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Symmetric bi-derivation on bitonic algebras

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Abstract

In this study, we give definition of symmetric bi-derivation on bitonic algebras and investigate its properties.

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1. Introduction

The notion of the symmetric bi-derivation was defined by Maksa in [4]. Firstly, many investigaters studied the symmetric bi-derivation in rings and near rings. In [2], Çeven applied to notion of the symmetric bi-derivation in ring and near ring theory to lattices. In [3], Ilbira, Firat and Jun introduced the notion of the left-right (resp. right-left) symmetric bi-derivation of BCI algebras and investigated its properties.

In [1], Özbal and Yon defined bitonic algebras. Bitonic algebra is a generalization of dual BCC-algebras. They introduced the notion of (r,l)-derivations and (l,r)-derivations on the bitonic algebras.

In this study, we give the definition of symmetric bi-derivation on bitonic algebras and investigate its properties.

2. Preliminaries

Definition 1. *[1] Let B be a set, $1 \in B$ and $*$ be a binary operation on B. If the following axioms hold then algebraic system $(B, *, 1)$ is called bitonic algebra.*

(B1) For any $x \in B$, $x * 1 = 1$,
(B2) For any $x \in B$, $1 * y = y$,
(B3) For any $x, y \in B$, $x * y = 1$ and $y * x = 1$ implies $x = y$,
(B4) For any $x, y, z \in B$, $x * y = 1$ implies $(z * x) * (z * y) = 1$ and $(y * z) * (x * z) = 1$.

Lemma 1. *[1] Let $(B, *, 1)$ be a bitonic algebra. For any $x, y, z \in B$, we have the following statements*

(1) $x * x = 1$,
(2) $x * y = y * z = 1$ implies $x * z = 1$,
(3) $x * (y * x) = 1$.

Let $(B, *, 1)$ be a bitonic algebra and define a binary relation "$\leq$"on B by

$$x \leq y \Leftrightarrow x * y = 1, \text{ for any } x, y \in B,$$

then $\leq$ is a partial order on B from (B3) and Lemma 1. Hence $(B, \leq)$ is a poset and 1 is the greatest element in B from (B3).

Lemma 2. *[1] Let $(B, *, 1)$ be a bitonic algebra. Then for any $x, y, z \in B$,*

(1) $x \leq y$ implies $z * x \leq z * y$ and $y * z \leq x * z$,
(2) $x \leq y * x$.

Example 1. *[1]The set $A = \{1, a, b, c, d\}$ is a bitonic algebra by the following table:*

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	c	d
b	1	a	1	c	d
c	1	1	1	1	d
Γ	1	1	1	c	1

Let $(B, *, 1)$ be a bitonic algebra. For every $x, y \in B$, the operation "$\vee$"on B is defined by $x \vee y = (x * y) * y$.

Lemma 3. *[1] Let $(B, *, 1)$ be a bitonic algebra. We have the following statements:*

(1) For any $x, y \in B$, $y \leq x \vee y$
(2) For any $x, y \in B$, $x \leq y$ implies $x \vee y = y$,
(3) For any $x \in B$, $1 \vee x = 1$ and $x \vee 1 = 1$.

Definition 2. *Let B be a bitonic algebra and $\Gamma : B \times B \to B$ mapping. We say that Γ is a symmetric mapping if $\Gamma(x, y) = \Gamma(y, x)$ for all $x, y \in B$.*

Definition 3. *Let B be a bitonic algebra. A mapping $\gamma : B \to B$ defined by $\gamma(x) = \Gamma(x, x)$ is called trace of Γ, where $\Gamma : B \times B \to B$ is a symmetric mapping.*

3. The symmetric bi-derivations on bitonic algebras

Definition 4. *Let B be a bitonic algebra and $\Gamma : B \times B \to B$ be a symmetric mapping. For every $x, y, z \in B$,*

(i) Γ is called (l,r)-symmetric bi-derivation if

$$\Gamma(x * y, z) = (\Gamma(x, z) * y) \vee (x * \Gamma(y, z)),$$

(ii) Γ is called (r,l)-symmetric bi-derivation if

$$\Gamma(x * y, z) = (x * \Gamma(y, z)) \vee (\Gamma(x, z) * y),$$

(*iii*) Γ is called symmetric bi-derivation if Γ are both (r,l) and (l,r)-symmetric bi-derivation.

