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Enumeration of spanning trees in prisms of some graphs

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Abstract

In graph theory, a prism over a graph G is the cartesian product of the graph G with P_2. The purpose of this work is to investigate the complexity of the prisms of some path and cycle-related graphs. In particular, we obtain simpler and more explicit formulas for the complexity of a special class of prisms of path-related graphs: fan graph, ladder graph, the composition $P_n[P_2]$ graph, and book graph. Moreover, we obtain straightforward formulas for the complexity of a special class of prisms of cycle-related graphs: wheel graph, gear graph, prism graph, $n-$crossed prism graph, mirror graph $M(C_n)$ of even cycle C_n, twisted prism, total graph $T(C_n)$ of the cycle C_n, the friendship graph, the flower graph, and planner sunflower graph. These closed formulas are deduced using some basic properties of block matrix, recurrence relation, eigenvalues of circulant matrices, and orthogonal polynomials.

Keywords: *Complexity; Recurrence relation; Prism over a graph; Chebyshev polynomials.*

2010 Mathematics Subject Classification: *05C05, 05C50, 05C90.*

1. Introduction

In graph theory, numerous graphs can generate from given ones by using graph operation. In this paper we are dealing with simple connected graphs. For a graph $G(V, E)$, let $|V(G)|$ be the cardinality of vertices $V(G)$ and $|E(G)|$ of $E(G)$. **The cartesian product** of two graphs G and H, denoted $G\Box H$, is the graph with vertex set $V(G\Box H) = V(G) \times V(H)$ and the edge $\{(u_1, v_1), (u_2, v_2)\}$ is presented in the product whenever $u_1u_2 \in E(G)$ and $v_1 = v_2$ or symmetrically $v_1v_2 \in E(H)$ and $u_1 = u_2$ [5]. The cartesian product of a graph G with P_2 is called **a prism** over G.

For a connected graph G, **a spanning tree** is a subgraph with $|V(G)|-1$ edges and contains no cycle. The number of spanning trees **(the complexity)** in G denoted by $\tau(G)$ is an attractive studied area, being interesting both for its own purpose and because it has many applications in various fields in engineering, bioinformatics, chemistry, and network reliability.

Let $A(G)$ be the adjacency matrix of G and $D(G)$ the diagonal matrix whose diagonal entries x_i are the vertex degrees of G. Let $L(G)$ be the Laplace matrix defined by $L(G) = D(G) - A(G)$, that is

$$L_{ij} = \begin{cases} x_i & \text{if i=j ;} \\ -1 & \text{if } i \neq j \text{ and } v_i \text{ is adjacent to } v_j; \\ 0 & \text{otherwise.} \end{cases}$$

Kirchhoff [14] presented the first method to calculate the number of spanning trees, that is Matrix tree theorem, which says that: all the cofactors of the L are equal and their common value is equal to the complexity $\tau(G)$ of G. Another method for enumerating the number of spanning trees $\tau(G)$ of G employs the eigenvalues of the Laplacian matrix L. Let $0 = \mu_1 \leq \mu_2 \leq \cdots \leq \mu_n$ denoting the eigenvalues of the L matrix. Kelmans and Chelnokov [13], reached the equation,

$$\tau(G) = \frac{1}{n}\prod_{i=2}^{n} \lambda_i. \tag{1.1}$$

After that, Temperley [21] showed the following

$$\tau(G) = \frac{1}{n^2} det(L + J), \tag{1.2}$$

where J is the $n \times n$ matrix whose elements are all ones.

From Temperley's equation (1.2) and by straightforward steps, we can prove the following lemma which express $\tau(G)$ directly as a determinant rather than in terms of cofactors or eigenvalues.

Lemma 1.1. *Let G be a graph with n vertices. Then,*

$$\tau(G) = \frac{1}{n^2} det(nI - \overline{D} + \overline{A}), \tag{1.3}$$

where $\overline{D}, \overline{A}$ are the degree and adjacency matrices, respectively, of $\overline{G}$, the complement of G.

For some distinctive categories of graphs, there exist simple closed formulas that make it much easier to enumerate the number of corresponding spanning trees, especially when these numbers are very large. Cayley showed that a complete graph K_n has $n^{n-2}, n \geq 2$, spanning trees [2]. Furthermore, Sedlacek [18] derived a formula for the wheel with $n+1$ vertices, W_{n+1}, whose number of spanning trees is $\tau(W_{n+1}) = (\frac{3+\sqrt{5}}{2})^n + (\frac{3-\sqrt{5}}{2})^n - 2, n \geq 2$. In [3], Boesch and Bogdanowicz derived a closed formula for spanning trees in a prism $\Pi_n, \tau(\Pi_n) = [\frac{n}{2}(2+\sqrt{3})^n + (2-\sqrt{3})^n - 2], n \geq 2$. Later Boesch and Prodinger [4] derived a new formula for spanning trees in a complete prism $\Pi_n(m) = K_m \Box C_n$. In [19], Sun and others used the electrically equivalent transformations and rules of weighted generating function to obtain a relationship for the weighted number of spanning trees. In [20], the authors introduced a formula for the circulant graphs. Nowadays, lots of closed formulas have been introduced on calculating the number of spanning trees for some categories of graphs, see [7, 8, 9, 10, 15, 22, 23].

2. Basic proof tools

Let $\Psi_n(x)$ be an $n \times n$ matrix such that:

$$\Psi_n(x) = \begin{pmatrix} 2x & -1 & 0 & 0 & \dots & \dots & 0 \\ -1 & 2x & -1 & 0 & \dots & \dots & 0 \\ 0 & -1 & 2x & -1 & \dots & \dots & 0 \\ \vdots & \vdots & \dots & \dots & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & \dots & -1 & 2x & -1 \\ \\ 0 & \dots & \dots & \dots & 0 & -1 & 2x \end{pmatrix}$$

There exist strong relations concerning orthogonal polynomials, especially Chebyshev polynomials of the first and second kind, and the determinants used in our computations. For a positive integer n, the Chebyshev Polynomials of the first kind are defined by [16]:

$$T_n(x) \quad = \cos(n \arccos x). \tag{2.1}$$

Chebyshev Polynomials of the first kind satisfy the recursion relation

$$T_{n+1}(x) - 2xT_n(x) + T_{n-1}(x) = 0;\ T_0(x) = 1, T_1(x) = x. \tag{2.2}$$

Using standard methods for solving the recursion (2.2) we get

$$T_n(x) \quad = \tfrac{1}{2}[(x + \sqrt{x^2-1})^n + (x - \sqrt{x^2-1})^n],\ n \geq 1. \tag{2.3}$$

The Chebyshev Polynomials of the second kind are defined by :

$$U_{n-1}(x) \quad = \tfrac{1}{n}\tfrac{d}{dx}T_n(x) = \tfrac{\sin(n \arccos x)}{\sin(\arccos x)}. \tag{2.4}$$

It satisfies the recursion relation

$$U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x);\ U_0(x) = 1, U_1(x) = 2x. \tag{2.5}$$

From this recursion relation (2.5) and by expanding $\det\Psi_n(x)$ one gets

$$U_n(x) \quad = det[\Psi_n(x)],\ n \geq 1. \tag{2.6}$$

Furthermore, by using standard methods for solving the recursion (2.5), one obtains the explicit formula

$$U_n(x) = \frac{1}{2\sqrt{x^2-1}}[(x + \sqrt{x^2-1})^{n+1} - (x - \sqrt{x^2-1})^{n+1}],\ n \geq 1, \tag{2.7}$$

where the identity is true for all complex x (except at $x = \pm 1$).

Lemma 2.1. *[17] Suppose P, Q, R, and S are block matrices of dimension $n \times n, n \times m, m \times n$, and $m \times m$, respectively. Then, when P and S are nonsingular,*

$$det \begin{pmatrix} P & Q \\ R & S \end{pmatrix} = det(P) \times det(S - RP^{-1}Q) = det(S) \times det(P - QS^{-1}R).$$

Lemma 2.2. *[11, 12]*

(i) $\forall x \geq 4, n \geq 3$, *let $B_n(x)$ be an $n \times n$ circulant matrix given by*

$$B_n(x) = \begin{pmatrix} x & 0 & 1 & 1 & \dots & 1 & 0 \\ 0 & x & 0 & 1 & \dots & 1 & 1 \\ 1 & 0 & x & 0 & \dots & \dots & 1 \\ \vdots & \vdots & \dots & \ddots & \ddots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & \ddots & x & 0 \\ 0 & 1 & 1 & \dots & 1 & 0 & x \end{pmatrix},$$

then $det[B_n(x)] = \frac{2(x+n-3)}{x-3}[T_n(\frac{x-1}{2}) - 1]$.

(ii) *Let $C_n(x)$ be an $n \times n$ circulant matrix given by*

$$C_n(x) = \begin{pmatrix} x & 1 & 1 & 1 & \dots & 1 \\ 1 & x & 1 & 1 & \dots & 1 \\ 1 & 1 & x & 1 & \dots & 1 \\ \vdots & \vdots & \dots & \ddots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & x & 1 \\ 1 & 1 & 1 & \dots & 1 & x \end{pmatrix},$$

then $det[C_n(x)] = (x + n - 1)(x - 1)^{n-1}$.

(iii) *If*

$$D_n(x) = \begin{pmatrix} x & -1 & 0 & \dots & \dots & 0 \\ -1 & (x+1) & -1 & \dots & \dots & 0 \\ 0 & -1 & (x+1) & \dots & \dots & 0 \\ \vdots & \vdots & \dots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \dots & \dots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & (x+1) & -1 \\ 0 & 0 & 0 & \dots & -1 & x \end{pmatrix},$$

then $det[D_n(x)] = (x-1)U_{n-1}(\frac{1+x}{2})$.

(iv) $\forall x > 2, n \geq 3$, *if*

$$W_n(x) = \begin{pmatrix} x & 0 & 1 & \dots & \dots & 1 \\ 0 & (x+1) & 0 & 1 & \dots & 1 \\ 1 & 0 & (x+1) & \dots & \dots & 1 \\ \vdots & \vdots & \dots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \dots & \dots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & (x+1) & 0 \\ 1 & 1 & 1 & \dots & 0 & x \end{pmatrix},$$

then $det[W_n(x)] = (n+x-2)U_{n-1}(\frac{x}{2})$.

Lemma 2.3. *Let* P, Q *and* R *be block matrices of dimension* $n \times n$, *then*

$$det \begin{pmatrix} P & Q & Q & R \\ Q & P & R & Q \\ Q & R & P & Q \\ R & Q & Q & P \end{pmatrix} = [det(P-R)]^2 det(P+R+2Q) det(P+R-2Q).$$

Proof. Using the properties of determinants and matrix row and column operations yields:

$$\begin{pmatrix} P & Q & Q & R \\ Q & P & R & Q \\ Q & R & P & Q \\ R & Q & Q & P \end{pmatrix} = \begin{pmatrix} P-R & O & O & R-P \\ O & P-R & R-P & O \\ Q & R & P & Q \\ R & Q & Q & P \end{pmatrix}$$

$$= \begin{pmatrix} P-R & O & O & O \\ O & P-R & O & O \\ Q & R & P+R & 2Q \\ R & Q & 2Q & P+R \end{pmatrix}.$$

Therefore,

$$det \begin{pmatrix} P & Q & Q & R \\ Q & P & R & Q \\ Q & R & P & Q \\ R & Q & Q & P \end{pmatrix} = det \begin{pmatrix} P-R & O \\ O & P-R \end{pmatrix} \times det \begin{pmatrix} P+R & 2Q \\ 2Q & R+P \end{pmatrix}$$

$$= [det(P-R)]^2 det(P+R+2Q) det(P+R-2Q).$$

□

3. Complexity of the prism $P(B_{n,2})$ of the book graph

Definition 3.1. **A star graph** S_n, *which is a tree with one internal vertex and* n *leaves, has number of vertices* $|V(S_n)| = n+1$ *and edges* $|E(S_n)| = n$. **The book graph** $B_{n,2}$ *is defined as the cartesian product* $B_{n,2} = S_n \square P_2$, *where* S_n *is a star graph, and* P_2 *is the path graph on two vertices. The book graph* $B_{n,2}$ *has number of vertices* $|V(B_{n,2})| = 2n+2$ *and edges* $|E(B_{n,2})| = 3n+1$. **The prism of the book graph** $P(B_{n,2}) = B_{n,2} \square P_2$ *has number of vertices* $|V[P(B_{n,2})]| = 4n+4$ *and edges* $|E[P(B_{n,2})]| = 8n+4$ *[5], see Fig. 1.*

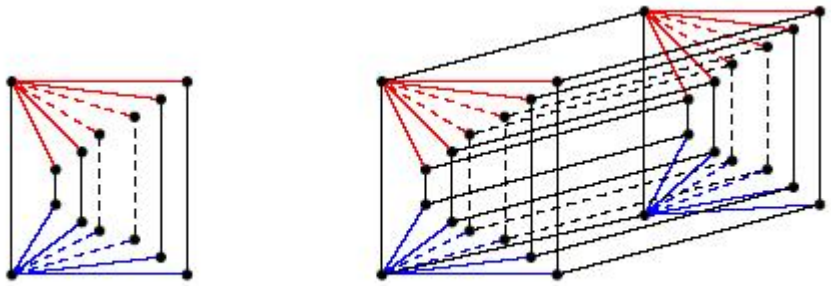

Figure 1: (a) The book graph $B_{n,2}$. (b) The prism $P(B_{n,2})$ of the book graph $B_{n,2}$.

