \sqrt{\frac{2m}{n}} + \sqrt{(n-2)(2m - \frac{2m}{n} - \frac{4m^2}{n^2})}$. *Equality holds iff G is $\frac{n}{2}K_2$.*

In the following section, basic reproduction number, graph energy and its bound for n times corona of graphs are determined.

3. Energy of Increasing and Decreasing n-corona product of order 1 for complete graph

Theorem 3.1. *Let $K_{r-1} \circ_{+1}^{n} K_r, 2 \leq r \leq n$ be increasing n-corona product of order 1 of complete graphs then its energy $E \leq 4n! + 4(n-1)! - 4$.*

Proof. The increasing n-corona product of order 1 of complete graphs $G = K_{r-1} \circ_{+1}^{n} K_r, 2 \leq r \leq n$ consists of p vertices and q edges where $p = 1 + \sum_{s=2}^{n} \left[2 \prod_{m=1}^{s-1} (1+m) \right]$ and $q = 6 + \sum_{s=3}^{n} \left[2 \prod_{m=1}^{s-2} (1+m). \frac{s(s-1)}{2} \right] + \sum_{s=3}^{n} \left[2 \prod_{m=1}^{s-1} (1+m) \right]$ respectively. Its adjacency matrix is

$$A(G) = \begin{cases} 1, \ if \ i \text{ and } j \text{ are adjacent} \\ 0, \ if \ i \text{ and } j \text{ are non- adjacent.} \end{cases}$$

The adjacency situations of $G = K_{r-1} \circ_{+1}^{n} K_r$ are with 1's on
$i = 1, j = 2, 3, 4, 5$
$j = 1, i = 2, 3, 4, 5$
$i = 2, j = 3$
$i = 4, j = 5$
$i = 3, j = 2$
$i = 5, j = 4$
Let $\nu = 2 \prod_{m=1}^{n-2} (1+m)$, $\mu_0 = 1 + \sum_{s=1}^{n-2} \left[2 \prod_{m=1}^{s} (1+m) \right]$ and $\mu_1 = 1 + \sum_{s=1}^{n-3} \left[2 \prod_{m=1}^{s} (1+m) \right]$

For $1 \leq k \leq \nu$ and if $r = 2$,

$i = k+1, k(1+r)+3 \leq j \leq (k+1)(1+r)+2$

$j = k+1, k(1+r)+3 \leq i \leq (k+1)(1+r)+2$

for $3 \leq r \leq n-1$

$i = k + \mu_1$

$\mu_0 - (1+r) + k(1+r) + 1 \leq j \leq \mu_0 - (1+r) + (k+1)(1+r)$

Likewise if $j = k + \mu_1$

$\mu_0 - (1+r) + k(1+r) + 1 \leq i \leq \mu_0 - (1+r) + (k+1)(1+r)$

Also if $r = 2$

$k(1+r)+3 \leq i \leq (k+1)(1+r)+2,$

$k(1+r)+3 \leq j \leq (k+1)(1+r)+2$

for $3 \leq r \leq n-1$

$\mu_0 - (1+r) + k(1+r) + 1 \leq i \leq \mu_0 - (1+r) + (k+1)(1+r)$

$\mu_0 - (1+r) + k(1+r) + 1 \leq j \leq \mu_0 - (1+r) + (k+1)(1+r)$

The non-adjacency situations are with 0's on $i = j$ and elsewhere. Based on this situations the adjacency matrix of G is formulated as follows.

$$\begin{array}{c c}
 & \begin{array}{cccccccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & \cdots & v_{p-1} & v_p \end{array} \\
\begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ \vdots \\ v_{p-1} \\ v_p \end{array} &
\begin{pmatrix}
0 & 1 & 1 & 1 & 1 & 0 & 0 & \ldots & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 1 & 1 & \ldots & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & \ldots & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & \ldots & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots & 1 & 0
\end{pmatrix}
\end{array}$$

Set $det\,(A(G) - \lambda I) = 0$. The characteristic equation of this adjacency matrix with order p is of the form $(-\lambda)^p + tr(-\lambda)^{p-1} + \ldots + det(A) = 0$ which has exactly p roots. Therfore there should be p eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_p$. From lemma 2.6, $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_p$.