Example 2. *Let $A = \{1, a, b, c, d\}$ be a bitonic algebra in Example 1. If we define a map $\Gamma : A \times A \to A$ by*

$$\Gamma(x, y) = \left\{ \begin{array}{ll} b, & x = y = b \\ 1, & otherwise \end{array} \right\}$$

then, Γ is a (l,r)-symmetric bi-derivation.

Example 3. *Let $B = \{1, x, y, 0\}$ be a set. If we define a binary operation $*$ on B by the following table:*

$*$	1	x	y	0
1	1	x	y	0
x	1	1	y	y
y	1	x	1	0
0	1	1	1	1

Then $(B, *, 1)$ is a bitonic algebra. If we define a map $\Gamma : B \times B \to B$ by

$$\Gamma(a, b) = \left\{ \begin{array}{ll} x, & a = b = x \\ 0, & a = b = 0 \\ 1, & \text{otherwise} \end{array} \right\}$$

then, Γ is a (r,l)-symmetric bi-derivation.

Lemma 4. *Let B be a bitonic algebra, $\Gamma : B \times B \to B$ be a (r,l)-symmetric bi-derivation and γ be a trace of Γ. For all $x \in B$,*

(1) $\gamma(1) = 1$,
(2) $\Gamma(1, x) = 1$,
(3) $\gamma(x) = \gamma(x) \vee x$,
(4) $x \leq \gamma(x)$,
(5) $\gamma(x) = x \vee \gamma(x)$.

Proof. (1) $\gamma(1) = \Gamma(1,1) = \Gamma(1*1,1) = (1*\Gamma(1,1)) \vee (\Gamma(1,1)*1) = \Gamma(1,1) \vee 1 = 1$.

(2) $\Gamma(1,x) = \Gamma(1*1,x) = (1*\Gamma(1,x)) \vee (\Gamma(1,x)*1) = \Gamma(1,x) \vee 1 = 1$, for all $x \in B$.

(3) For all $x \in B$,

$\gamma(x) = \Gamma(x,x) = \Gamma(1*x,x) = (1*\Gamma(x,x)) \vee (\Gamma(1,x)*x) = \gamma(x) \vee x$.

(4) From (3), we get $\gamma(x) = \gamma(x) \vee x$. From Lemma 3(1), we have $x \leq \gamma(x) \vee x = \gamma(x)$.

(5) From (4), we get $x \leq \gamma(x)$. From Lemma 3(2), we have $x \vee \gamma(x) = \gamma(x)$ □

If Γ be a (r,l)-symmetric bi-derivation on B with trace γ, then we get $\gamma(x) = \gamma(x) \vee x = x \vee \gamma(x)$ from Lemma 4(3) and (5).

Lemma 5. *Let B be a bitonic algebra and γ be a trace of Γ where Γ is a (r,l)-symmetric bi-derivation on B. Then for all $x, y \in B$, $\gamma(x)*y \leq x*\gamma(y)$.*

Proof. Let Γ be a (r,l)-symmetric bi-derivation on B and $x, y \in B$. From Lemma 4(4), we get $x \leq \gamma(x)$ and $y \leq \gamma(y)$. We obtain that $\gamma(x)*y \leq x*y \leq x*\gamma(y)$ from Lemma 2(1). Thus, we have $\gamma(x)*y \leq x*\gamma(y)$ for all $x, y \in B$. □

Lemma 6. *Let B be a bitonic algebra and γ be a trace of Γ where Γ is a (l,r)-symmetric bi-derivation on B. For all $x \in B$,*

(1) $\gamma(1) = 1$,
(2) $\Gamma(1,x) = 1$,
(3) $\gamma(x) = x \vee \gamma(x)$.