Theorem 3.2. *For $n \geq 2$, the number of spanning trees of the prism $P(B_{n,2})$ of the book graph $B_{n,2}$ is given by:*

$$\tau[P(B_{n,2})] = 4(n+5)(n+3)^2 3^{2n-2} 5^{n-1}.$$

Proof. Applying Lemma 1.1, we have:

$$\tau[P(B_{n,2})] = \frac{1}{(4n+4)^2} det[(4n+4)I - \overline{D} + \overline{A}]$$

$$= \frac{1}{(4n+4)^2} det \begin{pmatrix} P & Q & Q & J \\ Q & P & J & Q \\ Q & J & P & Q \\ J & Q & Q & P \end{pmatrix},$$

where

$$P = \begin{pmatrix} n+3 & 0 & 0 & \dots & \dots & 0 & 0 \\ 0 & 4 & 1 & 1 & \dots & \dots & 1 \\ 0 & 1 & 4 & 1 & \dots & \dots & 1 \\ 0 & 1 & 1 & 4 & \dots & \dots & 1 \\ \vdots & \dots & \dots & \dots & \ddots & \ddots & \vdots \\ 0 & 1 & \dots & \dots & 1 & 4 & 1 \\ 0 & 1 & 1 & \dots & \dots & 1 & 4 \end{pmatrix}_{(n+1)\times(n+1)}$$

and

$$Q = \begin{pmatrix} 0 & 1 & 1 & \dots & \dots & 1 & 1 \\ 1 & 0 & 1 & 1 & \dots & \dots & 1 \\ 1 & 1 & 0 & 1 & \dots & \dots & 1 \\ 1 & 1 & 1 & 0 & \dots & \dots & 1 \\ \vdots & \dots & \dots & \dots & \ddots & \ddots & \vdots \\ 1 & 1 & \dots & \dots & 1 & 0 & 1 \\ 1 & 1 & 1 & \dots & \dots & 1 & 0 \end{pmatrix}_{(n+1)\times(n+1)}$$

Applying Lemma 2.3, we have:

$$\tau[P(B_{n,2})] = \frac{1}{(4n+4)^2}[det(P-J)]^2(P+J+2Q) \times det(P+J-2Q)]. \tag{3.1}$$

From Lemma 2.2, we obtain:

$$\text{det(P-J)} = \det \begin{pmatrix} n+2 & -1 & -1 & -1 & \dots & \dots & -1 \\ -1 & 3 & 0 & 0 & \dots & \dots & 0 \\ -1 & 0 & 3 & 0 & \dots & \dots & 0 \\ \vdots & \dots & \dots & \dots & \ddots & \ddots & \vdots \\ -1 & 0 & 0 & 0 & \dots & 3 & 0 \\ -1 & 0 & 0 & 0 & \dots & 0 & 3 \end{pmatrix}_{(n+1)\times(n+1)}$$

$$= 2 \det \begin{pmatrix} 4 & 1 & 1 & 1 & \dots & \dots & 1 \\ 1 & 4 & 1 & 1 & \dots & \dots & 1 \\ 1 & 1 & 4 & 1 & \dots & \dots & 1 \\ \vdots & \vdots & \dots & \dots & \ddots & \ddots & \vdots \\ 1 & 1 & 1 & 1 & \dots & 4 & 1 \\ 1 & 1 & 1 & 1 & \dots & 1 & 4 \end{pmatrix}_{n\times n}$$

$$= 2(x+n-1)(x-1)^{n-1} = 2(n+3)3^{n-1}. \tag{3.2}$$

$$\det(\mathrm{P+J+2Q}) = \det \begin{pmatrix} n+4 & 3 & 3 & 3 & \dots & \dots & 3 \\ 3 & 5 & 4 & 4 & \dots & \dots & 4 \\ 3 & 4 & 5 & 4 & \dots & \dots & 4 \\ 3 & 4 & 4 & 5 & \dots & \dots & 4 \\ \vdots & \dots & \dots & \dots & \ddots & \ddots & \vdots \\ 3 & 4 & 4 & 4 & \dots & 5 & 4 \\ 3 & 4 & 4 & 0 & \dots & 4 & 5 \end{pmatrix}_{(n+1)\times(n+1)}$$

$$= (4\ \mathrm{n}{+}4) \quad \det \begin{pmatrix} 2 & 1 & 1 & 1 & \dots & \dots & 1 \\ 1 & 2 & 1 & 1 & \dots & \dots & 1 \\ 1 & 1 & 2 & 1 & \dots & \dots & 1 \\ 1 & 1 & 1 & 2 & \dots & \dots & 1 \\ \vdots & \dots & \dots & \dots & \ddots & \ddots & \vdots \\ 1 & 1 & 1 & 1 & \dots & 2 & 1 \\ 1 & 1 & 1 & 1 & \dots & 1 & 2 \end{pmatrix}_{n\times n}{}_{(n+1)\times(n+1)}$$

$$= (4n+4)(x+n-1)(x-1)^{n-1} = (4n+4)(n+1). \tag{3.3}$$

$$\det(\text{P+J-2Q}) = \det \begin{pmatrix} n+4 & -1 & -1 & -1 & \dots & \dots & -1 \\ -1 & 5 & 0 & 0 & \dots & \dots & 0 \\ -1 & 0 & 5 & 0 & \dots & \dots & 0 \\ -1 & 0 & 0 & 5 & \dots & \dots & 0 \\ \vdots & \dots & \dots & \dots & \ddots & \ddots & \vdots \\ -1 & 0 & 0 & 0 & \dots & 5 & 0 \\ -1 & 0 & 0 & 0 & \dots & 0 & 5 \end{pmatrix}_{(n+1)\times(n+1)}$$

$$= 4 \det \begin{pmatrix} 6 & 1 & 1 & 1 & \dots & \dots & 1 \\ 1 & 6 & 1 & 1 & \dots & \dots & 1 \\ 1 & 1 & 6 & 1 & \dots & \dots & 1 \\ 1 & 1 & 1 & 6 & \dots & \dots & 1 \\ \vdots & \dots & \dots & \dots & \ddots & \ddots & \vdots \\ 1 & 1 & 1 & 1 & \dots & 6 & 1 \\ 1 & 1 & 1 & 1 & \dots & 1 & 6 \end{pmatrix}_{n\times n}$$

$$= 4(x+n-1)(x-1)^{n-1} = 4(n+5)5^{n-1}. \tag{3.4}$$

Substituting the equations (3.2), (3.3) and (3.4) in equation (3.1), we get the result. □

4. Complexity of the prism $P(\Pi_n)$ of the prism graph

The prism $P(\Pi_n) = \Pi_n \square P_2$ of the prim graph $\Pi_n = C_n \square P_2$ has number of vertices $|V[P(\Pi_n)]| = 4n$ and edges $|E[P(\Pi_n)]| = 8n, n \geq 3$, see Fig. 2.

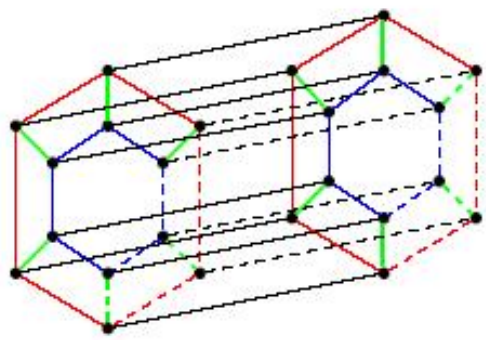

Figure 2. The prism $P(\Pi_n)$ of the prism graph Π_n.

Theorem 4.1. *For a positive integer $n \geq 2$, the number of spanning trees of the prism $P(\Pi_n)$ of the prism graph Π_n is given by:*

$$\tau[P(\Pi_n)] = \frac{n}{4}[(2+\sqrt{3})^n + (2-\sqrt{3})^n - 2]^2[(3+2\sqrt{2})^n + (3-2\sqrt{2})^n - 2].$$

Proof. From Lemma 1.1, we have:

$$\tau[P(\Pi_n)] = \frac{1}{(4n)^2} det[4nI - \overline{D} + \overline{A}] = \frac{1}{(4n)^2} det \begin{pmatrix} P & Q & Q & J \\ Q & P & J & Q \\ Q & J & P & Q \\ J & Q & Q & P \end{pmatrix},$$

where $P = \begin{pmatrix} 5 & 0 & 1 & \dots & \dots & 1 & 0 \\ 0 & 5 & 0 & 1 & \dots & \dots & 1 \\ 1 & 0 & 5 & 0 & \dots & \dots & 1 \\ \vdots & \dots & \dots & \dots & \ddots & \ddots & \vdots \\ 1 & 1 & \dots & \dots & 0 & 5 & 0 \\ 0 & 1 & 1 & \dots & \dots & 0 & 5 \end{pmatrix}_{n \times n}$

$$\text{and } Q = \begin{pmatrix} 0 & 1 & 1 & \dots & \dots & 1 & 1 \\ 1 & 0 & 1 & 1 & \dots & \dots & 1 \\ 1 & 1 & 0 & 1 & \dots & \dots & 1 \\ \vdots & \dots & \dots & \ddots & \ddots & \ddots & \vdots \\ 1 & 1 & \dots & \dots & 1 & 0 & 1 \\ 1 & 1 & 1 & \dots & \dots & 1 & 0 \end{pmatrix}_{n\times n}.$$

Applying Lemma 2.3, we have:

$$\det(\text{P-J}) = \det \begin{pmatrix} 4 & -1 & 0 & \dots & \dots & -1 \\ -1 & 4 & -1 & \dots & \dots & 0 \\ 0 & -1 & 4 & \dots & \dots & 0 \\ \vdots & \dots & \vdots & \ddots & \dots & \vdots \\ 0 & 0 & \dots & \dots & 4 & -1 \\ -1 & 0 & \dots & \dots & -1 & 4 \end{pmatrix}$$

$$= 2\,[T_n(2) - 1\,] = [\,(2+\sqrt{3})^n + (2-\sqrt{3}\,)^n - 2]. \tag{4.1}$$

$$\det(\text{P+J-2Q}) = \det \begin{pmatrix} 6 & -1 & 0 & \dots & -1 \\ -1 & 6 & -1 & \dots & 0 \\ 0 & -1 & 6 & \dots & 0 \\ \vdots & \dots & \dots & \dots & \vdots \\ 0 & 0 & \dots & 6 & -1 \\ -1 & 0 & \dots & -1 & 6 \end{pmatrix}$$

$$= 2\,[T_n(3) - 1] = [\,(3+2\sqrt{2})^n + (3-2\sqrt{2}\,)^n - 2]. \tag{4.2}$$

$$\det(\text{P+J+2Q}) = \det \begin{pmatrix} 6 & 3 & 4 & \dots & 4 & 3 \\ 3 & 6 & 3 & 4 & \dots & 4 \\ 4 & 3 & 6 & 3 & \dots & 4 \\ \vdots & \dots & \dots & \ddots & \ddots & \vdots \\ 4 & 4 & 4 & \dots & 6 & 3 \\ 3 & 4 & 4 & \dots & 3 & 6 \end{pmatrix}$$

$$= 4det \begin{pmatrix} 3 & 0 & 1 & \dots & 1 & 0 \\ 0 & 3 & 0 & \dots & 1 & 1 \\ 1 & 0 & 3 & 0 & \dots & 1 \\ \vdots & \dots & \dots & \ddots & \ddots & \vdots \\ 1 & 1 & \dots & \dots & 3 & 0 \\ 0 & 1 & \dots & \dots & 0 & 3 \end{pmatrix} = 4\ n^3. \tag{4.3}$$

Substituting the equations (4.1), (4.2) and (4.3) in Lemma 2.3, we get the result. □

5. Complexity of the prism $P(L_n)$ of the ladder graph

The prism $P(L_n) = L_n \times P_2$ of the ladder graph $L_n = P_n \times P_2$ has number of vertices $|V(P(L_n))| = 4n$ and edges $|E(P(L_n)| = 4(2n-1), n \geq 2$, see Fig. 3.

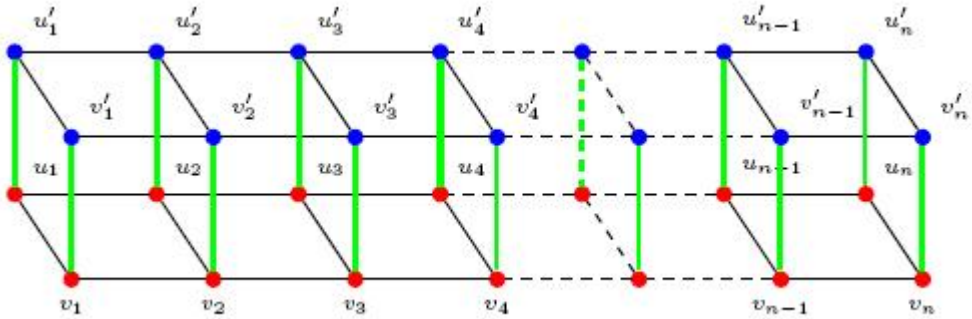

Figure 3. The prism $P(L_n)$ of the Ladder L_n.

Theorem 5.1. *For $n \geq 2$, the number of spanning trees of the prism $P(L_n)$ of the ladder graph L_n is given by:*

$$\tau[P(L_n)] = \frac{1}{12\sqrt{2}}[(3+2\sqrt{2})^n - (3-2\sqrt{2})^n][(2+\sqrt{3})^n - (2-\sqrt{3})^n]^2.$$

Proof. Applying the same method in Theorem 4.1, we have:

$$P = \begin{pmatrix} 4 & 0 & 1 & \dots & \dots & 1 & 1 \\ 0 & 5 & 0 & 1 & \dots & \dots & 1 \\ 1 & 0 & 5 & 0 & \dots & \dots & 1 \\ \vdots & \dots & \dots & \dots & \ddots & \ddots & \vdots \\ 1 & 1 & \dots & \dots & 0 & 5 & 0 \\ 1 & 1 & 1 & \dots & \dots & 0 & 4 \end{pmatrix} \quad \text{and}$$

$$Q = \begin{pmatrix} 0 & 1 & 1 & \dots & \dots & 1 & 1 \\ 1 & 0 & 1 & 1 & \dots & \dots & 1 \\ 1 & 1 & 0 & 1 & \dots & \dots & 1 \\ \vdots & \dots & \dots & \ddots & \ddots & \ddots & \vdots \\ 1 & 1 & \dots & \dots & 1 & 0 & 1 \\ 1 & 1 & 1 & \dots & \dots & 1 & 0, \end{pmatrix}$$

$$det(P - J) = 2U_{n-1}(2) = \frac{1}{\sqrt{3}}[(2+\sqrt{3})^n - (2-\sqrt{3})^n]. \tag{5.1}$$

$$det(P + J - 2Q) = 4U_{n-1}(3) = \frac{1}{\sqrt{2}}[(3+2\sqrt{2})^n - (3-2\sqrt{2})^n]. \tag{5.2}$$

$$det(P + J + 2Q) = 4n^2. \tag{5.3}$$

Substituting the equations (5.1), (5.2) and (5.3) in Lemma 2.3 yields the result. □

6. Complexity of the prism $P(R_n)$ of the $n-$crossed prism graph

Definition 6.1. *For an even integer $n \geq 4$,* **the $n-$crossed prism** *graph is defined as a graph obtained by taking two disjoint cycles C_n with n vertex, that is C_n^1 and C_n^2 where $V(C_n^1) = \{x_1, x_2, \cdots, x_n\}$ and $V(C_n^2) = \{y_1, y_2, \cdots, y_n\}$ and adding edge $y_i x_{i+1}$for $i \in \{1, 3, \cdots, n-1\}$ and $y_k x_{k-1}$*

for $k \in \{2, 4, \cdots, n\}$. **The prism** $P(R_n)$ **of the n- crossed prism** *graph* R_n *has number of vertices* $|V[P(R_n)]| = 4n$ *and edges* $|E[P(R_n)]| = 8n$, *see Fig. 4.*

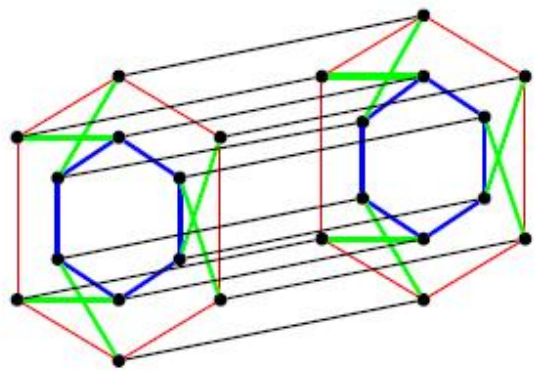

Figure 4. The prism $P(R_6)$ of the 6- crossed prism graph R_6.