The largest eigenvalue λ_1 is the basic reproduction number. i.e., $R_0 = \lambda_1$

The energy $E = \sum_{i=1}^{p} |\lambda_i|$.

It is clear that for n-corona of complete graph $E_1 < E_2 < \ldots < E_p$.
By Cauchy Schwarz inequality

$$\left(\sum_{i=1}^{p} |\lambda_i|\right)^2 \leq \sum_{i=1}^{p} |1| \ \sum_{i=1}^{p} |\lambda_i|^2$$

$$\left(\sum_{i=2}^{p-1} |\lambda_i| - |\lambda_1| - |\lambda_p|\right)^2 \leq \left(\sum_{i=2}^{p-1} |1| - 2\right) \left(\sum_{i=2}^{p-1} |\lambda_i|^2 - |\lambda_1|^2 - |\lambda_p|^2\right)$$

$$\sum_{i=2}^{p-1} |\lambda_i| \leq |\lambda_1| + |\lambda_p| + \sqrt{(p-2)\left(\sum_{i=2}^{p-1} |\lambda_i|^2 - |\lambda_1|^2 - |\lambda_p|^2\right)}$$

$$E(G) \leq \left[|\lambda_1| + |\lambda_p| + \sqrt{(p-2)\left(\sum_{i=2}^{p-1} |\lambda_i|^2 - |\lambda_1|^2 - |\lambda_p|^2\right)}\ \right]$$

Now let $|\lambda_1| = x$ and $|\lambda_p| = y$

$$E(G) \leq \left[x + y + \sqrt{(p-2)\left(\sum_{i=2}^{p-1} |\lambda_i|^2 - x^2 - y^2\right)}\right]$$

Construct the function

$$f(x,y) = \frac{1}{\sqrt{p}} \left[x + y + \sqrt{(p-2)\left\{(4n! + 4(n-1)! - 4)^2 - x^2 - y^2\right\}}\ \right]$$

The first and second order derivatives of the function $f(x,y)$ are

$$f_x = \frac{1}{\sqrt{p}} - \frac{x(p-2)}{\sqrt{p}\sqrt{(p-2)\left\{(4n! + 4(n-1)! - 4)^2 - x^2 - y^2\right\}}},$$

$$f_y = \frac{1}{\sqrt{p}} - \frac{y(p-2)}{\sqrt{p}\sqrt{(p-2)\left\{(4n! + 4(n-1)! - 4)^2 - x^2 - y^2\right\}}},$$

$$f_{xx} = -\frac{\sqrt{p-2}\left[\{4n! + 4(n-1)! - 4\}^2 - y^2\right]}{\sqrt{p}\left[\{4n! + 4(n-1)! - 4\}^2 - x^2 - y^2\right]^{\frac{3}{2}}},$$

$$f_{yy} = -\frac{\sqrt{p-2}\left[\{4n! + 4(n-1)! - 4\}^2 - x^2\right]}{\sqrt{p}\left[\{4n! + 4(n-1)! - 4\}^2 - x^2 - y^2\right]^{\frac{3}{2}}},$$

and

$$f_{xy} = f_{yx} = -\frac{xy\sqrt{p-2}}{\sqrt{p}\left[\{4n! + 4(n-1)! - 4\}^2 - x^2 - y^2\right]^{\frac{3}{2}}}.$$

The Hessian matrix is defined by

$$\Delta^2 f(x,y) = \begin{cases} \frac{\sqrt{p-2}\left[\{4n!+4(n-1)!-4\}^2-y^2\right]}{\sqrt{p}\left[\{4n!+4(n-1)!-4\}^2-x^2-y^2\right]^{\frac{3}{2}}} \\ -\frac{xy\sqrt{p-2}}{\sqrt{p}\left[\{4n!+4(n-1)!-4\}^2-x^2-y^2\right]^{\frac{3}{2}}} \\ -\frac{xy\sqrt{p-2}}{\sqrt{p}\left[\{4n!+4(n-1)!-4\}^2-x^2-y^2\right]^{\frac{3}{2}}} \\ -\frac{\sqrt{p-2}\left[\{4n!+4(n-1)!-4\}^2-x^2\right]}{\sqrt{p}\left[\{4n!+4(n-1)!-4\}^2-x^2-y^2\right]^{\frac{3}{2}}} \end{cases}$$

