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The impact of time delay in the transmission of Japanese encephalitis without vaccination

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Abstract

In this manuscript, the influence of time delay in the transmission of Japanese encephalitis without vaccination model has been studied. The time delay is because of the existence of an incubation period during which the Japanese encephalitis virus reproduces enough in the mosquitoes with the goal that it tends to be transmitted by the mosquitoes to people. The motivation behind this manuscript is to assess the influence of the time delay it takes to infect susceptible human populations after interacting with infected mosquitoes. The steady-state and the threshold value R_0 *of the delay model were resolved. This value assists with setting up the circumstance that ensures the asymptotic stability of relating equilibrium points. Utilizing the delay as a bifurcation parameter, we built up the circumstance for the presence of a Hopf bifurcation. Moreover, we infer an express equation to decide the stability and direction of Hopf bifurcation at endemic equilibrium by using center manifold theory and normal structure strategy. It has been seen that delay plays a vital role in stability exchanging. Furthermore, the presence of Hopf bifurcation is affected by larger values of virus transmission rate from an infected mosquito to susceptible individuals and the natural mortality of humans in a model. Finally, to understand some analytical outcomes, the delay framework is simulated numerically.*

Keywords: *JEV. DDEs. Intrinsic incubation period. Delay differential model. Bifurcation analysis*

1. Introduction

In the current year, the incidence of Japanese encephalitis (JE) has risen significantly across the world. The disease is currently prevalent in over 24 countries in Southeast Asia and the Western Pacific. It is evaluated that over 3 billion individuals are in danger from the Japanese encephalitis virus (JEV) and this disease in numerous nations of Asia is 68,000 medical cases per year, with roughly 13,600 to 20,400 deaths, [1]. The case casualty rate among those with encephalitis can be as high as 30%, [27] and permanent neurologic can happen in 30%-50% of those of encephalitis, Centers for Disease Control, (CDC) 2018. JE is an infection caused by a virus that has been linked to West Nile and St. Louis encephalitis. People are infected with JEV through the bite of infected Culex species mosquitoes, especially Culex tritaeniorhynchus. Mosquitoes and pigs keep the infection going in a cycle. People are coincidental or dead-end hosts because their circulation systems do not contain enough high convergences of JE infection to infect feeding mosquitoes. The transmission of the JE virus occurs primarily in rural agrarian areas, and is frequently linked to rice production and flooding water systems. These conditions can occur near urban centers in some Asian regions. In milder regions of Asia, JE infection transmission is occasional. Human infection normally tops in the mid-year and fall.

Infected mosquitoes can spread the infection for an unprecedented amount of time after being infected. After being bitten by infected mosquitoes, an infected person will show symptoms after a 5-15 day incubation period, [1]. There is no cure for such infection. Treatment is centered around soothing extreme medical signs and supporting the patient to conquer the infection. Protected and valuable immunizations are accessible to prevent JE. WHO suggests that JE inoculation be incorporated into national vaccination plans for all territories where JE disease is perceived as a general medical problem. We can view that the intrinsic incubation period of the infection is impressively long and thus huge. There have been a few JE mathematical models published to date, in which the dynamics of JE infection have been explored and analyzed. First of all Mukhopadhyay et al., [10] have considered a mathematical framework on Japanese encephalitis, in which they talked about the basic properties like positivity, boundedness, and stability of the framework. The basic reproduction number, which is further specified, is a significant quantity that provides information about the potential spread of Japanese encephalitis. Following that, some manuscripts on the mathematical model of Japanese encephalitis infection were produced, including [3, 5, 9, 10]. Likewise, mathematical frameworks have been generally used to quantify the transmission of different infectious infections, such as, [2, 4, 7, 11, 12, 14, 33, 34]. These models are compartmental models where a population is separated into various compartments relating to the various courses of infection. Since transmission dynamics of JEV is still now unexplored, thus due to this pandemic each year numerous individuals are died, because of the attack of JEV. In this way, an investigation of the dynamics of JEV transmission and the methodology for its control is fundamental. The modified model of Japanese encephalitis has been constructed by the work of Tapaswi et al., [9], De et al., [16] and Baniya et al., [13, 17].