Conjecture 6.2. *For an even integer* $n \geq 4$, *the number of spanning trees of the prism* $P(R_n)$ *of the* $n-$*crossed prism graph* R_n *is given by:*

$\tau[P(R_n)] = 2^{4n-2}3^{(n+2)/2}n[(49+20\sqrt{6})(10+4\sqrt{6})^{(n/2)-2}+(49-20\sqrt{6})\times(10-4\sqrt{6})^{(n/2)-2} - 2^{(n-2)/2}]$.

Proof. Applying Lemma 1.1, we have:

$$\tau[P(R_n)] = \frac{1}{(4n)^2}det[4nI - \overline{D} + \overline{A}] = \frac{1}{(4n)^2}det \begin{pmatrix} P & Q & R & J \\ Q & P & J & R \\ R & J & P & Q \\ J & R & Q & P \end{pmatrix},$$

where

$$P = \begin{pmatrix} 5 & 0 & 1 & \dots & \dots & 0 \\ 0 & 5 & 0 & 1 & \dots & 1 \\ 1 & 0 & 5 & \dots & \dots & 1 \\ \vdots & \dots & \dots & \ddots & \ddots & \vdots \\ 1 & 1 & \dots & \dots & 5 & 0 \\ 0 & 1 & \dots & \dots & 0 & 5 \end{pmatrix}_{n\times n}, Q = \begin{pmatrix} 1 & 0 & 1 & \dots & \dots & 1 \\ 0 & 1 & 1 & \dots & \dots & 1 \\ 1 & 1 & 1 & \dots & \dots & 1 \\ \vdots & \dots & \ddots & \ddots & \ddots & \vdots \\ 1 & 1 & \dots & \dots & 1 & 0 \\ 1 & 1 & \dots & \dots & 0 & 1 \end{pmatrix}_{n\times n}$$

$$\text{and R} = \begin{pmatrix} 0 & 1 & \dots & \dots & 1 \\ 1 & 0 & 1 & \dots & 1 \\ 1 & 1 & 0 & \dots & 1 \\ \vdots & \dots & \ddots & \ddots & \vdots \\ 1 & \dots & \dots & 0 & 1 \\ 1 & \dots & \dots & 1 & 0 \end{pmatrix}_{n\times n}$$

Doing $L = det \begin{pmatrix} P & Q \\ Q & P \end{pmatrix}$ and $M = det \begin{pmatrix} R & J \\ J & R \end{pmatrix}$, we have

$$(6.1)\, \tau[P(R_n)] = \frac{1}{(4n)^2} det \begin{pmatrix} L & M \\ M & L \end{pmatrix} = \frac{1}{(4n)^2}[det(L+M)(L-M)].$$

$$\begin{aligned} det(L+M) &= det \begin{pmatrix} P+R & Q+J \\ Q+J & P+R \end{pmatrix} \\ &= det(P+R+Q+J)(P+R-Q-J) \end{aligned}$$

$$= det \begin{pmatrix} 7 & 2 & 4 & 4 & \dots & \dots & 3 \\ 2 & 7 & 3 & 4 & \dots & \dots & 4 \\ 4 & 2 & 7 & 3 & \dots & \dots & 4 \\ 4 & 4 & 2 & 7 & \dots & \dots & 4 \\ \vdots & \vdots & \dots & \dots & \vdots & \ddots & \vdots \\ 4 & 4 & \dots & \dots & 3 & 7 & 2 \\ 3 & 4 & 4 & \dots & \dots & 2 & 7 \end{pmatrix} \times det \begin{pmatrix} 3 & 0 & 0 & 0 & \dots & \dots & -1 \\ 0 & 3 & -1 & 0 & \dots & \dots & 0 \\ 0 & -1 & 3 & 0 & \dots & \dots & 0 \\ 0 & 0 & 0 & 3 & \dots & \dots & 0 \\ \vdots & \vdots & \dots & \vdots & \ddots & \dots & \vdots \\ 0 & 0 & 0 & \dots & \dots & 3 & 0 \\ -1 & 0 & 0 & \dots & \dots & 0 & 3 \end{pmatrix}$$

$$= 4ndet \begin{pmatrix} 1 & 0 & 2 & 2 & \dots & \dots & 1 \\ 1 & 5 & 1 & 2 & \dots & \dots & 2 \\ 1 & 1 & 5 & 3 & \dots & \dots & 2 \\ 1 & 2 & 1 & 5 & \dots & \dots & 2 \\ \vdots & \vdots & \dots & \dots & \vdots & \ddots & \vdots \\ 1 & 2 & \dots & \dots & \dots & 5 & 0 \\ 1 & 2 & 2 & \dots & \dots & 0 & 5 \end{pmatrix} \times det \begin{pmatrix} 2 & 0 & 0 & 0 & \dots & \dots & 2 \\ 0 & 2 & 2 & 0 & \dots & \dots & 0 \\ 0 & -1 & 3 & 0 & \dots & \dots & 0 \\ 0 & 0 & 0 & 2 & \dots & \dots & 0 \\ \vdots & \vdots & \dots & \vdots & \ddots & \dots & \vdots \\ 0 & 0 & 0 & \dots & \dots & 3 & 0 \\ -1 & 0 & 0 & \dots & \dots & 0 & 3 \end{pmatrix}$$

$$= \frac{4n}{2n} det \begin{pmatrix} 2n & 0 & 2 & 2 & \dots & \dots & 1 \\ 2n & 5 & 1 & 2 & \dots & \dots & 2 \\ 2n & 1 & 5 & 1 & \dots & \dots & 2 \\ 2n & 2 & 1 & 5 & \dots & \dots & 2 \\ \vdots & \vdots & & & \vdots & \ddots & \vdots \\ 2n & 2 & \dots & \dots & \dots & 5 & 0 \\ 2n & 2 & 2 & \dots & \dots & 0 & 5 \end{pmatrix} \times \; 2^{\frac{n}{2}} det \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & \dots & 1 \\ 0 & 1 & 1 & 0 & \dots & \dots & 0 \\ 0 & 0 & 4 & 0 & \dots & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & \dots & 0 \\ \vdots & \vdots & & \vdots & \ddots & \dots & \vdots \\ 0 & 0 & 0 & \dots & \dots & 4 & 0 \\ 0 & 0 & 0 & \dots & \dots & 0 & 4 \end{pmatrix}$$

$$= 2det \begin{pmatrix} 5 & 0 & 2 & 2 & \dots & \dots & 1 \\ 0 & 5 & 1 & 2 & \dots & \dots & 2 \\ 2 & 1 & 5 & 0 & \dots & \dots & 2 \\ 2 & 2 & 1 & 5 & \dots & \dots & 2 \\ \vdots & \vdots & & & \vdots & \ddots & \vdots \\ 2 & 2 & 2 & \dots & 1 & 5 & 0 \\ 1 & 2 & 2 & \dots & \dots & 0 & 5 \end{pmatrix} \times 2^{\frac{n}{2}} \; det \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & \dots & 1 \\ 0 & 1 & 1 & 0 & \dots & \dots & 0 \\ 0 & 0 & 4 & 0 & \dots & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & \dots & 0 \\ \vdots & \vdots & & \vdots & \ddots & \dots & \vdots \\ 0 & 0 & 0 & \dots & \dots & 4 & 0 \\ 0 & 0 & 0 & \dots & \dots & 0 & 4 \end{pmatrix}$$

$$= 2 \times (\frac{3n^3}{8} \times 2^{\frac{n+4}{2}}) \times 2^{\frac{n}{2}} \times 4^{\frac{n}{2}} = 3n^3 2^{2n}. \tag{6.2}$$

In the same way, we can find

$$\begin{aligned} det(L-M) &= det \begin{pmatrix} P-R & Q-J \\ Q-J & P-R \end{pmatrix} \\ &= det(P-R+Q-J)(P-R-Q+J) \end{aligned}$$

$$= det \begin{pmatrix} 5 & -2 & 0 & 0 & \dots & \dots & -1 \\ -2 & 5 & -1 & 0 & \dots & \dots & 0 \\ 0 & -1 & 5 & -2 & \dots & \dots & 0 \\ 0 & 0 & -2 & 5 & \dots & \dots & 0 \\ \vdots & \vdots & \dots & \vdots & \ddots & \dots & \vdots \\ 0 & 0 & 0 & \dots & \dots & 5 & -2 \\ -1 & 0 & 0 & \dots & \dots & -2 & 5 \end{pmatrix} \times 4^{\frac{n}{2}} det \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & \dots & 1 \\ 0 & 1 & 1 & 0 & \dots & \dots & 0 \\ 0 & 0 & 6 & 0 & \dots & \dots & 0 \\ 0 & 0 & 0 & 1 & 1 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots & \ddots & \dots & \vdots \\ 0 & 0 & 0 & \dots & \dots & 6 & 0 \\ 0 & 0 & 0 & \dots & \dots & 0 & 6 \end{pmatrix}$$

$$= det \begin{pmatrix} 5 & -2 & 0 & 0 & \dots & \dots & -1 \\ -2 & 5 & -1 & 0 & \dots & \dots & 0 \\ 0 & -1 & 5 & -2 & \dots & \dots & 0 \\ 0 & 0 & -2 & 5 & \dots & \dots & 0 \\ \vdots & \vdots & \dots & \vdots & \ddots & \dots & \vdots \\ 0 & 0 & 0 & \dots & \dots & 5 & -2 \\ -1 & 0 & 0 & \dots & \dots & -2 & 5 \end{pmatrix} \times 4^{\frac{n}{2}} \times 6^{\frac{n}{2}} = det(H) \times 4^{\frac{n}{2}} \times 6^{\frac{n}{2}}.$$

Expanding the first seven determinants of the H matrix gives the values $384 = 4^3 \times 6,7744 = 4^3 \times 121,153600 = 4^3 \times 2400,3041536 = 4^3 \times$

$47524, 60217344 = 4^3 \times 940896, 1192182784 = 4^3 \times 186227856 \cdots$ for $n = 4, 6, 8, 10, 12.14 \cdots$. These values can be written in the form

$$2400 = 20 \times 121 - 4 \times 6 + 4,$$
$$47524 = 20 \times 2400 - 4 \times 121 + 8,$$
$$940896 = 20 \times 47524 - 4 \times 2400 + 16,$$
$$20625085 = 20 \times 940896 - 4 \times 47524 + 32.$$

Consequently, we have the following non-homogeneous recurrence relation

$$a_{n+2} = 20a_{n+1} - 4a_n + 2^n. \tag{6.3}$$

Let $a_n = b_n + k2^n$ be the solution of the non-homogeneous recurrence relation, where b_n is the solution of the homogeneous recurrence relation $b_{n+2} - 20b_{n+1} + 4b_n = 0$. Substituting in Eq.(6.3), we get $k = -2^{-5}$, then $a_n = b_n - 2^{n-5}$. The characteristic equation of the homogeneous recurrence relation is $r^2 - 20r + 4 = 0$ with two roots being $10 + 4\sqrt{6}$ and $10 - 4\sqrt{6}$, so the general solution of the non-homogeneous recurrence relation is

$$a_n = \alpha(10 + 4\sqrt{6})^{(n/2)-2} + \beta(10 - 4\sqrt{6})^{(n/2)-2} - 2^{(n-10)/2}.$$

Using the initial conditions $detH = 6, 121$ at $n = 4, 6$, respectively, we get $\alpha = \dfrac{49 + 20\sqrt{6}}{16}, \beta = \dfrac{49 - 20\sqrt{6}}{16}$.

Therefore,

$$\begin{aligned} det(H) &= 4^3 a_n = 4[(49 + 20\sqrt{6})(10 + 4\sqrt{6})^{(n/2)-2} \\ &+ (49 - 20\sqrt{6}) \times (10 - 4\sqrt{6})^{(n/2)-2} - 2^{(n-2)/2}]. \end{aligned}$$

(6.5)

$$\begin{aligned} det(L - M) &= 2^{2n+2}3^{n/2}[(49 + 20\sqrt{6})(10 + 4\sqrt{6})^{(n/2)-2} \\ &+ (49 - 20\sqrt{6}) \times (10 - 4\sqrt{6})^{(n/2)-2} - 2^{(n-2)/2}]. \end{aligned} \tag{6.6}$$

Substituting the equations (6.2) and (6.4) in the equation (6.1), we obtain the claimed result. □

7. Complexity of the prism $P[M(C_n)]$ of the mirror graph

Definition 7.1. *Let C_n be a bipartite cycle with bipartite sets V_1 and V_2 and let C'_n be a copy of C_n with corresponding bipartite sets V'_1 and V'_2.* **The mirror graph** *$M(C_n)$ is the graph obtained from C_n and C'_n by joining each vertex of V_1 to its corresponding vertex in V'_1 by an edge.* **The prism** *$P[M(C_n)]$* **of the mirror graph** *$M(C_n)$ of even cycle C_n has number of vertices $|V[P(M(C_n)])| = 4n$ and edges $|E[P(M(C_n))]| = 7n$, see Fig. 5.*

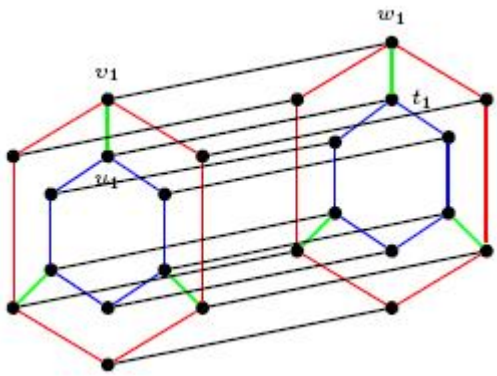

Figure 5. The Prism $P[M(C_n)]$ of the mirror graph $M(C_6)$ of even cycle C_6.