Equating the partial derivatives f_x and f_y of the function to zero.
$1-\frac{x(p-2)}{\sqrt{(p-3)\{(4n!+4(n-1)!-4)^2-x^2-y^2\}}} = 0 \Rightarrow x^2(p-1)+y^2 = \{4n! + 4(n-1)! - 4\}^2$
$1-\frac{y(p-2)}{\sqrt{(p-3)\{(4n!+4(n-1)!-4)^2-x^2-y^2\}}} = 0 \Rightarrow x^2+y^2(p-1) = \{4n! + 4(n-1)! - 4\}^2$
solving the above two equations

$$x = y = \pm\frac{1}{\sqrt{p}}\{4n! + 4(n-1)! - 4\}.$$

Therefore the critical points are $\left(\pm\frac{4n!+4(n-1)!-4}{\sqrt{p}}, \pm\frac{4n!+4(n-1)!-4}{\sqrt{p}}\right)$

The Hessian at the critical point is

$$\Delta^2 f(x,y) = \begin{cases} \frac{(p-1)}{(p-2)[4n!+4(n-1)!-4]} \\ -\frac{1}{(p-2)[4n!+4(n-1)!-4]} \\ -\frac{1}{(p-2)[4n!+4(n-1)!-4]} \\ -\frac{(p-1)}{(p-2)[4n!+4(n-1)!-4]} \end{cases}$$

which is a negative definite and makes a local maximum of the function.

So $f\left(\pm\frac{4n!+4(n-1)!-4}{\sqrt{p}}, \pm\frac{4n!+4(n-1)!-4}{\sqrt{p}}\right) \geq f(x,y)$ which implies

$$f(x,y) \leq 4n! + 4(n-1)! - 4$$

Thus it is successfully bounded above function with a point of local maximum at $\left(\frac{4n!+4(n-1)!-4}{\sqrt{p}}, \frac{4n!+4(n-1)!-4}{\sqrt{p}}\right)$.

In consequence $E(G) \leq 4n! + 4(n-1)! - 4$. $\square$

Illustration: The following table illustrates the energy bounds and basic reproduction number for increasing n-Corona of complete graph with order 1.

Graphs $K_{r-1} \circ_{+1}^{n} K_r$, $2 \leq r \leq n$	Vertices p	Edges q	Energy ε	Energy bound E	Dominant Eigenvalue R_0
$K_{r-1} \circ_{+1}^{3} K_r$, $2 \leq r \leq 3$,	17	30	27.5489	28	3.7637
$K_{r-1} \circ_{+1}^{4} K_r$, $2 \leq r \leq 4$	65	150	114.5435	116	4.9735
$K_{r-1} \circ_{+1}^{5} K_r$, $2 \leq r \leq 5$	305	870	562.9675	572	6.2860
$K_{r-1} \circ_{-1}^{6} K_r$, $2 \leq r \leq 6$	1745	5910	3307.9	3556	7.6078
$K_{r-1} \circ_{-1}^{7} K_r$, $2 \leq r \leq 7$	11825	46230	22762	23036	8.9137

Table 3.1: E and R_0 of increasing n-Corona of complete graph with order 1.