Proof. (1) $\gamma(1) = \Gamma(1,1) = \Gamma(1*1,1) = (\Gamma(1,1)*1) \vee (1*\Gamma(1,1)) = 1$.

(2) For all $x \in B$,

$\Gamma(1,x) = \Gamma(1*1,x) = (\Gamma(1,x)*1) \vee (1*\Gamma(1,x)) = 1$.

(3) $\gamma(x) = \Gamma(x,x) = \Gamma(1*x,x) = (\Gamma(1,x)*x) \vee (1*\Gamma(x,x)) = x \vee \gamma(x)$.
□

Proposition 1. *Let $(B,*,1)$ be a bitonic algebra and $\Gamma : B \times B \to B$ be a symmetric mapping. Then we have the following*

(i) If Γ be a (l,r)-symmetric bi-derivation,

$$\Gamma(x,y) = x \vee \Gamma(x,y) \text{ for all } x, y \in B.$$

(*ii*) If Γ be a (r,l)-symmetric bi-derivation,

$$\Gamma(x,y) = \Gamma(x,y) \vee x.$$

Proof. (*i*) Suppose that Γ be a (l,r)-symmetric bi-derivation. For all $x, y \in B$,

$$\Gamma(x,y) = \Gamma(1 * x, y) = (\Gamma(1,y) * x) \vee (1 * \Gamma(x,y)) = x \vee \Gamma(x,y),$$

since $\Gamma(1,y) = 1$ and $1 * x = x$.

(*ii*) Let Γ be a (r,l)-symmetric bi-derivation and $x, y \in B$. Hence,

$$\Gamma(x,y) = \Gamma(1 * x, y) = (1 * \Gamma(x,y)) \vee (\Gamma(1,y) * x) = \Gamma(x,y) \vee x.$$

□

Remark 1. *If Γ be a (r,l)-symmetric bi-derivation, then we have $x \leq \Gamma(x,y)$ for all $x, y \in B$.*

Proof. Since $y \leq x \vee y$ from Lemma 3(1), we get $x \leq \Gamma(x,y) \vee x = \Gamma(x,y)$. □

Theorem 1. *Let $(B, *, 1)$ be a bitonic algebra and $\Gamma : B \times B \to B$ be a mapping. If Γ is a symmetric bi-derivation, then we have $\Gamma(x * y, z) = x * \Gamma(y,z)$ for all $x, y, z \in B$.*

Proof. Let Γ be a symmetric bi-derivation and $x, y, z \in B$. Since $x \leq \Gamma(x,z)$ and $y \leq \Gamma(y,z)$, we have

$$\Gamma(x,z) * y \leq x * y \leq x * \Gamma(y,z)$$

from Lemma 2(1). Thus we have

$$\Gamma(x * y, z) = (\Gamma(x,z) * y) \vee (x * \Gamma(y,z)) = x * \Gamma(y,z)$$

from Lemma 3(2). □

Corollary 1. *If Γ is a symmetric bi-derivation on bitonic algebra B, then $\Gamma(x, y * z) = y * \Gamma(x,z)$ for all $x, y, z \in B$.*

Theorem 2. *Let B be a bitonic algebra and $\Gamma : B \times B \to B$ be a symmetric mapping. If for all $x, y, z \in B$, $\Gamma(x * y, z) = \Gamma(x,z) * y$, then Γ is a (r,l)-symmetric bi-derivation.*

Proof. Since $\Gamma(x,z)*x = \Gamma(x*x,z) = \Gamma(1,z) = 1$ for all $x,z \in B$, we get $\Gamma(x,z) \le x$. Similarly, we have $\Gamma(y,z) \le y$ for all $y,z \in B$. Therefore,

$$x*\Gamma(y,z) \le \Gamma(x,z)*\Gamma(y,z) \le \Gamma(x,z)*b,$$

and so $x*\Gamma(y,z) \le \Gamma(x,z)*y$. Since

$$\Gamma(x*y,z) = \Gamma(x,z)*y = (x*\Gamma(y,z)) \vee (\Gamma(x,z)*y),$$

Γ is a (r,l)-symmetric bi-derivation. □

Let B be a bitonic algebra and $D : B \times B \to B$ be any mapping on B. If for all $x \in B$, $D(x,x) \vee x = x \vee D(x,x)$, then D is called commutative. If for all $x \in B$, $(D(x,x)*x)*D(x,x) = D(x,x)$, then D is called implicative.