Conjecture 7.2. *For an even integer $n \geq 4$, the number of spanning trees of the prism $P[M(C_n)]$ of the mirror graph $M(C_n)$ of a cycle C_n is given by:*

$$\tau[P(M(C_n))] = \frac{n}{2}[T_n(2) - 1] \times [(17 + 12\sqrt{2})(3 + 2\sqrt{2})^{\frac{n}{2}-2}$$
$$+(17 - 12\sqrt{2})(3 - 2\sqrt{2})^{\frac{n}{2}-2} - 2] \times [(241 + 44\sqrt{30})(11 + 2\sqrt{30})^{\frac{n}{2}-2}$$
$$+(241 - 44\sqrt{30})(11 - 2\sqrt{30})^{\frac{n}{2}-2} - 2].$$

Proof. From Lemma 1.1,

$$\tau[P(M(C_n))] = \frac{1}{(4n)^2} det[4nI - \overline{D} + \overline{A}] = \frac{1}{(4n)^2} det \begin{pmatrix} P & Q & R & J \\ Q & P & J & R \\ R & J & P & Q \\ J & R & Q & P \end{pmatrix}_{4n \times 4n},$$

where

$$P = \begin{pmatrix} 5 & 0 & 1 & \dots & \dots & 0 \\ 0 & 4 & 0 & \dots & \dots & 1 \\ 1 & 0 & 5 & 0 & \dots & 1 \\ 1 & 1 & 0 & 4 & \dots & 1 \\ \vdots & \dots & \dots & \ddots & \ddots & \vdots \\ 1 & 1 & \dots & \dots & 5 & 0 \\ 0 & 1 & \dots & \dots & 0 & 4 \end{pmatrix}, Q = \begin{pmatrix} 0 & 1 & \dots & \dots & 1 & 1 \\ 1 & 1 & 1 & \dots & \dots & 1 \\ 1 & 1 & 0 & \dots & \dots & 1 \\ 1 & 1 & 1 & \dots & \dots & 1 \\ \vdots & \dots & \ddots & \ddots & \ddots & \vdots \\ 1 & 1 & \dots & \dots & 0 & 1 \\ 1 & 1 & \dots & \dots & 1 & 1 \end{pmatrix}$$

and

$$R = \begin{pmatrix} 0 & 1 & \dots & \dots & 1 \\ 1 & 0 & 1 & \dots & 1 \\ 1 & 1 & 0 & \dots & 1 \\ \vdots & \dots & \ddots & \ddots & \vdots \\ 1 & \dots & \dots & 0 & 1 \\ 1 & \dots & \dots & 1 & 0 \end{pmatrix}$$

By using the same method in Conjecture 6.2, and Eq.(6.1), we have:

$$det(L+M) = \det \begin{pmatrix} 6 & 3 & 4 & \dots & \dots & 3 \\ 3 & 6 & 3 & 4 & \dots & 4 \\ 4 & 3 & 6 & \dots & \dots & 4 \\ 4 & 4 & 3 & \dots & \dots & 4 \\ \vdots & \dots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \dots & \ddots & \ddots & \ddots & \vdots \\ 4 & 4 & \dots & \dots & 6 & 3 \\ 3 & 4 & \dots & \dots & 3 & 6 \end{pmatrix} \times \det \begin{pmatrix} 4 & -1 & 0 & 0 & \dots & \dots & -1 \\ -1 & 2 & -1 & 0 & \dots & \dots & 0 \\ 0 & -1 & 4 & -1 & \dots & \dots & 0 \\ 0 & 0 & -1 & 2 & \dots & \dots & 0 \\ \vdots & \dots & \dots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \dots & \ddots & 2 & -1 & \vdots \\ 0 & 0 & 0 & \dots & -1 & 4 & -1 \\ -1 & 0 & 0 & \dots & \dots & -1 & 2 \end{pmatrix}$$

$$= 4n^3(\Psi) \ .$$

Expanding the first seven determinants of the Ψ matrix gives the values $32 = 4 \times 8, 196 = 4 \times 49, 1152 = 4 \times 288, 6724 = 4 \times 1681, 39200 = 4 \times 9800, 228484 = 4 \times 57121, \cdots$ for $n = 4, 6, 8, 10, 12, 14, \cdots$. The values $8, 49, 288, 1681, 9800, 57121, \cdots$ have the following non-homogeneous recurrence relation

$$a_{n+2} = 6a_{n+1} - a_n + 2.$$

The general solution of the non-homogeneous recurrence relation is

$$a_n = \alpha(3+2\sqrt{2})^{\frac{n}{2}-2} + \beta(3-2\sqrt{2})^{\frac{n}{2}-2} - \tfrac{1}{2}.$$

Using the initial conditions $a_n = 8, 49$ at $n = 4, 6$, respectively, we get $\alpha = \frac{17+12\sqrt{2}}{4}$ and $\beta = \frac{17-12\sqrt{2}}{4}$.

Then, $a_n = (\dfrac{17+12\sqrt{2}}{4})(3+2\sqrt{2})^{\frac{n}{2}-2} + (\dfrac{17-12\sqrt{2}}{4})(3-2\sqrt{2})^{\frac{n}{2}-2} - \dfrac{1}{2}$.

$det(\Psi) = 4a_n = (17+12\sqrt{2})(3+2\sqrt{2})^{\frac{n}{2}-2} + (17-12\sqrt{2})(3-2\sqrt{2})^{\frac{n}{2}-2} - 2.$

$$det(L+M) = 4n^3 \times [(17+12\sqrt{2})(3+2\sqrt{2})^{\frac{n}{2}-2} + (17-12\sqrt{2})(3-2\sqrt{2})^{\frac{n}{2}-2} - 2]. \quad (7.1)$$

Also, we can see that:

$$det(L-M) = det \begin{pmatrix} 4 & -1 & 0 & 0 & \dots & \dots & -1 \\ -1 & 4 & -1 & 0 & \dots & \dots & 0 \\ 0 & -1 & 4 & -1 & \dots & \dots & 0 \\ 0 & 0 & -1 & 4 & \dots & \dots & 0 \\ \vdots & \vdots & \dots & \dots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \dots & 4 & -1 \\ -1 & 0 & 0 & \dots & \dots & -1 & 4 \end{pmatrix}$$

$$\times det \begin{pmatrix} 6 & -1 & 0 & \dots & \dots & \dots & -1 \\ -1 & 4 & -1 & 0 & \dots & \dots & 0 \\ 0 & -1 & 6 & -1 & \dots & \dots & 0 \\ 0 & 0 & -1 & 4 & \dots & \dots & 0 \\ \vdots & \vdots & \dots & \dots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & \dots & -1 & 6 & -1 \\ -1 & 0 & \dots & \dots & \dots & -1 & 4 \end{pmatrix}$$

$$= 2\ [T_n(2) - 1] \times \quad det \begin{pmatrix} 6 & -1 & 0 & \dots & \dots & \dots & -1 \\ -1 & 4 & -1 & 0 & \dots & \dots & 0 \\ 0 & -1 & 6 & -1 & \dots & \dots & 0 \\ 0 & 0 & -1 & 4 & \dots & \dots & 0 \\ \vdots & \vdots & \dots & \dots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & \dots & -1 & 6 & -1 \\ -1 & 0 & \dots & \dots & \dots & -1 & 4 \end{pmatrix}$$

$$= 2[T_n(2) - 1] \times det(\Phi).$$

Expanding the first seven determinants of the Φ matrix gives the values $480 = 20 \times 24, 10580 = 20 \times 529, 232320 = 20 \times 11616, 5100500 = 20 \times 255025, 111978720 = 20 \times 5598936, 2458431380 = 20 \times 122921569$ for $n = 4, 6, 8, 10, 12, 14, \cdots$. The values $24, 529, 11616, 255025, 5598936, 122921569, \cdots$ have the following non-homogeneous recurrence relation

$$a_{n+2} = 22a_{n+1} - a_n + 2.$$

The general solution of the non-homogeneous recurrence relation is $a_n = \alpha(11 + 2\sqrt{30}) + \beta(11 - 2\sqrt{30}) - \frac{1}{10}$.

Using the initial conditions we get $\alpha = \dfrac{241 + 44\sqrt{30}}{20}$ and $\beta = \dfrac{241 - 44\sqrt{30}}{20}$. Therefore,

$$a_n = (\frac{241 + 44\sqrt{30}}{20})(11+2\sqrt{30})^{\frac{n}{2}-2} + (\frac{241 - 44\sqrt{30}}{20})(11-2\sqrt{30})^{\frac{n}{2}-2} - \frac{1}{10}.$$

$det(\Phi) = 20a_n = [(241 + 44\sqrt{30})(11 + 2\sqrt{30})^{\frac{n}{2}-2} + (241 - 44\sqrt{30})(11 - 2\sqrt{30})^{\frac{n}{2}-2} - 2]$.

$$det(L-M) = 2[T_n(2)-1] \times [(241+44\sqrt{30})(11+2\sqrt{30})^{\frac{n}{2}-2} + (241-44\sqrt{30})(11-2\sqrt{30})^{\frac{n}{2}-2} - 2]. \tag{7.2}$$

Equations (7.1) and (7.2) are substituted into equation (6.1) to generate the result. □

8. Complexity of the prism $P[T(\Pi_n)]$ of the twisted prism

Definition 8.1. The twisted prism *graph (Möbius ladder)* $T(\Pi_n)$ *is obtained by introducing a twist in a prism graph* Π_n *of order* n. **The prism** $P[T(\Pi_n)]$ **of the twisted prism** $T(\Pi_n)$ *has number of vertices* $|V[P(T(\Pi_n))]| = 4n$ *and edges* $|E[P(T(\Pi_n))]| = 8n$, *see Fig. 6*

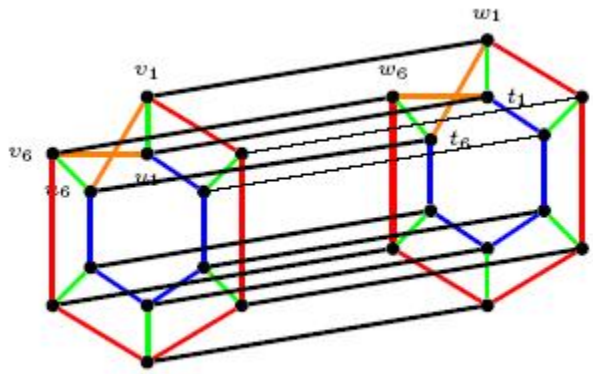

Figure 6. The Prism $P[T(\Pi_6)]$ of of the twisted prism $T(\Pi_6)$.

Conjecture 8.2. *For an even integer* $n \geq 4$, *the number of spanning trees of the prism* $P[T(\Pi_n)]$ *of the twisted prism* $T(\Pi_n)$ *is given by:*
$\tau[P(T(\Pi_n))] = \frac{n}{4}[(3+2\sqrt{2})^n + (3-2\sqrt{2})^n + 2] \times [[(2+\sqrt{3})^n + (2-\sqrt{3})^n]^2 - 4]$.

Proof. Applying Lemma 1.1, we have:

$$\tau[P(T(\Pi_n))] = \frac{1}{(4n)^2} det[4nI - \overline{D} + \overline{A}] = \frac{1}{(4n)^2} det \begin{pmatrix} P & Q & R & J \\ Q & P & J & R \\ R & J & P & Q \\ J & R & Q & P \end{pmatrix}, \text{ where}$$

$$P = \begin{pmatrix} 5 & 0 & 1 & \dots & \dots & 0 \\ 0 & 5 & 0 & \dots & \dots & 1 \\ 1 & 0 & 5 & \dots & \dots & 1 \\ \vdots & & & \ddots & \ddots & \vdots \\ 1 & 1 & \dots & \dots & 5 & 0 \\ 0 & 1 & \dots & \dots & 0 & 5 \end{pmatrix}, Q = \begin{pmatrix} 0 & 1 & 1 & \dots & \dots & 0 \\ 1 & 0 & 1 & \dots & \dots & 1 \\ 1 & 1 & 0 & \dots & \dots & 1 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 1 & 1 & \dots & \dots & 0 & 1 \\ 0 & 1 & \dots & \dots & 1 & 0 \end{pmatrix}$$

$$\text{and } R = \begin{pmatrix} 0 & 1 & \dots & \dots & \dots & 1 \\ 1 & 0 & 1 & \dots & \dots & 1 \\ 1 & 1 & 0 & \dots & \dots & 1 \\ \vdots & \dots & \dots & \ddots & \ddots & \vdots \\ 1 & \dots & \dots & 1 & 0 & 1 \\ 1 & \dots & \dots & \dots & 1 & 0 \end{pmatrix}$$

Straightforward computation and use of the same methodology as in Conjecture 6.2 leads to

$$\begin{aligned} det(L+M) &= det \begin{pmatrix} 6 & 3 & 4 & \dots & \dots & 3 \\ 3 & 6 & 3 & 4 & \dots & 4 \\ 4 & 3 & 6 & \dots & \dots & 4 \\ \vdots & \vdots & \dots & \ddots & \ddots & \vdots \\ 4 & 4 & \dots & \dots & 6 & 3 \\ 3 & 4 & \dots & \dots & 3 & 6 \end{pmatrix} \times det \begin{pmatrix} 4 & -1 & 0 & \dots & \dots & 1 \\ -1 & 4 & -1 & 0 & \dots & 0 \\ 0 & -1 & 4 & -1 & \dots & 0 \\ \vdots & \dots & \dots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & \dots & 4 & -1 \\ 1 & 0 & \dots & \dots & -1 & 4 \end{pmatrix} \\ &= 4n^3 det(X). \end{aligned}$$

Expanding the first seven determinants of the X matrix gives the values $54, 196, 726, 2704, 10086, 37636, \cdots$ for $n = 3, 4, 5, 6, 7, 8, \cdots$ which have the following non-homogeneous recurrence relation $a_{n+2} = 4a_{n+1} - a_n - 4$. The general solution of the non-homogeneous recurrence relation is $a_n = \alpha(2+\sqrt{3})^n + \beta(2-\sqrt{3})^n + 2$. Using the initial conditions $a_n = 54, 196$ at $n = 3, 4$ respectively, we get $\alpha = 1$ and $\beta = 1$; therefore, $det(X) = a_n = (2+\sqrt{3})^n + (2-\sqrt{3})^n + 2$, then

$$det(L+M) = 4n^3 \times [(2+\sqrt{3})^3 + (2-\sqrt{3})^n + 2]. \tag{8.1}$$

$$det(L-M) = det \begin{pmatrix} 4 & -1 & 0 & 0 & \dots & \dots & -1 \\ -1 & 4 & -1 & 0 & \dots & \dots & 0 \\ 0 & -1 & 4 & -1 & \dots & \dots & 0 \\ 0 & 0 & -1 & 4 & \dots & \dots & 0 \\ \vdots & \vdots & \dots & \dots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \dots & 4 & -1 \\ -1 & 0 & 0 & \dots & \dots & -1 & 4 \end{pmatrix}$$

$$\times det \begin{pmatrix} 6 & -1 & 0 & 0 & \dots & \dots & 1 \\ -1 & 6 & -1 & 0 & \dots & \dots & 0 \\ 0 & -1 & 6 & -1 & \dots & \dots & 0 \\ 0 & 0 & -1 & 6 & \dots & \dots & 0 \\ \vdots & \vdots & \dots & \dots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & -1 & 6 & -1 \\ 1 & 0 & 0 & \dots & \dots & -1 & 6 \end{pmatrix}$$

$$= 2[T_n(2)-1]\times \ det \begin{pmatrix} 6 & -1 & 0 & 0 & \dots & \dots & 1 \\ -1 & 6 & -1 & 0 & \dots & \dots & 0 \\ 0 & -1 & 6 & -1 & \dots & \dots & 0 \\ 0 & 0 & -1 & 6 & \dots & \dots & 0 \\ \vdots & \vdots & \dots & \vdots & \ddots & \dots & \vdots \\ 0 & 0 & 0 & \dots & -1 & 6 & -1 \\ 1 & 0 & 0 & \dots & \dots & -1 & 6 \end{pmatrix} = 2[T_n(2)-1](Y).$$

Expanding the first seven determinants of the Y matrix gives the values $200, 1156, 6728, 39204, 228488, 1331716, \cdots$ for $n = 3, 4, 5, 6, 7, 8, \cdots$ which have the non-homogeneous recurrence relation $a_{n+2} = 6a_{n+1} - a_n - 8$. The general solution of the non-homogeneous recurrence relation is $det(Y) = a_n = (3+2\sqrt{2})^n + (3-2\sqrt{2})^n + 2$,

$$det(L-M) = 2[T_n(2)-1] \times [(3+2\sqrt{2})^n + (3-2\sqrt{2})^n + 2]. \tag{8.2}$$

Substituting the equations (8.1) and (8.2) in the equation (6.1), we obtain the claimed result. □

9. Complexity of the prism $P(P_n[P_2])$ of the composition $P_n[P_2]$

Definition 9.1. The composition *of two graphs G and H, written as $G[H]$, is the graph with vertex set $V(G) \times V(H)$ and with (u_1, v_1) adjacent to (u_2, v_2) if either u_1 is adjacent to u_2 in G or $u_1 = u_2$ and v_1 is adjacent to v_2 in H.* **The prism** $P(P_n[P_2]) = P_n[P_2] \times P_2$ **of the composition graph** $P_n[P_2]$ *has number of vertices $|V(P(P_n[P_2]))| = 4n$ and edges $|E(P(P_n[P_2]))| = 4(3n-2), n \geq 3$, see, Fig. 7.*

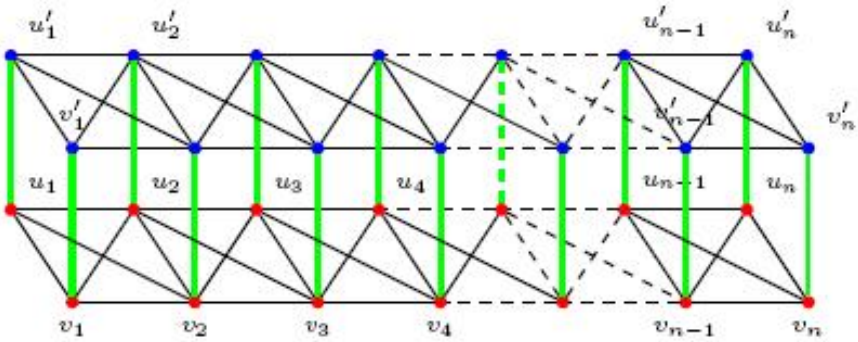

Figure 7. The prism $P(P_n[P_2]) = P_n[P_2] \times P_2$ of the composition graph $P_n[P_2]$.