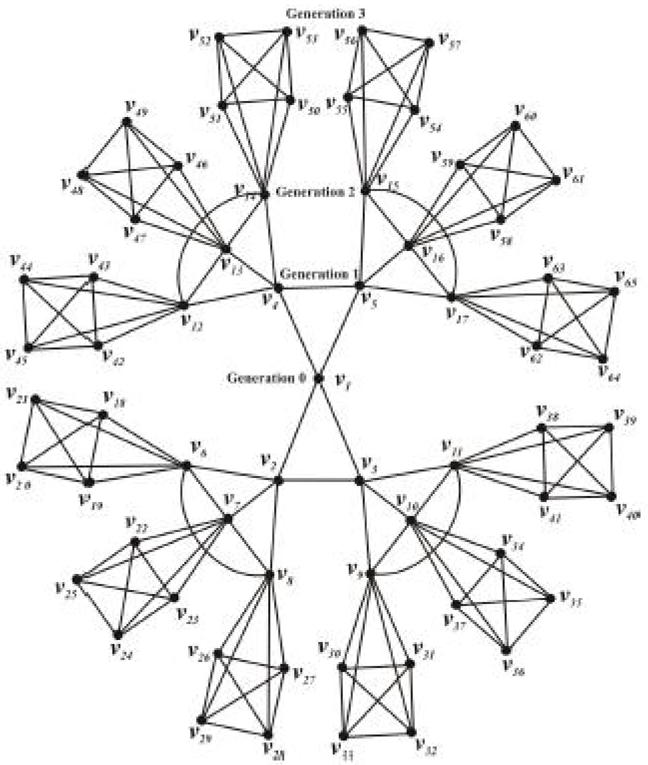

Figure 3: Increasing 4-Corona of complete graph with order 1.

Theorem 3.2. *Let $K_{r-1} \circ_{-1}^{n} K_r, 2 \leq r \leq n$ be decreasing n-corona product of order 1 of complete graphs then its energy $E \leq 4n! - 2(n-1)!$.*

Proof. The decreasing n-corona product of order 1 of complete graphs $G = K_{r-1} \circ_{-1}^{n} K_r, 2 \leq r \leq n$ consists of p vertices and q edges respectively as $p = n + \sum_{s=3}^{n} \left[n \prod_{m=1}^{s-2} (n-m) \right] + U$ while $U = \begin{cases} \quad , \; if \; n = 3 \\ n \prod_{m=1}^{n-3} (n-m), \; for \; n > 3. \end{cases}$

and $q = \sum_{s=1}^{n-2} \left[n \prod_{m=1}^{s} \frac{(n-m)}{2} + n \prod_{m=1}^{s} (n-m) \right] + n \prod_{m=1}^{n-2} \frac{(n-m)}{2} + n \prod_{m=1}^{n-2} (n-m)$.

Its adjacency matrix is

$$A(G) = \begin{cases} 1, & if\ i \text{ and } j \text{ are adjacent} \\ 0, & if\ i \text{ and } j \text{ are non- adjacent.} \end{cases}$$

The adjacency of $G = K_{r-1} \circ_{-1}^{n} K_r$ are with 1's on
$i = 1, j = 2, 3, \ldots, n$
$j = 1, i = 2, 3, \ldots, n$
$2 \leq i \leq n, 2 \leq j \leq n$ and $i \neq j$
if $i \leq n, (i-1)(n-1) + 1 + n \leq j \leq i(n-1) + n$
if $j \leq n, (j-1)(n-1) + 1 + n \leq i \leq j(n-1) + n$
for $1 \leq l \leq n$,
$n + (l-1)(n-1) + 1 \leq i \leq l(n-1) + n$,
$n + (l-1)(n-1) + 1 \leq j \leq l(n-1) + n$
for $1 \leq m \leq n \prod_{r=1}^{n-3} (n-r)$,
if $r = 2, n + n(n-1) + (m-1)(n-2) + 1 \leq i \leq n + n(n-1) + m(n-2)$,
$n + n(n-1) + (m-1)(n-2) + 1 \leq j \leq n + n(n-1) + m(n-2)$

if $r > 2$,
$n + n(n-1) + n \prod_{r=1}^{n-3} (n-r) + (m-1)(n-r) + 1 \leq i \leq n + n(n-1) + n \prod_{r=1}^{n-3} (n-r) + m(n-r)$,
$i \leq n + \sum_{s=1}^{n-2} \left[\prod_{r=1}^{s} n(n-r) \right]$
$n + n(n-1) + n \prod_{r=1}^{n-3} (n-r) + (m-1)(n-r) + 1 \leq j \leq n + n(n-1) + n \prod_{r=1}^{n-3} (n-r) + m(n-r)$,
$j \leq n + \sum_{s=1}^{n-2} \left[\prod_{r=1}^{s} n(n-r) \right]$