Lemma 7. *Let Γ be a (l,r)-symmetric bi -derivation on a bitonic algebra B. If Γ is commutative, then $x \le \gamma(x)$ for all $x \in B$.*

Proof. Let $x \in B$. From Lemma 6(3), we have $\gamma(x) = x \vee \gamma(x)$. Since Γ is commutative, $\gamma(x) \vee x = x \vee \gamma(x)$. From here, we get $\gamma(x) = \gamma(x) \vee x$. From Lemma 3(1), we get $x \le \gamma(x)$. □

Definition 5. *Let Γ be a symmetric bi -derivation on a bitonic algebra B and γ be a trace of Γ. The kernel of γ is defined by*

$$Ker\gamma := \{x \in B \mid \Gamma(x,x) = \gamma(x) = 1\}.$$

Lemma 8. *Let B be a bitonic algebra, Γ be a symmetric bi -derivation on B and γ be a trace of Γ. Then the folowing properties are hold:*

(1) For all $x \in B$, $x*\gamma(x) \in Ker\gamma$,
(2) $Ker\gamma = \{\gamma(x)*x \mid x \in B\}$.

Proof. (1) Let $x \in B$.

$$\begin{aligned} \Gamma(x*\Gamma(x)) &= \Gamma(x*\gamma(x), x*\gamma(x)) \\ &= x*\Gamma(\gamma(x), x*\gamma(x)) \\ &= x*\Gamma(x*\gamma(x), \gamma(x)) \\ &= x*(x*\gamma(\gamma(x))) = x*1 = 1. \end{aligned}$$

Thus $x*\gamma(x) \in Ker\gamma$.

(2) Let $x \in B$.

$$\begin{aligned}\gamma(\gamma(x) * x) &= \Gamma(\gamma(x) * x, \gamma(x) * x)\\ &= \gamma(x) * \Gamma(x, \gamma(x) * x)\\ &= \gamma(x) * \Gamma(\gamma(x) * x, x))\\ &= \gamma(x) * (\gamma(x) * \gamma(x)) = \gamma(x) * 1 = 1.\end{aligned}$$

Thus $\{\gamma(x) * x \mid x \in B\} \subseteq Ker\gamma$.

Let $x \in Ker\gamma$. Since $x = 1 * x = \gamma(x) * x$, we get $x \in \{\gamma(x) * x \mid x \in B\}$. Therefore, $Ker\gamma = \{\gamma(x) * x \mid x \in B\}$. □

Lemma 9. *Let γ be a trace of Γ where Γ is (r,l)-symmetric bi-derivation on a bitonic algebra B. If $Ker\gamma = \{1\}$, then γ is identity map.*

Proof. Let $x \in B$. From Lemma 8(2), we get $\gamma(x) * x \in Ker\gamma = \{1\}$. Then $\gamma(x) * x = 1$ and $\gamma(x) \leq x$. Moreover, we have $x \leq \gamma(x)$ for all $x \in B$ from Lemma 4(4). Therefore $\gamma(x) = x$. □

Theorem 3. *Let γ be a trace of Γ where Γ is a symmetric bi-derivation on a bitonic algebra B. If γ is implicative, then $\gamma^2 = \gamma$.*

Proof. Let $x \in B$. Thus we have

$$\begin{aligned}\gamma^2(x) &= \gamma(\gamma(x) \vee x) = \gamma((\gamma(x) * x) * x)\\ &= \Gamma((\gamma(x) * x) * x, (\gamma(x) * x) * x)\\ &= (\gamma(x) * x) * \Gamma((\gamma(x) * x) * x, x)\\ &= (\gamma(x) * x) * ((\gamma(x) * x) * \gamma(x))\\ &= (\gamma(x) * x) * \gamma(x)\\ &= \gamma(x).\end{aligned}$$

□

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