Theorem 9.2. *For $n \geq 3$, the number of spanning trees of the prism $P(P_n[P_2])$ of the composition $P_n[P_2]$ is given by:*
$\tau(P(P_n[P_2])) = \dfrac{(2^{6n-5}3^n)}{\sqrt{5}}[(3+\sqrt{5})^n - (3-\sqrt{5})^n]$.

Proof. Applying Lemma 1.1, we have:

$$\tau(P(P_n[P_2])) = \frac{1}{(4n)^2} det[4nI - \overline{D} + \overline{A}] = \frac{1}{(4n)^2} det \begin{pmatrix} P & Q & R & J \\ Q & P & J & R \\ R & J & P & Q \\ J & R & Q & P \end{pmatrix}$$

where

$$P = \begin{pmatrix} 5 & 0 & 1 & \dots & \dots & 1 & 0 \\ 0 & 5 & 0 & 1 & \dots & \dots & 1 \\ 1 & 0 & 5 & 0 & \dots & \dots & 1 \\ \vdots & \dots & \dots & \dots & \ddots & \ddots & \vdots \\ 1 & 1 & \dots & \dots & 0 & 5 & 0 \\ 0 & 1 & 1 & \dots & \dots & 0 & 5 \end{pmatrix}_{n \times n}$$

and

$$Q = \begin{pmatrix} 0 & 1 & 1 & \dots & \dots & 1 & 1 \\ 1 & 0 & 1 & 1 & \dots & \dots & 1 \\ 1 & 1 & 0 & 1 & \dots & \dots & 1 \\ \vdots & \dots & \dots & \ddots & \ddots & \ddots & \vdots \\ 1 & 1 & \dots & \dots & 1 & 0 & 1 \\ 1 & 1 & 1 & \dots & \dots & 1 & 0 \end{pmatrix}_{n\times n}.$$

Applying the same technique in Conjecture 6.2, we can prove that:

$$det(L+M) = 2^n 2n^2 \times 2^{n+2}3^{n-2} = 2^{2n+3}n^2 3^{n-2}. \tag{9.1}$$

$$det(L-M) = [2^n U_{n-1}(\frac{3}{2})] \times 2^{3n-4}3^2 = \frac{(2^{4n-4}3^2)}{\sqrt{5}}[(3+\sqrt{5})^n - (3-\sqrt{5})^n]. \tag{9.2}$$

Substituting the equations (9.1) and (9.2) in the equation (6.1), we get the result. □

10. Complexity of the prism $P(G_n)$ of the gear graph G_n

Definition 10.1. The gear graph G_n *is a wheel graph with a graph vertex added between each pair of adjacent graph vertices of the outer cycle. The gear graph* G_n *has* $2n+1$ *vertices and* $3n$ *edges.* **The prism** $P(G_n)$ **of the gear graph** G_n *has number of vertices* $|V(P(G_n))| = 4n+2$ *and edges* $|E(P(G_n))| = 8n+1$, *see Fig. 8.*

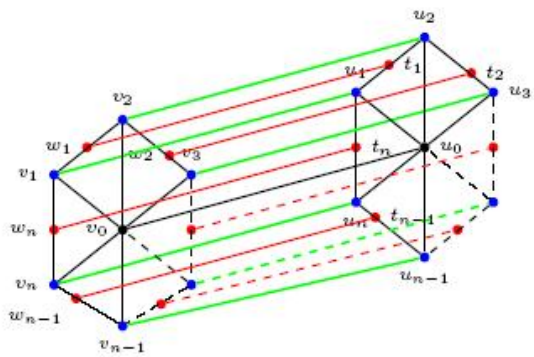

Figure 8. The prism $P(G_n)$ of the gear graph.

Theorem 10.2. *For $n \geq 2$, the number of spanning trees of the prism $P(G_n)$ of the gear graph G_n is given by:* $\tau[P(G_n)] = \dfrac{(3n+8)}{8}[(2+\sqrt{3})^n + (2-\sqrt{3})^n - 2][(9+4\sqrt{5})^n + (9-4\sqrt{5})^n - 2]$.

Proof. Applying Lemma 1.1, we have:

$$\tau[P(G_n)] = \frac{1}{(4n+2)^2}[(4n+2)I - \overline{D} + \overline{A}]$$

$$= \frac{1}{(4n+2)^2} det \begin{pmatrix} P & Q & R & E \\ Q^t & S & E & R \\ R & E & P & Q \\ E & R & Q^t & S \end{pmatrix} = \frac{1}{(4n+2)^2} det \begin{pmatrix} L & M \\ M & L \end{pmatrix},$$

where

$$P = \begin{pmatrix} n+2 & 0 & 0 & \dots & \dots & 0 & 0 \\ 0 & 5 & 1 & 1 & \dots & \dots & 1 \\ 0 & 1 & 5 & 1 & \dots & \dots & 1 \\ \vdots & \dots & \dots & \dots & \ddots & \ddots & \vdots \\ 0 & 1 & \dots & \dots & 1 & 5 & 1 \\ 0 & 1 & 1 & \dots & \dots & 1 & 5 \end{pmatrix}_{(n+1)\times(n+1)},$$

$$Q = \begin{pmatrix} 1 & 1 & 1 & \dots & \dots & 1 & 1 \\ 0 & 1 & 1 & 1 & \dots & \dots & 0 \\ 0 & 0 & 1 & 1 & \dots & \dots & 1 \\ \vdots & \dots & \dots & \ddots & \ddots & \ddots & \vdots \\ 1 & 1 & \dots & \dots & 0 & 0 & 1 \\ 1 & 1 & 1 & \dots & \dots & 0 & 0 \end{pmatrix}_{(n+1)\times n},$$

$$R = \begin{pmatrix} 0 & 1 & \dots & \dots & 1 \\ 1 & 0 & 1 & \dots & 1 \\ 1 & 1 & 0 & \dots & 1 \\ \vdots & \dots & \ddots & \ddots & \vdots \\ 1 & \dots & \dots & 0 & 1 \\ 1 & \dots & \dots & 1 & 0 \end{pmatrix}_{(n+1)\times(n+1)},$$

$$S = \begin{pmatrix} 4 & 1 & \dots & 1 & 0 \\ 1 & 4 & 1 & \dots & 1 \\ 1 & 1 & 4 & \dots & 1 \\ \vdots & \dots & \dots & \ddots & \vdots \\ 1 & \dots & \dots & 4 & 1 \\ 1 & \dots & \dots & 1 & 4 \end{pmatrix}_{n\times n} \quad \text{and}$$

$$E = \begin{pmatrix} 1 & 1 & \dots & \dots & 1 \\ 1 & 1 & \dots & \dots & 1 \\ 1 & 1 & \dots & \dots & 1 \\ \vdots & \dots & \ddots & \ddots & \vdots \\ 1 & \dots & \dots & 1 & 1 \\ 1 & \dots & \dots & 1 & 1 \end{pmatrix}_{(n+1)\times n}.$$

$$\tau[P(G_n)] = \frac{1}{(4n+2)^2}[det(L+M)(L-M)]. \tag{10.1}$$

$$det(L+M) = det \begin{pmatrix} P+R & Q+E \\ Q^t+E & S+R \end{pmatrix}$$

$$= det \begin{pmatrix} n+2 & 1 & 1 & 1 & \dots & \dots & 1 & 2 & 2 & 2 & \dots & \dots & 2 \\ 1 & 5 & 2 & 2 & \dots & \dots & 2 & 1 & 2 & 2 & \dots & \dots & 1 \\ 1 & 2 & 5 & 2 & \dots & \dots & 2 & 1 & 1 & 2 & \dots & \dots & 2 \\ 1 & 2 & 2 & 5 & \dots & \dots & 2 & 2 & 1 & 1 & \dots & \dots & 2 \\ \vdots & \vdots & \dots & \vdots & \ddots & \dots & \vdots & \dots & \vdots & \dots & \ddots & \ddots & \vdots \\ 1 & 2 & 2 & \dots & \dots & 5 & 2 & 2 & 2 & \dots & \dots & 1 & 2 \\ 1 & 2 & 2 & \dots & \dots & 2 & 5 & 2 & 2 & \dots & \dots & 1 & 1 \\ 2 & 1 & 1 & 2 & \dots & \dots & 2 & 4 & 2 & 2 & \dots & \dots & 2 \\ 2 & 2 & 1 & 1 & \dots & \dots & 2 & 2 & 4 & 2 & \dots & \dots & 2 \\ 2 & 2 & 2 & 1 & \dots & \dots & 2 & 2 & 2 & 4 & \dots & \dots & 2 \\ \vdots & \vdots & \dots & \ddots & \dots & \vdots & \vdots & \dots & \ddots & \dots & \ddots & \ddots & \vdots \\ 2 & 2 & 2 & \dots & \dots & 1 & 1 & 2 & 2 & \dots & \dots & 4 & 2 \\ 2 & 1 & 2 & \dots & \dots & 2 & 1 & 2 & 2 & \dots & \dots & 2 & 4 \end{pmatrix}_{(2n+1)\times(2n+1)} .$$

$$= (4n+2) \times det \begin{pmatrix} 8 & 3 & 4 & 4 & \dots & \dots & 3 \\ 3 & 8 & 3 & 4 & \dots & \dots & 4 \\ 4 & 3 & 8 & 3 & \dots & \dots & 4 \\ 4 & 4 & 3 & 8 & \dots & \dots & 4 \\ \vdots & \vdots & \dots & \dots & \ddots & \ddots & \vdots \\ 4 & 4 & 4 & \dots & \ddots & 8 & 3 \\ 3 & 4 & 4 & \dots & \dots & 3 & 8 \end{pmatrix}$$

$$= \frac{(4n+2)^2}{n+2} \times det \begin{pmatrix} 5 & 0 & 1 & 1 & \dots & \dots & 0 \\ 0 & 5 & 0 & 1 & \dots & \dots & 1 \\ 1 & 0 & 5 & 0 & \dots & \dots & 1 \\ 1 & 1 & 0 & 5 & \dots & \dots & 1 \\ \vdots & \vdots & \dots & \dots & \ddots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & \ddots & 5 & 0 \\ 0 & 1 & 1 & \dots & \dots & 0 & 5 \end{pmatrix}_{n\times n}$$

$$= \frac{(4n+2)^2}{2}[(2+\sqrt{3})^n + (2-\sqrt{3})^n - 2]. \tag{10.2}$$

Similarly

$$det(L-M) = 2 \times det \begin{pmatrix} 24 & 5 & 6 & 6 & \dots & \dots & 5 \\ 5 & 24 & 5 & 6 & \dots & \dots & 6 \\ 6 & 5 & 24 & 5 & \dots & \dots & 6 \\ 6 & 6 & 5 & 24 & \dots & \dots & 6 \\ \vdots & \vdots & \dots & \vdots & \ddots & \dots & \dots \\ 6 & 6 & 6 & \dots & \ddots & 24 & 5 \\ 5 & 6 & 6 & \dots & \dots & 5 & 24 \end{pmatrix}$$

$$= \frac{2(6n+16)}{n+16} \times det \begin{pmatrix} 19 & 0 & 1 & 1 & \dots & \dots & 0 \\ 0 & 19 & 0 & 1 & \dots & \dots & 1 \\ 1 & 0 & 19 & 0 & \dots & \dots & 1 \\ 1 & 1 & 0 & 19 & \dots & \dots & 1 \\ \vdots & \vdots & \dots & \vdots & \ddots & \dots & \dots \\ 1 & 1 & 1 & \dots & \ddots & 19 & 0 \\ 0 & 1 & 1 & \dots & \dots & 0 & 19 \end{pmatrix}_{n \times n}$$

$$= \frac{(6n+16)}{8} \times [(9+4\sqrt{5})^n + (9-4\sqrt{5})^n - 2]. \tag{10.3}$$

Equations (10.2) and (10.3) are substituted into equation (10.1) to produce the result. □

11. Complexity of the Prisms $P(C_3^n)$ of the friendship graph C_3^n

Definition 11.1. The friendship graph C_3^n *is a planner undirected graph constructed by joining* n *copies of cycle graph* C_3 *with a common vertex, so* $|V(C_3^n)| = 2n+1$ *and* $|E(C_3^n)| = 3n$. **The prism** $P(C_3^n)$ **of the friendship graph** C_3^n *has number of vertices* $|V[P(C_3^n)]| = 4n+2$ *and edges* $|E[P(C_3^n)]| = 8n+1$, *see Fig. 9.*

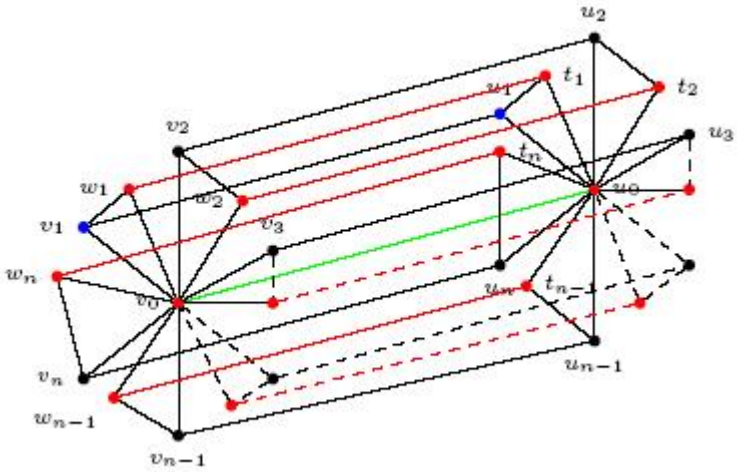

Figure 9. The prism $P(C_3^n)$ of the friendship graph.