When $n > 4$ and $r = 2$,
$i = n+m, n+n \prod_{r=1}^{n-3} (n-r) + (m-1)(n-r) + 1 \leq j \leq n+n \prod_{r=1}^{n-3} (n-r) + m(n-r)$
$j = n+m, n+n \prod_{r=1}^{n-3} (n-r) + (m-1)(n-r) + 1 \leq i \leq n+n \prod_{r=1}^{n-3} (n-r) + m(n-r)$

Also if $r > 2, i = n + n \prod_{r=1}^{n-4} (n-r) + m$,
$n + n(n-1) + n \prod_{r=1}^{n-3} (n-r) + (m-1)(n-r) + 1 \leq j \leq n + n(n-1) + n \prod_{r=1}^{n-3} (n-r) + m(n-r)$
and $j = n + n \prod_{r=1}^{n-4} (n-r) + m$,
$n + n(n-1) + n \prod_{r=1}^{n-3} (n-r) + (m-1)(n-r) + 1 \leq i \leq n + n(n-1) + n \prod_{r=1}^{n-3} (n-r) + m(n-r)$

For $1 \leq x \leq n \prod_{r=1}^{n-3} (n-r)$ and $j = n + \sum_{s=1}^{n-2} \left[\prod_{r=1}^{s} n(n-r) \right] + x$, it can have both
$i = n + \sum_{s=1}^{n-2} \left[n \prod_{r=1}^{s} (n-r) \right] - n \prod_{r=1}^{n-2} (n-r) + 2x - 1$
and $i = n + \sum_{s=1}^{n-2} \left[n \prod_{r=1}^{s} (n-r) \right] - n \prod_{r=1}^{n-2} (n-r) + 2x$.
Similarly for $i = n + \sum_{s=1}^{n-2} \left[\prod_{r=1}^{s} n(n-r) \right] + x$,
$j = n + \sum_{s=1}^{n-2} \left[n \prod_{r=1}^{s} (n-r) \right] - n \prod_{r=1}^{n-2} (n-r) + 2x - 1$
and $j = n + \sum_{s=1}^{n-2} \left[n \prod_{r=1}^{s} (n-r) \right] - n \prod_{r=1}^{n-2} (n-r) + 2x$.

The non-adjacency situations are with 0's on $i = j$ and elsewhere. Based on this situations the adjacency matrix of G is formulated as follows.

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_n \\ v_{n+1} \\ v_{n+2} \\ \vdots \\ v_{p-1} \\ v_p \end{array}
\begin{array}{c}
\begin{array}{cccccccccc} v_1 & v_2 & v_3 & \cdots & v_n & v_{n+1} & v_{n+2} & \cdots & v_{p-1} & v_p \end{array} \\
\begin{pmatrix}
0 & 1 & 1 & \dots & 1 & 1 & 1 & \dots & 0 & 0 \\
1 & 0 & 1 & \dots & 1 & 0 & 0 & \dots & 0 & 0 \\
1 & 1 & 0 & \dots & 1 & 0 & 0 & \dots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1 & 1 & 1 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\
1 & 0 & 0 & \dots & 0 & 0 & 1 & \dots & 0 & 0 \\
1 & 0 & 0 & \dots & 0 & 1 & 0 & \dots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\
0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0
\end{pmatrix}
\end{array}$$

Set $det\,(A(G) - \lambda I) = 0$. The characteristic equation of this adjacency matrix with order p is of the form $(-\lambda)^p + tr(-\lambda)^{p-1} + \ldots + det(A) = 0$ which has exactly p roots. Therfore there should be p eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_p$. By lemma 2.6, $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_p$.

Here the dominant eigenvalue λ_1 is identified as the basic reproduction number. i.e., $R_0 = \lambda_1$

The energy $E = \sum_{i=1}^{p} |\lambda_i|$.

It is clear that for n-corona of complete graph $E_1 < E_2 < \ldots < E_p$.

By applying Cauchy Schwarz inequality as in the proof of theorem 3.1, it follows that

$$E(G) \leq \left[|\lambda_1| + |\lambda_p| + \sqrt{(p-2)\left(\sum_{i=2}^{p-1} |\lambda_i|^2 - |\lambda_1|^2 - |\lambda_p|^2 \right)} \right]$$

Now let $|\lambda_1| = x$ and $|\lambda_p| = y$

$$E(G) \leq \left[x + y + \sqrt{(p-2)\left(\sum_{i=2}^{p-1} |\lambda_i|^2 - x^2 - y^2 \right)} \right]$$

Consider the function

$$(3.1) f(x,y) = \frac{1}{\sqrt{p}} \left[x + y + \sqrt{(p-2)\left\{ (4n! - 2(n-1)!)^2 - x^2 - y^2 \right\}} \right]$$

Differentiation of the function $f(x,y)$ partially with respect to x and y upto second order are

$$f_x = \frac{1}{\sqrt{p}} - \frac{x(p-2)}{\sqrt{p}\sqrt{(p-2)\left\{(4n!-2(n-1)!)^2 - x^2 - y^2\right\}}},$$

$$f_y = \frac{1}{\sqrt{p}} - \frac{y(p-2)}{\sqrt{p}\sqrt{(p-2)\left\{(4n!-2(n-1)!)^2 - x^2 - y^2\right\}}},$$

$$f_{xx} = -\frac{\sqrt{p-2}\left[\{4n!-2(n-1)!\}^2 - y^2\right]}{\sqrt{p}\left[\{4n!-2(n-1)!\}^2 - x^2 - y^2\right]^{\frac{3}{2}}},$$

$$f_{yy} = -\frac{\sqrt{p-2}\left[\{4n!-2(n-1)!\}^2 - x^2\right]}{\sqrt{p}\left[\{4n!-2(n-1)!\}^2 - x^2 - y^2\right]^{\frac{3}{2}}},$$

and

$$f_{xy} = -\frac{xy\sqrt{p-2}}{\sqrt{p}\left[\{4n!-2(n-1)!\}^2 - x^2 - y^2\right]^{\frac{3}{2}}}.$$

For the detection of maxima or minima equate the partial derivatives f_x and f_y of the function to zero.

$$1 - \frac{x(p-2)}{\sqrt{(p-3)\left\{(4n!-2(n-1)!)^2 - x^2 - y^2\right\}}} = 0$$

$$\Rightarrow x^2(p-1) + y^2 = \{4n! - 2(n-1)!\}^2$$

$$1 - \frac{y(p-2)}{\sqrt{(p-3)\left\{(4n!-2(n-1)!)^2 - x^2 - y^2\right\}}} = 0$$

$$\Rightarrow x^2 + y^2(p-1) = \{4n! - 2(n-1)!\}^2$$

solving the above two equations, the stationary points are

$$x = y = \pm\frac{1}{\sqrt{p}}\{4n! - 2(n-1)!\}.$$

At this point the values are

$$f_{xx} = f_{yy} = -\frac{p-1}{(p-2)(4n!-2(n-1)!)} \leq 0,$$

$$f_{xy} = -\frac{1}{(p-2)\,(4n! - 2(n-1)!)} \leq 0.$$

But

$$\Delta = f_{xx}.f_{yy} - (f_{xy})^2 = \frac{p}{(p-2)\,(4n! - 2(n-1)!)^2} \geq 0.$$

Hence $f(x,y)$ attains the maximum value at $x = y = \frac{1}{\sqrt{p}}\,(4n! - 2(n-1)!)$.

The maximum value of the function is obtained by switching the values of x and y in (1).

So $f\left(\pm\frac{4n!+2(n-1)!}{\sqrt{p}}, \pm\frac{4n!+2(n-1)!}{\sqrt{p}}\right) = 4n! - 2(n-1)!$ which implies

$$E(G) \leq 4n! + 2(n-1)!$$

Thus it proves to be the upper bounded. □

Illustration:

The following table shows the energy bounds and basic reproduction number for decreasing n-Corona of complete graph with order 1.