Theorem 11.2. *For $n \geq 2$, the number of spanning trees of the prism $P(C_3^n)$ of the friendship graph C_3^n is given by: $\tau[P(C_3^n)] = (2n+3)3^{2n-1}5^n$*

Proof. Applying Lemma 1.1, we have:

$$\tau[P(C_3^n)] = \frac{1}{(4n+2)^2} det \begin{pmatrix} P_{n+1.n+1} & Q_{n+1.n} & S_{n+1.n+1} & E_{n+1.n} \\ Q^t & R_{n.n} & E^t & F_{n.n} \\ S & E & P & Q \\ E^t & F & Q^t & R \end{pmatrix}, \text{ where}$$

$$P = \begin{pmatrix} 2n+2 & 0 & 0 & \dots & \dots & 0 & 0 \\ 0 & 4 & 1 & 1 & \dots & \dots & 1 \\ 0 & 1 & 4 & 1 & \dots & \dots & 1 \\ \vdots & \dots & \dots & \dots & \ddots & \ddots & \vdots \\ 0 & 1 & \dots & \dots & 1 & 4 & 1 \\ 0 & 1 & 1 & \dots & \dots & 1 & 4 \end{pmatrix}, \; Q = \begin{pmatrix} 0 & 0 & 0 & \dots & \dots & 0 \\ 0 & 1 & 1 & \dots & \dots & 0 \\ 1 & 0 & 1 & \dots & \dots & 1 \\ \vdots & \dots & \ddots & \ddots & \ddots & \vdots \\ 1 & \dots & \dots & 1 & 0 & 1 \\ 1 & 1 & \dots & \dots & 1 & 0 \end{pmatrix},$$

$$R = \begin{pmatrix} 4 & 1 & \dots & \dots & 1 \\ 1 & 4 & 1 & \dots & 1 \\ 1 & 1 & 4 & \dots & 1 \\ \vdots & \dots & \dots & \ddots & \vdots \\ 1 & \dots & \dots & 4 & 1 \\ 1 & \dots & \dots & 1 & 4 \end{pmatrix},$$

$$S = \begin{pmatrix} 0 & 1 & \dots & \dots & 1 & 1 \\ 1 & 0 & 1 & \dots & \dots & 1 \\ 1 & 1 & 0 & \dots & \dots & 1 \\ \vdots & \dots & \dots & \ddots & \ddots & \vdots \\ 1 & 1 & \dots & \dots & 0 & 1 \\ 1 & 1 & \dots & \dots & 1 & 0 \end{pmatrix}, \quad E = \begin{pmatrix} 1 & 1 & \dots & \dots & 1 \\ 1 & 1 & \dots & \dots & 1 \\ 1 & 1 & \dots & \dots & 1 \\ \vdots & \dots & \ddots & \ddots & \vdots \\ 1 & 1 & \dots & \dots & 1 \\ 1 & 1 & \dots & \dots & 1 \end{pmatrix} \text{ and}$$

$$F = \begin{pmatrix} 0 & 1 & 1 & \dots & \dots & 1 \\ 1 & 0 & 1 & \dots & \dots & 1 \\ 1 & 1 & 0 & \dots & \dots & 1 \\ \vdots & \dots & \dots & \ddots & \ddots & \vdots \\ 1 & 1 & \dots & \dots & 0 & 1 \\ 1 & 1 & \dots & \dots & 1 & 0. \end{pmatrix}$$

By using the same method in Theorem 10.2, and equation (10.1), we have:

$$det(L+M) = (4n+2) \times det \begin{pmatrix} 3 & 2 & 2 & 2 & \dots & \dots & 2 \\ 2 & 3 & 2 & 2 & \dots & \dots & 2 \\ 2 & 2 & 3 & 2 & \dots & \dots & 2 \\ 2 & 2 & 2 & 3 & \dots & \dots & 2 \\ \vdots & \vdots & \dots & \dots & \ddots & \ddots & \vdots \\ 2 & 2 & 2 & \dots & \ddots & 3 & 2 \\ 2 & 2 & 2 & \dots & \dots & 2 & 3 \end{pmatrix}_{n\times n}$$

$$\times det \begin{pmatrix} 3 & 0 & 0 & 0 & \dots & \dots & 0 \\ 0 & 3 & 0 & 0 & \dots & \dots & 0 \\ 0 & 0 & 3 & 0 & \dots & \dots & 0 \\ 0 & 0 & 0 & 3 & \dots & \dots & 0 \\ \vdots & \vdots & \dots & \dots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \ddots & 3 & 0 \\ 0 & 0 & 0 & \dots & \dots & 0 & 3 \end{pmatrix}_{n\times n}$$

$$= (4n+2)(2n+1)3^n. \tag{11.1}$$

Similarly

$$det(L-M) = 2 \times det \begin{pmatrix} 5 & 2 & 2 & 2 & \dots & \dots & 2 \\ 2 & 5 & 2 & 2 & \dots & \dots & 2 \\ 2 & 5 & 5 & 2 & \dots & \dots & 2 \\ 2 & 2 & 2 & 5 & \dots & \dots & 2 \\ \vdots & \vdots & \dots & \vdots & \ddots & \ddots & \vdots \\ 2 & 2 & 2 & \dots & \ddots & 5 & 2 \\ 2 & 2 & 2 & \dots & \dots & 2 & 5 \end{pmatrix}_n$$

$$\times n \times det \begin{pmatrix} 5 & 0 & 0 & 0 & \dots & \dots & 0 \\ 0 & 5 & 0 & 0 & \dots & \dots & 0 \\ 0 & 0 & 5 & 0 & \dots & \dots & 0 \\ 0 & 0 & 0 & 5 & \dots & \dots & 0 \\ \vdots & \vdots & \dots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \ddots & 5 & 0 \\ 0 & 0 & 0 & \dots & \dots & 0 & 5 \end{pmatrix}_{n \times n}$$

$$= 2(2n+3)3^{n-1}5^n. \tag{11.2}$$

Substituting the equations (11.1) and (11.2) in the equation (10.1), we obtain the result. □

12. Complexity of the Prisms $P(FL_n)$ of the flower graph FL_n

Definition 12.1. The flower graph FL_n, $n \geq 3$ *is the graph obtained from a helm* H_n *by joining each pendant vertex to the center of the helm [5].* **The prism** $P(FL_n)$ **of the flower graph** FL_n *has number of vertices* $|V[P(FL_n)]| = 4n+2$ *and edges* $|E[P(FL_n)]| = 10n+1$, *see Fig. 10.*

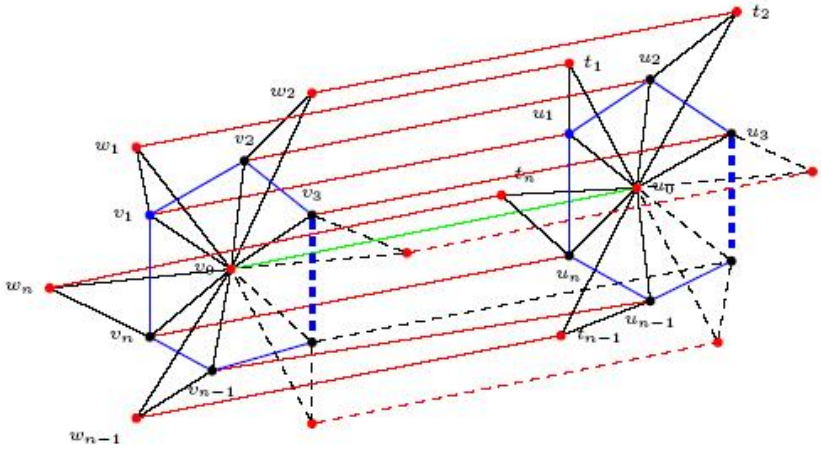

Figure 10. The prism $P(FL_n)$ of the flower graph FL_n.

Theorem 12.2. *For a positive integer $n \geq 2$, the number of spanning trees of the prism $P(FL_n)$ of the flower graph FL_n is given by:*

$$\tau[P(FL_n)] = \frac{7^{n-1}(8n+7)(2n+3)}{2^n(6n+3)}[(23+\sqrt{465})^n+(23-\sqrt{465})^n-2^{1+3n}]$$

Proof. Applying Lemma 1.1, we have:

$$\tau[P(FL_n)] = \frac{1}{(4n+2)^2} det \begin{pmatrix} P_{n+1.n+1} & Q_{n+1.n} & S_{n+1.n+1} & E_{n+1.n} \\ Q^t & R_{n.n} & E^t & F_{n.n} \\ S & E & P & Q \\ E^t & F & Q^t & R \end{pmatrix},$$

where

$$P = \begin{pmatrix} 2n+2 & 0 & 0 & \dots & \dots & 0 \\ 0 & 6 & 0 & 1 & \dots & 0 \\ 0 & 0 & 6 & 0 & \dots & 1 \\ \vdots & \dots & \dots & \ddots & \ddots & \vdots \\ 0 & 1 & \dots & \dots & 6 & 0 \\ 0 & 0 & 1 & \dots & 0 & 6 \end{pmatrix}, \; Q = \begin{pmatrix} 0 & 0 & 0 & \dots & \dots & 0 \\ 0 & 1 & 1 & \dots & \dots & 0 \\ 1 & 0 & 1 & \dots & \dots & 1 \\ \vdots & \dots & \dots & \ddots & \ddots & \vdots \\ 1 & 1 & \dots & \dots & 0 & 1 \\ 1 & 1 & \dots & \dots & 1 & 0, \end{pmatrix}$$

$$R = \begin{pmatrix} 4 & 1 & 1 & \dots & \dots & 1 \\ 1 & 4 & 1 & \dots & \dots & 1 \\ 1 & 1 & 4 & \dots & \dots & 1 \\ \vdots & \dots & \dots & \ddots & \ddots & \vdots \\ 1 & 1 & \dots & \dots & 4 & 1 \\ 1 & 1 & \dots & \dots & 1 & 4 \end{pmatrix},$$

$$S = \begin{pmatrix} 0 & 1 & 1 & \dots & \dots & 1 \\ 1 & 0 & 1 & \dots & \dots & 1 \\ 1 & 1 & 0 & \dots & \dots & 1 \\ \vdots & & & \ddots & \ddots & \vdots \\ 1 & 1 & \dots & \dots & 0 & 1 \\ 1 & 1 & \dots & \dots & 1 & 0 \end{pmatrix}, \; E = \begin{pmatrix} 1 & 1 & 1 & \dots & \dots & 1 \\ 1 & 1 & 1 & \dots & \dots & 1 \\ 1 & 1 & 1 & \dots & \dots & 1 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 1 & 1 & \dots & \dots & 1 & 1 \\ 1 & 1 & 1 & \dots & \dots & 1 \end{pmatrix} \text{and}$$

$$F = \begin{pmatrix} 0 & 1 & 1 & \dots & \dots & 1 \\ 1 & 0 & 1 & \dots & \dots & 1 \\ 1 & 1 & 0 & \dots & \dots & 1 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 1 & 1 & \dots & \dots & 0 & 1 \\ 1 & 1 & \dots & \dots & 1 & 0 \end{pmatrix}.$$

By using the same method in Theorem 10.2, and Eq.(10.1), we have:

$$det(L+M) = (4n+2) \times det \begin{pmatrix} 15 & 8 & 8 & 8 & \dots & \dots & 8 \\ 8 & 15 & 8 & 8 & \dots & \dots & 8 \\ 8 & 8 & 15 & 8 & \dots & \dots & 8 \\ 8 & 8 & 8 & 15 & \dots & \dots & 8 \\ \vdots & \vdots & \dots & \vdots & \ddots & \dots & \dots \\ 8 & 8 & 8 & \dots & \dots & 15 & 8 \\ 8 & 8 & 8 & \dots & \dots & 8 & 15 \end{pmatrix}_{n \times n}$$

$$= (4n+2)(8n+7)7^{n-1}. \tag{12.1}$$

Similarly,

$$det(L-M) = 2 \times det \begin{pmatrix} 33 & 6 & 10 & 10 & \dots & \dots & 6 \\ 6 & 33 & 6 & 10 & \dots & \dots & 10 \\ 10 & 6 & 33 & 6 & \dots & \dots & 10 \\ 10 & 10 & 6 & 33 & \dots & \dots & 10 \\ \vdots & \vdots & \dots & \dots & \ddots & \ddots & \vdots \\ 10 & 10 & \dots & \dots & \dots & 33 & 6 \\ 6 & 10 & \dots & \dots & \dots & 6 & 33 \end{pmatrix}_{n \times n}$$

$$= \frac{2^{2n+1}(10n+15)}{(4n+15)} \det \begin{pmatrix} 27/4 & 0 & 1 & \dots & \dots & 0 \\ 0 & 27/4 & 0 & \dots & \dots & 1 \\ 1 & 0 & 27/4 & 0 & \dots & 1 \\ 1 & 1 & 0 & 27/4 & \dots & 1 \\ \vdots & \dots & \dots & \dots & \ddots & \vdots \\ 1 & 1 & \dots & \dots & 27/4 & 0 \\ 0 & 1 & \dots & \dots & 0 & 27/4 \end{pmatrix}_{n\times n}$$

$$= \frac{2^{2n+1}(10n+15)}{(4n+15)} \times \frac{2}{15}(4n+15)[T_n(\frac{23}{8})-1]$$

$$= \frac{2^{2n+1}(2n+3)}{3}[(\frac{23+\sqrt{465}}{8})^n + (\frac{23-\sqrt{465}}{8})^n - 2]. \tag{12.2}$$

Substituting the equations (12.1) and (12.2) in the equation (10.1), we get the result. □

13. Complexity of the prism $P(SF_n)$ of the planner sunflower graph

For calculating the numbers of spanning trees of the following two graphs, we need the following theorem.