Graphs $K_{n-r+2} \circ_{-1}^{n} K_{n-r+1}$, $2 \leq r \leq n$	Vertices p	Edges q	Energy ε	Energy bound E	Dominant Eigenvalue R_0
$K_{3-r+2} \circ_{-1}^{3} K_{3-r+1}$, $2 \leq r \leq 3$,	12	18	18.1290	20	3.2361
$K_{4-r+2} \circ_{-1}^{4} K_{4-r+1}$, $2 \leq r \leq 4$	52	90	83.2330	84	4.5616
$K_{5-r+2} \circ_{-1}^{5} K_{5-r+1}$, $2 \leq r \leq 5$	265	480	431.5432	432	5.9304
$K_{6-r+2} \circ_{-1}^{6} K_{6-r+1}$, $2 \leq r \leq 6$	1596	2925	2609	2640	7.2808
$K_{7-r+2} \circ_{-1}^{7} K_{7-r+1}$, $2 \leq r \leq 7$	11179	20538	18287	18720	8.6073

Table 3.2: E and R_0 of decreasing n-Corona of complete graph with order 1.

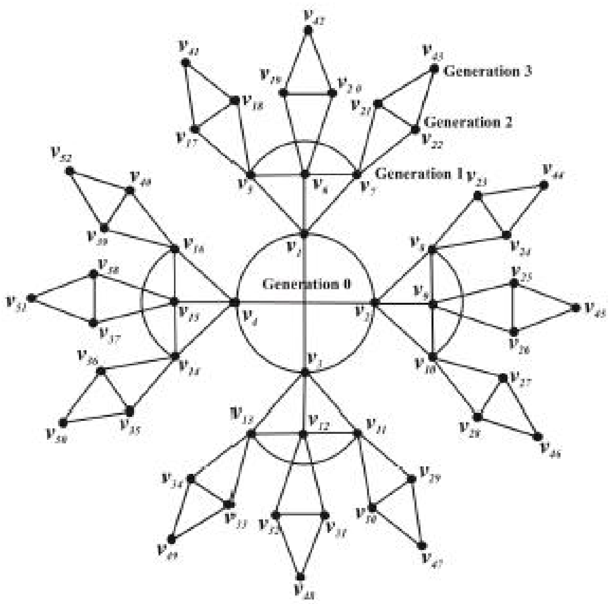

Figure 4: Diminishing 4-Corona of complete graph with order 1.

4. Conclusion

The phenomena of corona graphs is observed as a unit of some basic graph being duplicated and joined at every vertex of the existing graph. Such duplication of random graphs can be extended n times and joined successively to form n-corona product of graphs and this area of research is wide open. In this paper, the n-corona product of complete graphs with increasing and decreasing order 1 was employed. It is hard to find out the bounds of these graphs due to mass and complexity of data. However, the matrix formation of such graphs helps in determining the energy and basic reproduction number, which has extensive applications in the field of mathematics, chemistry and biological sciences.

References

[1] P. K. Nayak, Y. K. Rao and T. C. Panda, "Calculation of basic reproduction number by graph reduction method and stability analysis in SEIQR e-epidemic model in computer network", *Journal of Engineering and Applied Sciences*, vol. 12, no. 23, pp. 7332-7338, 2017. [On line]. Available: https://bit.ly/3Bf0AmL

[2] M.J. Wanjau, R. Titus, C. Isaac, "Mathematical Modeling of Infectious Disease Transmission Dynamics in a Metapopulation", *IOSR Journal of Mathematics*, vol. 15, no. 6, pp. 18-36, 2019. [On line]. Available: https://bit.ly/3yVWVYi

[3] R. Frucht and F. Harary, "On the corona of two graphs", *Aequationes Mathematicae*, vol. 4, pp. 322-325, 1970. doi: 10.1007/BF01844162

[4] M. Tavakoli, F. Rahbarnia, A. R. Ashrafi, "Studying the corona product of graphs under some graph invariants", *Transactions on Combinatorics*, vol. 3, no. 3, pp. 43-49, 2014. doi: 10.22108/toc.2014.5542

[5] K. Kaliraj, V. J. Veninstine, V. J. Vernold, "Equitable coloring on corona graph of graphs", *Journal of Combinatorial Mathematics and Combinatorial Computing*, vol. 81, pp. 191-197, 2012.