Theorem 13.1. *For a positive integer $n \geq 5$, let the symmetric circulant matrix E_n be defined as*

$$E_n = \begin{pmatrix} a & b & 1 & 0 & \dots & \dots & 0 & 1 & b \\ b & a & b & 1 & \dots & \dots & 0 & 0 & 1 \\ 1 & b & a & b & \dots & \dots & \dots & 0 & 0 \\ \vdots & \dots & \dots & \dots & \ddots & \dots & \ddots & \ddots & \vdots \\ 1 & 0 & 0 & \dots & \dots & \dots & b & a & b \\ b & 1 & 0 & \dots & \dots & \dots & 1 & b & a \end{pmatrix}_{n\times n},$$

then

$$det(E_n) = \begin{cases} [a+2(b+1)] \prod_{i=1}^{\lfloor \frac{n}{2} \rfloor} (a + 2b\cos\frac{2\pi i}{n} + 2\cos\frac{4\pi i}{n})^2, & \text{if } n \text{ odd;} \\ [a^2+4(a-b^2+1)] \prod_{i=1}^{\frac{n}{2}-1} (a + 2b\cos\frac{2\pi i}{n} + 2\cos\frac{4\pi i}{n})^2, & \text{if } n \text{ even.} \end{cases}$$

(13.1)

Proof. Let F be the symmetric circulant matrix with the first row $(t_0, t_1, t_2, \ldots, t_2, t_1)$ i.e.,

$$F = \begin{pmatrix} t_0 & t_1 & t_2 & \ldots & \ldots & t_2 & t_1 \\ t_1 & t_0 & t_1 & \ldots & \ldots & t_3 & t_2 \\ t_2 & t_1 & t_0 & \ldots & \ldots & t_2 & t_3 \\ \vdots & \ldots & \ldots & \ddots & \ddots & \ddots & \vdots \\ t_2 & t_3 & \ldots & \ldots & \ldots & t_0 & t_1 \\ t_1 & t_2 & t_3 & \ldots & \ldots & t_1 & t_0 \end{pmatrix}_{n\times n}.$$

For any $n \times n$ matrix F with eigenvalues $\lambda_0, \lambda_1, \cdots, \lambda_{n-1}$ we have $det(F) = \prod_{i=0}^{n-1} \lambda_i$ [1]. The eigenvectors of the symmetric circulant matrix F are $(1, w, w^2, \cdots, w^{n-1})^t$ where w is the primitive $n-$th root of unity. In addition, the eigenvalues of this symmetric circulant matrix are known explicitly [6]. Defined $w_k = \frac{2\pi k}{n} for k \geq 0$, then the eigenvalues of this symmetric circulant matrix are given by:

(a) for n odd:

$$\lambda_0 = t_0 + 2\sum_{j=1}^{\lfloor \frac{n}{2} \rfloor} t_j,$$

$$\lambda_k = t_0 + 2\sum_{j=1}^{\lfloor \frac{n}{2} \rfloor} t_j \cos(w_k j), for 1 \leq k \leq \lfloor \frac{n}{2} \rfloor,$$

with $\lambda_{n-k} = \lambda_k for 1 \leq k \leq \lfloor \frac{n}{2} \rfloor$, where $\lfloor x \rfloor$ denotes the greatest integer that is less than or equal to x.
(b) for n even

$$\lambda_0 = t_0 + 2\sum_{j=1}^{\frac{n}{2}-1} t_j + t_{n/2},$$

$$\lambda_k = t_0 + 2\sum_{j=1}^{\frac{n}{2}-1} t_j \cos(w_k j) + (-1)^k t_{n/2}, for\ 1 \leq k \leq \frac{n}{2},$$

with $\lambda_{n-k} = \lambda_k for 1 \leq k \leq \frac{n}{2}$.

Comparing the matrix E_n with the matrix F_n we get the following: $t_j = 0 \quad for \quad 3 \leq j \leq \frac{n}{2}$, so

$$\lambda_0 = t_0 + 2(t_1 + t_2) = a + 2(b+1) \quad if \quad n \geq 5.$$

For $n \geq 5$ and $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$ or $1 \leq k \leq \frac{n}{2}$, we infer the following:

$\lambda_k = \lambda_{n-k} = t_0 + 2[t_1 \cos(w_k) + t_2 \cos(2w_k)] = a + 2b\cos(\frac{2\pi k}{n}) + 2\cos(\frac{4\pi k}{n})$ then,

$$det(E_n) = \prod_{i=0}^{n-1} \lambda_i = \begin{cases} \lambda_0 \prod_{i=1}^{\lfloor \frac{n}{2} \rfloor} (\lambda_i)^2 & \text{if } n \text{ odd} \\ \lambda_0 \lambda_{\frac{n}{2}} \prod_{i=1}^{\frac{n}{2}-1} (\lambda_i)^2 & \text{if } n \text{ even} \end{cases}$$

then,

$$det(E_n) = \begin{cases} [a + 2(b+1)] \prod_{i=1}^{\lfloor \frac{n}{2} \rfloor} (a + 2b\cos\frac{2\pi i}{n} + 2\cos\frac{4\pi i}{n})^2 & \text{if } n \text{ odd} \\ [a^2 + 4(a - b^2 + 1)] \prod_{i=1}^{\frac{n}{2}-1} (a + 2b\cos\frac{2\pi i}{n} + 2\cos\frac{4\pi i}{n})^2 & \text{if } n \text{ even} \end{cases}$$

• For $n = 3$, the symmetric circulant matrix E_3 has a special form with the first row $(a, b+1, b+1)$, then $\lambda_0 = t_0 + 2t_1 = a + 2(b+1) and \lambda_1 = t_0 + 2t_1 \cos(\frac{2\pi}{3}) = t_0 - t_1 = a - b - 1$.

Thus, $det(E_3) = \prod_{i=0}^{2} \lambda_i = \lambda_0(\lambda_1)^2 = (a+2b+2)(a-b-1)^2$.

• For $n = 4$, the symmetric circulant matrix E_4 has a special form with the first row $(a, b, 2, b)$, then:

$\lambda_0 = t_0 + 2t_1 + t_2 = a + 2b + 2$,

$\lambda_1 = t_0 + 2t_1 \cos(\frac{\pi}{2}) - t_2 = t_0 - t_2 = a - 2$,

$\lambda_2 = t_0 + 2t_1 \cos(\pi) + t_2 = t_0 - 2t_1 + 2 = a - 2b + 2$.

Thus, $det(E_4) = \prod_{i=0}^{3} \lambda_i = \lambda_0\lambda_2(\lambda_1)^2 = (a^2 + 4a - 4b^2 + 4)(a-2)^2$. □

Definition 13.2. The planner sunflower graph SF_n, $n \geq 3$ *is the graph obtained from a wheel* $W_n = \{v_0, v_1, v_2, \cdots, v_n\}$ *by adding a new vertex* $w_i, i \in [1, n]$ *corresponding to each vertex on the rim of* W_n *and joining each new vertex* w_i *to the vertices* v_i *and* v_{i+1} *of the wheel. The planner sunflower graph* SF_n *has number of vertices* $|V(SF_n)| = 2n+1$, *edges* $|E(SF_n)| = 4n$ *and the center vertex* v_0 *has degree* $d(v_0) = n$ *[5].* **The prism** $P(SF_n)$ **of the planner sunflower graph** *has number of vertices* $|V[P(SF_n)]| = 4n+2$ *and edges* $|E[P(SF_n)]| = 8n+1$, *see Fig. 11.*

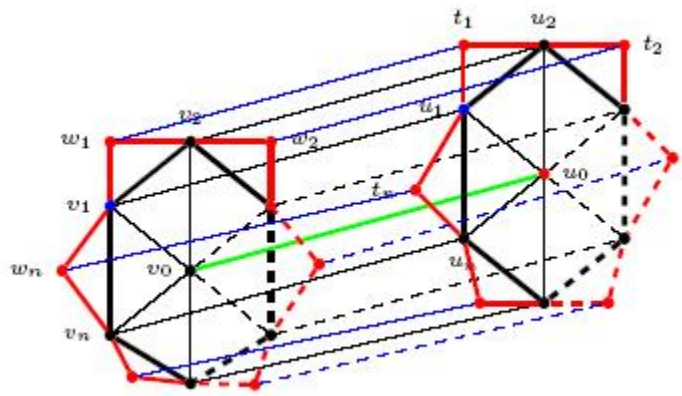

Figure 11. The prism $P(SF_n)$ of the planner sunflower graph SF_n.

Theorem 13.3. *For a positive integer* $n \geq 3$, *the number of spanning trees of the prism* $P(SF_n)$ *of the planner sunflower graph* SF_n *is given by:*

$$\tau[P(SF_n)] = \begin{cases} \dfrac{2^{2n+2}(5n+11)}{11}[T_n(\dfrac{19}{8})-1]\displaystyle\prod_{i=1}^{\lfloor\frac{n}{2}\rfloor}[8-6\cos(\dfrac{2\pi i}{n})]^2 & \textit{if } n \textit{ odd;} \\ \dfrac{72^{2n+3}(5n+11)}{11}[T_n(\dfrac{19}{8})-1]\displaystyle\prod_{i=1}^{\frac{n}{2}-1}[8-6\cos(\dfrac{2\pi i}{n})]^2 & \textit{if } n \textit{ even.} \end{cases}$$

Proof. $$\tau[P(SF_n)] = \frac{1}{(4n+2)^2} det \begin{pmatrix} P_{n+1.n+1} & Q_{n+1.n} & S_{n+1.n+1} & E_{n+1.n} \\ Q^t & R_{n.n} & E^t & F_{n.n} \\ S & E & P & Q \\ E^t & F & Q^t & R \end{pmatrix},$$

where

$$P = \begin{pmatrix} n+2 & 0 & 0 & \dots & \dots & 0 \\ 0 & 7 & 1 & \dots & \dots & 1 \\ 0 & 1 & 7 & \dots & \dots & 1 \\ \vdots & & & \ddots & \ddots & \vdots \\ 0 & 1 & \dots & \dots & 7 & 1 \\ 0 & 1 & \dots & \dots & 1 & 7 \end{pmatrix}, \; Q = \begin{pmatrix} 1 & 1 & 1 & \dots & \dots & 1 \\ 0 & 1 & 1 & \dots & \dots & 0 \\ 0 & 0 & 1 & \dots & \dots & 1 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 1 & 1 & \dots & \dots & 0 & 1 \\ 1 & 1 & \dots & \dots & 0 & 0, \end{pmatrix}$$

$$R = \begin{pmatrix} 4 & 1 & 1 & \dots & \dots & 1 \\ 1 & 4 & 1 & \dots & \dots & 1 \\ 1 & 1 & 4 & \dots & \dots & 1 \\ \vdots & & & \ddots & \ddots & \vdots \\ 1 & 1 & \dots & \dots & 4 & 1 \\ 1 & 1 & \dots & \dots & 1 & 4 \end{pmatrix}, \; S = \begin{pmatrix} 0 & 1 & 1 & \dots & \dots & 1 \\ 1 & 0 & 1 & \dots & \dots & 1 \\ 1 & 1 & 0 & \dots & \dots & 1 \\ \vdots & & & \ddots & \ddots & \vdots \\ 1 & 1 & \dots & \dots & 0 & 1 \\ 1 & 1 & \dots & \dots & 1 & 0 \end{pmatrix},$$

$$E = \begin{pmatrix} 1 & 1 & 1 & \dots & \dots & 1 \\ 1 & 1 & 1 & \dots & \dots & 1 \\ 1 & 1 & 1 & \dots & \dots & 1 \\ \vdots & & & \ddots & \ddots & \vdots \\ 1 & 1 & \dots & \dots & 1 & 1 \\ 1 & 1 & \dots & \dots & 1 & 1 \end{pmatrix} \text{ and } F = \begin{pmatrix} 0 & 1 & 1 & \dots & \dots & 1 \\ 1 & 0 & 1 & \dots & \dots & 1 \\ 1 & 1 & 0 & \dots & \dots & 1 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 1 & 1 & \dots & \dots & 0 & 1 \\ 1 & 1 & \dots & \dots & 1 & 0 \end{pmatrix}.$$

$$\tau[P(SF_n)] = \frac{1}{(4n+2)^2} det \begin{pmatrix} L & M \\ M & L \end{pmatrix} = \frac{1}{(4n+2)^2}[det(L+M)\times det(L-M)]. \tag{13.2}$$

$$det(L+M) = det \begin{pmatrix} n+2 & 1 & 1 & 1 & \dots & \dots & 1 & 2 & 2 & 2 & \dots & \dots & 2 \\ 1 & 7 & 1 & 2 & \dots & \dots & 1 & 1 & 2 & 2 & \dots & \dots & 1 \\ 1 & 1 & 7 & 1 & \dots & \dots & 2 & 1 & 1 & 2 & \dots & \dots & 2 \\ 1 & 2 & 1 & 7 & \dots & \dots & 2 & 2 & 1 & 1 & \dots & \dots & 2 \\ \vdots & \vdots & \dots & \vdots & \ddots & \dots & \vdots & \dots & \vdots & \dots & \ddots & \ddots & \vdots \\ 1 & 2 & 2 & \dots & \dots & 7 & 1 & 2 & 2 & \dots & \dots & 1 & 2 \\ 1 & 1 & 2 & \dots & \dots & 1 & 7 & 2 & 2 & \dots & \dots & 1 & 1 \\ 2 & 1 & 1 & 2 & \dots & \dots & 2 & 4 & 2 & 2 & \dots & \dots & 2 \\ 2 & 2 & 1 & 1 & \dots & \dots & 2 & 2 & 4 & 2 & \dots & \dots & 2 \\ 2 & 2 & 2 & 1 & \dots & \dots & 2 & 2 & 2 & 4 & \dots & \dots & 2 \\ \vdots & \vdots & \dots & \ddots & \dots & \vdots & \vdots & \dots & \ddots & \dots & \ddots & \ddots & \vdots \\ 2 & 2 & 2 & \dots & \dots & 1 & 1 & 2 & 2 & \dots & \dots & 4 & 2 \\ 2 & 1 & 2 & \dots & \dots & 2 & 1 & 2 & 2 & \dots & \dots & 2 & 4 \end{pmatrix}_{(2n+1)\times(2n+1)}$$

$$= (4n+2) \times det \begin{pmatrix} 12 & 1 & 4 & 4 & \dots & \dots & 1 \\ 1 & 12 & 1 & 4 & \dots & \dots & 4 \\ 4 & 1 & 12 & 1 & \dots & \dots & 4 \\ \vdots & \dots & \dots & \dots & \ddots & \ddots & \vdots \\ 4 & 4 & 4 & \dots & \ddots & 12 & 1 \\ 1 & 4 & 4 & \dots & \dots & 1 & 12 \end{pmatrix}$$

$$= \frac{(4n+2)^2}{3n+2} \times det \begin{pmatrix} 11 & 0 & 3 & 3 & \dots & \dots & 0 \\ 0 & 11 & 0 & 3 & \dots & \dots & 3 \\ 3 & 0 & 11 & 0 & \dots & \dots & 3 \\ \vdots & \dots & \dots & \dots & \ddots & \ddots & \vdots \\ 3 & 3 & 3 & \dots & \ddots & 11 & 0 \\ 0 & 3 & 3 & \dots & \dots & 0 & 11 \end{pmatrix}_{n\times n}$$

$$= \frac{(4n+2)^2}{2} \times det \begin{pmatrix} 8 & -3 & 0 & 0 & \dots & \dots & -3 \\ -3 & 8 & -3 & 0 & \dots & \dots & 0 \\ 0 & -3 & 8 & -3 & \dots & \dots & 0 \\ \vdots & \dots & \dots & \dots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \ddots & 8 & -3 \\ -3 & 0 & 0 & \dots & \dots & -3 & 8 \end{pmatrix}_{n\times n} .$$

The matrix in $(L+M)$ is symmetric circulant. Applying Theorem 13.1, we have:

$$det(L+M) = \begin{cases} (4n+2)^2 \prod_{i=1}^{\lfloor \frac{n}{2} \rfloor} [8 - 6\cos(\frac{2\pi i}{n})]^2 & \text{if } n \text{ odd,} \\ 14(4n+2)^2 \prod_{i=1}^{\frac{n}{2}-1} [8 - 6\cos(\frac{2\pi i}{n})]^2 & \text{if } n \text{ even.} \end{cases}$$

(13.3)

Similarly, we can find

$$det(L-M) = 2 \times det \begin{pmatrix} 24 & 1 & 5 & 5 & \dots & \dots & 1 \\ 1 & 24 & 1 & 5 & \dots & \dots & 5 \\ 5 & 1 & 24 & 1 & \dots & \dots & 5 \\ 5 & 5 & 1 & 24 & \dots & \dots & 5 \\ \vdots & \vdots & \dots & \vdots & \ddots & \dots & \dots \\ 5 & 5 & 5 & \dots & \ddots & 24 & 1 \\ 1 & 5 & 5 & \dots & \dots & 1 & 24 \end{pmatrix}$$

$$=2\frac{4^n(5n+11)}{(4n+11)} \times det \begin{pmatrix} 23/4 & 0 & 1 & \dots & \dots & 0 \\ 0 & 23/4 & 0 & 1 & \dots & 1 \\ 1 & 0 & 23/4 & 0 & \dots & 1 \\ \vdots & \dots & \dots & \dots & \ddots & \dots \\ \vdots & \dots & \vdots & \ddots & \dots & \dots \\ 1 & 1 & \dots & \ddots & 23/4 & 0 \\ 0 & 1 & \dots & \dots & 0 & 23/4 \end{pmatrix}$$

$$det(L-M) = 2 \times \frac{4^n(5n+11)}{(4n+11)} \times \frac{2(4n+11)}{11}[T_n(\frac{19}{8}) - 1] = \frac{2^{2n+2}(5n+11)}{11}[T_n(\frac{19}{8}) - 1].$$

(13.4)

Substituting the equations (13.3) and (13.4) in the equation (13.2) we get the result. □

14. Complexity of the prism $P[T(C_n)]$ of the total graph $T(C_n)$ of a cycle C_n

Definition 14.1. **The total graph** $T(C_n)$ *of a cycle* C_n *is the graph whose vertex set is* $V(C_n) \cup E(C_n)$, *and two vertices are adjacent whenever they are either incident or adjacent in* C_n. **The prism** $P[T(C_n)]$ **of the total graph** $T(C_n)$ *of the cycle* C_n *has number of vertices* $|V[P((T(C_n))]| = 4n$ *and edges* $|E[P(T(C_n))]| = 10n$, *see Fig. 12.*

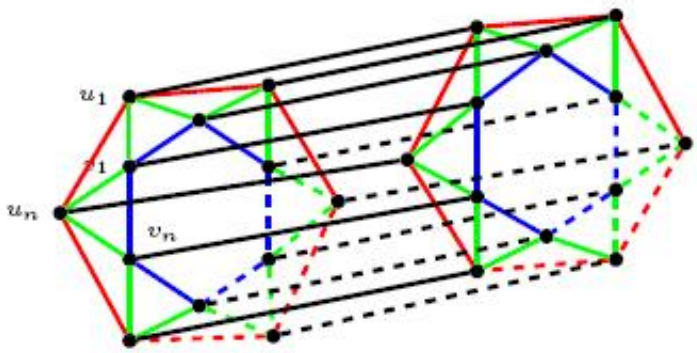

Figure 12. The prism $P[T(C_n)]$ of the total graph $T(C_n)$ of a cycle C_n

Conjecture 14.2. *For* $n \geq 5$, *the number of spanning trees of the prism* $P[T(C_n))$ *of the total graph* $T(C_n)$ *of a cycle* C_n *is given by:*

$$\tau[P(T(C_n))] = \begin{cases} (\frac{12n}{5})\Delta \prod_{i=1}^{\lfloor \frac{n}{2} \rfloor} [36 - 26\cos(\frac{2\pi i}{n}) + 2\cos(\frac{4\pi i}{n})]^2 & \text{if } n \text{ odd} \\ (\frac{768n}{5})\Delta \prod_{i=1}^{\frac{n}{2}-1} [36 - 26\cos(\frac{2\pi i}{n}) + 2\cos(\frac{4\pi i}{n})]^2 & \text{if } n \text{ even} \end{cases}$$

where $\Delta = [(\frac{7+3\sqrt{5}}{2})^n + (\frac{7-3\sqrt{5}}{2})^n - 2]$.

Proof. $\tau[P(T(C_n))] = \dfrac{1}{(4n)^2} det[4nI - \overline{D} + \overline{A}]$

$$= \frac{1}{(4n)^2} det \begin{pmatrix} P & Q & R & J \\ Q^t & P & J & R \\ R & J & P & Q \\ J & R & Q^t & P \end{pmatrix}, \text{ where}$$

$$P = \begin{pmatrix} 6 & 0 & 1 & \dots & \dots & 0 \\ 0 & 6 & 0 & \dots & \dots & 1 \\ 1 & 0 & 6 & \dots & \dots & 1 \\ \vdots & \dots & \dots & \ddots & \ddots & \vdots \\ 1 & 1 & \dots & \dots & 6 & 0 \\ 0 & 1 & \dots & \dots & 0 & 6 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & 1 & 1 & \dots & \dots & 1 & 0 \\ 0 & 0 & 1 & 1 & \dots & \dots & 1 \\ 1 & 0 & 0 & 1 & \dots & \dots & 1 \\ \vdots & \dots & \dots & \ddots & \ddots & \ddots & \vdots \\ 1 & 1 & \dots & \dots & 0 & 0 & 1 \\ 1 & 1 & 1 & \dots & \dots & 0 & 0 \end{pmatrix}$$

and $R = \begin{pmatrix} 0 & 1 & \dots & \dots & 1 \\ 1 & 0 & 1 & \dots & 1 \\ 1 & 1 & 0 & \dots & 1 \\ \vdots & \dots & \ddots & \ddots & \vdots \\ 1 & \dots & \dots & 0 & 1 \\ 1 & \dots & \dots & 1 & 0 \end{pmatrix}$

.

$$\tau[P(T(C_n))] = \frac{1}{(4n)^2} det \begin{pmatrix} L & M \\ M & L \end{pmatrix} = \frac{1}{(4n)^2} [det(L+M)(L-M)]. \tag{14.1}$$

$$det(L+M) = \det \begin{pmatrix} 6 & 1 & 2 & \dots & \dots & 1 & 1 & 2 & 2 & \dots & \dots & 1 \\ 1 & 6 & 1 & \dots & \dots & 2 & 1 & 1 & 2 & \dots & \dots & 2 \\ 2 & 1 & 6 & \dots & \dots & 2 & 2 & 1 & 1 & \dots & \dots & 2 \\ \vdots & \vdots & \dots & \ddots & \ddots & \dots & \vdots & \dots & \vdots & \ddots & \dots & \vdots \\ 2 & 2 & \dots & \dots & 6 & 1 & 2 & 2 & \dots & \dots & 1 & 2 \\ 1 & 2 & \dots & \dots & 1 & 6 & 2 & 2 & \dots & \dots & 1 & 1 \\ 1 & 1 & 2 & \dots & \dots & 2 & 6 & 1 & 2 & \dots & \dots & 1 \\ 2 & 1 & 1 & \dots & \dots & 2 & 1 & 6 & 1 & \dots & \dots & 2 \\ 2 & 2 & 1 & \dots & \dots & 2 & 2 & 1 & 6 & \dots & \dots & 2 \\ \vdots & \vdots & \dots & \ddots & \ddots & \vdots & \vdots & \dots & \ddots & \ddots & \vdots & \vdots \\ 2 & 2 & \dots & \dots & 1 & 1 & 2 & 2 & \dots & \dots & 6 & 1 \\ 1 & 2 & \dots & \dots & 2 & 1 & 1 & 2 & \dots & \dots & 1 & 6 \end{pmatrix}$$

$$= \det \begin{pmatrix} X & Y \\ Y^t & X \end{pmatrix}$$

$$= det[X^2 - Y \times Y^t] = det \begin{pmatrix} 32 & 7 & 17 & 16 & \dots & \dots & 16 & 17 & 7 \\ 7 & 32 & 7 & 17 & 16 & \dots & \dots & 16 & 17 \\ 17 & 7 & 32 & 7 & 17 & \dots & \dots & 16 & 16 \\ 16 & 17 & 7 & 32 & 7 & \dots & \dots & 16 & 16 \\ \vdots & \vdots & \dots & \vdots & \dots & \dots & \ddots & \dots & \vdots \\ 17 & 16 & 16 & \dots & \dots & 17 & 7 & 32 & 7 \\ 7 & 17 & 16 & \dots & \dots & 16 & 17 & 7 & 32 \end{pmatrix}_{n \times n}.$$

Expanding the first seven determinants of $L+M$, yields

$$\text{for } n=3, det(L+M) = 27648 = 2^4(n)^3 \times 64,$$
$$\text{for } n=4, det(L+M) = 451584 = 2^4(n)^3 \times 441,$$
$$\text{for } n=5, det(L+M) = 6050000 = 2^4(n)^3 \times 3025,$$
$$\text{for } n=6, det(L+M) = 71663616 = 2^4(n)^3 \times 20736,$$
$$\text{for } n=7, det(L+M) = 780003952 = 2^4(n)^3 \times 142129,$$
$$\text{for } n=8, det(L+M) = 7980392448 = 2^4(n)^3 \times 974169.$$

The values $64, 441, 3025, 20736, 142129, 974169, \cdots$ for $n = 3, 4, 5, 6, 7, 8 \cdots$ have the following non-homogeneous recurrence relation $a_{n+2} = 7a_{n+1} - a_n + 2$. Let $a_n = b_n + d$ be the solution of the non-homogeneous recurrence relation, where b_n is the solution of homogeneous recurrence relation.

Substituting in the non-homogeneous recurrence relation, we get $d = \frac{-2}{5}$. Then the general solution of the non-homogeneous recurrence relation is

$$a_n = \alpha(\frac{7+3\sqrt{5}}{2})^n + \beta(\frac{7-3\sqrt{5}}{2})^n - \frac{2}{5}.$$

Using the initial conditions $a_n = 64, 441$ at $n = 3, 4$, respectively, we get $\alpha = \beta = \frac{1}{5}$. Then,

$$(14.2)\ det(L+M) = 2^4 n^3 a_n = \frac{2^4 n^3}{5}[(\frac{7+3\sqrt{5}}{2})^n + (\frac{7-3\sqrt{5}}{2})^n - 2].$$

$$det(L-M) = det \begin{pmatrix} 6 & -1 & 0 & \dots & \dots & -1 & -1 & 0 & 0 & \dots & \dots & -1 \\ -1 & 6 & -1 & \dots & \dots & 0 & -1 & -1 & 0 & \dots & \dots & 0 \\ 0 & -1 & 6 & \dots & \dots & 0 & 0 & -1 & -1 & \dots & \dots & 0 \\ \vdots & \vdots & \dots & \ddots & \dots & \vdots & \dots & \vdots & \vdots & \ddots & \dots & \vdots \\ 0 & 0 & \dots & \dots & 6 & -1 & 0 & 0 & \dots & \dots & -1 & 0 \\ -1 & 0 & \dots & \dots & -1 & 6 & 0 & 0 & \dots & \dots & -1 & -1 \\ -1 & -1 & 0 & \dots & \dots & 0 & 6 & -1 & 0 & \dots & \dots & -1 \\ 0 & -1 & -1 & \dots & \dots & 0 & -1 & 6 & -1 & \dots & \dots & 0 \\ 0 & 0 & -1 & \dots & \dots & 0 & 0 & -1 & 6 & \dots & \dots & 0 \\ \vdots & \dots & \vdots & \ddots & \dots & \vdots & \dots & \dots & \vdots & \ddots & \dots & \vdots \\ 0 & 0 & \dots & \dots & -1 & -1 & 0 & 0 & \dots & \dots & 6 & -1 \\ -1 & 0 & \dots & \dots & 0 & -1 & -1 & 0 & \dots & \dots & -1 & 6 \end{pmatrix}_{2n\times 2n}$$

$$= det \begin{pmatrix} K & H \\ H^t & K \end{pmatrix}$$

$$= det(K^2 - H\times H^t) = det \begin{pmatrix} 36 & -13 & 1 & 0 & \dots & \dots & 0 & 1 & -13 \\ -13 & 36 & -13 & 1 & 0 & \dots & \dots & 0 & 1 \\ 1 & -13 & 36 & -13 & 1 & \dots & \dots & 0 & 0 \\ 0 & 1 & -13 & 36 & -13 & \dots & \dots & 0 & 0 \\ \vdots & \vdots & \dots & \vdots & \dots & \dots & \ddots & \dots & \vdots \\ 1 & 0 & 0 & \dots & \dots & 1 & -13 & 36 & -13 \\ -13 & 1 & 0 & \dots & \dots & 0 & 1 & -13 & 36 \end{pmatrix}_{n\times n}.$$

The matrix $(L - M)$ is symmetric circulant. Applying Theorem 13.1, we conclude the following:

$$det(L-M) = \begin{cases} 3 \times 2^2 \prod_{i=1}^{\lfloor \frac{n}{2} \rfloor} [36 - 26\cos(\frac{2\pi i}{n}) + 2\cos(\frac{4\pi i}{n})]^2 & \text{if } n \text{ odd,} \\ 3 \times 2^8 \prod_{i=1}^{\frac{n}{2}-1} [36 - 26\cos(\frac{2\pi i}{n}) + 2\cos(\frac{4\pi i}{n})]^2 & \text{if } n \text{ even.} \end{cases}$$

(14.3)

From the equations (14.2) and (14.3) in the equation (14.1), we get the result.

• For $n = 3$, the symmetric circulant matrix $(L-M)_{3\times 3}$ has a special form with the first row $(a, b+1, b+1)$, then $det((L-M)_{3\times 3}) = (a+2b+2)(a-b-1)^2 = 12 \times 48^2 = 27648$, therefore $\tau[P(T(C_3))] = 27648 \times 27648 = 27648^2$. • For $n = 4$, the symmetric circulant matrix $(L-M)_{4\times 4}$ has a special form with the first row $(a, b, 2, b)$, then $det((L-M)_{4\times 4}) = (a^2 + 4a - 4b^2 + 4)(a-2)^2 = 12 \times 64 \times 34^2 = 887808$, therefore $\tau[P(T(C_4))] = 887808 \times 887808 = 887808^2$. □

15. Conclusions

The number of spanning trees, complexity, is a very important algebraic invariant of a graph which has been widely studied in many aspects of mathematics, such as algebra and combinatorics. Therefore it has a lot of connections with networks. The greater the quality and perfection in the network, the greater the number of trees spanning found in this network. This leads to greater possibilities for the connection between two nodes, thus ensuring good rigidity and reliability. In this work, we obtained simpler and more explicit formulas for the number of spanning trees of prisms of some path-related graphs: fan graph, ladder graph, the composition $P_n[P_2]$ graph, and book graph. We also obtained straightforward formulas for the complexity of a special class of prisms of cycle-related graphs: wheel graph, gear graph, prism graph, $n-$crossed prism graph,

mirror graph $M(C_n)$ of even cycle C_n, twisted prism, total graph $T(C_n)$ of the cycle C_n, the friendship graph, the flower graph, and planner sunflower graph. These closed formulas are deduced using some basic properties of block matrix, recurrence relation, eigenvalues of circulant matrices, and orthogonal polynomials.

16. Declaration

Availability of data and material

Data used to support the findings of this study are included within the article.

Author's contributions

The author read and approved the final manuscript.

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