[6] H. Furmańczyk and M. Kubale, "Equitable coloring of corona products of cubic graphs is harder than ordinary coloring", *Ars Mathematica Contemporanea*, vol. 10, no. 2, pp. 333-347, 2016. doi: 10.26493/1855-3974.687.99b

[7] H. Furmańczyk and M. Kubale, "Equitable colorings on corona multiproducts of graphs", *Discussiones Mathematicae Graph Theory*, vol. 37, no. 4, pp. 1079-1094, 2017. doi: 10.7151/dmgt.1992

[8] I. Gutman, "The energy of a graph", *Ber. Math.-Statist. Sekt. Forschungsz. Graz*, vol. 103, pp. 1-22, 1978.

[9] R. Balakrishnan, "The energy of a graph", *Linear Algebra and its Applications*, vol. 387, pp. 287-295, 2004. doi: 10.1016/j.laa.2004.02.038

[10] V. Nikiforov, “The energy of graphs and matrices”, *Journal of Mathematical Analysis and Applications,* vol. 326, pp. 1472-1475, 2007. doi: 10.1016/j.jmaa.2006.03.072

[11] H. Liu, M. Lu and F. Tian, “Some upper bounds for the energy of graphs”, *Journal of Mathematical Chemistry*, vol. 41, no. 1, pp. 45-57, 2007. doi: 10.1007/s10910-006-9183-9

[12] K. C. Das, S. A. Mojallal, and I. Gutman, “Improving the McCelland inequality for energy”, *MATCH Communications in Mathematical and in Computer Chemistry*, vol. 70, pp. 663-668, 2013.

[13] G. Sridhara, M. R. Rajesh Kanna, R. Jagadeesh, and I. N. Cangul, “Improved McClelland and Koolen-Moulton bounds for the energy of graphs”, *Scientia Magna*, vol. 13, no. 1, pp. 1-10, 2018.

[14] O. Diekmann, J. A. P. Heesterbeek and J. A. J. Metz, “On the definition and the computation of the basic reproduction ratio R_0 in models for infectious diseases in heterogeneous populations”, *Journal of Mathematical Biology*, Vol. 28, pp. 365-382, 1990. doi: 10.1007/BF00178324

[15] K. Dietz, “The estimation of the basic reproduction number for infectious diseases”, *Statistical Methods in Medical Research,* vol 2: pp. 23-41, 1993. doi: 10.1177/096228029300200103

[16] S. Barik, S. Pati and B. K. Sarma, “The spectrum of the corona of two graphs”, *SIAM Journal on Discrete Mathematics,* vol. 21, pp. 47-56, 2007. doi: 10.1137/050624029

[17] C. McLeman and E.McNicholas, “Spectra of coronae”, *Linear Algebra and its Applications,* vol. 435, pp. 998-1007, 2011. doi: 10.1016/j.laa.2011.02.007

[18] O. Diekmann, J. A. P. Heesterbeek, and M. G. Roberts. “The construction of the next generation matrices for compartmental epidemic models”, *Journal of Royal Society Interface*, vol. 7, pp. 873-885, 2010. doi: 10.1098/rsif.2009.0386

[19] H. Wolkowicz and G. P. H. Styan, “Bounds for eigenvalues using traces”, *Linear Algebra and its Applications,* vol. 29, pp. 471-506, 1980. doi: 10.1016/0024-3795(80)90258-X

[20] J.V. Vivik and P. Xavier, "Energy of 2-Corona of Graphs, Mathematical Modeling", In: *Advances in Intelligent Systems and Computing. Computational Intelligence Techniques and Renewable Energy*, vol. 1405, pp. 67-75. Singapore: Springer, 2022. doi: 10.1007/978-981-16-5952-2_7

Veninstine Vivik J.
Department of Mathematics,
Karunya Institute of Technology and Sciences,
Coimbatore 641 114,
Tamil Nadu,
India
e-mail: vivikjose@gmail.com