Time delays have recently been extended to pandemic systems to gain a better understanding of an ever-increasing number of perplexing marvels for portraying a few aspects of disease transmission. We realize that the impressive characteristic intrinsic incubation time of JEV is available in JE transmission. Time delay has certainly been widely used to frame the incubation time of pathogens of several infectious infections such as malaria, dengue fever, Hepatitis C Virus, and so on, time delay has been extensively used, [8, 14, 15]. The delayed system experiences Hopf bifurcation, periodic solution and once in chaos may likewise happen, [11, 14, 19, 21, 22, 25, 26, 28]. Gandhi et al., [18] have discussed the dynamical characteristic of a VL framework with time delay and established criteria for stability and the presence of Hopf bifurcation. Kumar et al., [23] have constructed the SIR pandemic model with the deterministic time-delayed system. In his work they have taken the Holling type III function as a treatment rate and nonlinear functional as the incidence rate of infection, Finally, Hopf bifurcation occurs in his system. Qi et al., [24] have studied the SEIRS time-delayed framework with vertical transmission and nonlinear incidence rate. They have shown that the vertical transmission and immunity period can affect the dynamics dealing with the SEIRS system. Goel et al., [20] have constructed and analyzed a SIR epidemic framework with time-delayed, saturated functional-type treatment rate and Beddington-DeAngelis-type incidence rate. In his work, they have derived the circumstances for the occurrence of backward bifurcation and Hopf bifurcation.

The implementation of delays in the JE model has, for the most part, focused on the incorporation of the incubation period. Japanese encephalitis model with time delay will have gotten a lot of consideration, since time delay may change the subjective conduct of the frameworks, for example, it can destabilize steady states and therefore lead to periodic solutions by Hopf bifurcation. Baniya et al., [17] have studied the impact of vaccination on the control of JEV, by using basic reproduction number and sensitivity analysis, have discussed the impressive role of immunization on the control of JEV in the human population. Thus, considering the effect of time delay on the study of the DDEs without vaccination of the JEV system with a standard incidence rate is interesting. However, in the existing literature, there are no JEV frameworks with time delay. Inspired by this, the main point of this manuscript is to build up a compartmental framework with a time delay without vaccination that makes susceptible humans infectious after interaction with infected mosquitoes. We study the influence of such time delays on the transmission of Japanese encephalitis.

The rest part of this work is as per the following: In sect. 2, the mathematical formulation and explanation of the essential properties of the JE model with time delay are examined. Sect. 3 deals with the positivity and boundedness of the disease framework as they represent populations. The dynamical behavior of the model, such as the occurrences of steady states, basic reproduction number, and stability analysis of the model are discussed in sect. 4. Sect. 5 manages the presence of Hopf bifurcation and utilizing the Hopf bifurcation hypothesis, we show the event of Hopf bifurcation when time delay goes through the critical point. The direction and stability of Hopf bifurcation at endemic equilibrium are discussed in Sect. 6. Furthermore, numerical simulations and their biological involvement are presented in Sect. 7, and lastly, the manuscript ends with a conclusions section.

2. A mathematical formulation of Japanese encephalitis framework

In inferring the framework conditions, we previously considered a logistic growth rate for mosquito population and standard incidence rate in both human and pig populations. In the mosquito compartment, there is no recovered class, because it acts as a transmitter of JEV only. The mosquito population is exposed to quick change, the size is assumed to be fluctuating, Tapaswi et al., [9]. Without loss of any all-inclusive statement, we expect that in this manuscript there is no recovered class in the human population since we have just considered the time delay to makes susceptible humans infected after interacting with an infected mosquito. Following De et al., [16] it is expected that the human and pig population varying in constant size. According to WHO, JE infection does not spread through direct contact between pig-pig, pig-human, or human-human. Mosquitoes, which serve as transmitters from an infected pig population to both susceptible pig and human populations, help to spread the disease. Let $X_1(t)$, $X_2(t)$ and $X_3(t)$ denote the susceptible, vaccinated and infected human population respectively and N_1 be the total human population at time t which is constant, equal to $N_1 = X_1(t) + X_2(t) + X_3(t)$; $V_1(t)$ and $V_2(t)$ denote the susceptible and infected mosquito population respectively such that $V(t) = V_1(t) + V_2(t)$; $Z_1(t)$ and $Z_2(t)$ denote the susceptible and infected pig population respectively such that $Z_1(t) + Z_2(t) = N_2$ where N_2 is total pig population at time t. In human, suppose continuous vaccination rate v such that $v\mu_h N_1$ is recruitment rate of vaccinated humans and rest $(1-v)$ goes to susceptible class, hence $(1-v)\mu_h N_1$ becomes the recruitment of susceptible humans. Since vaccination does not give 100% protection [1], so δX_2 amount of vaccinated humans goes to susceptible class. The Japanese encephalitis spreads among X_1 and Z_1 from V_2, so disease transmission to human and pig populations should be in the form of $\frac{\beta_1 V_2 X_1}{N_1}$ and $\frac{\beta_3 V_2 Z_1}{N_2}$ respectively. Also, susceptible mosquitoes V_1 gets an infection from infected pig Z_2, so Japanese encephalitis transmission from pig to mosquitoes should be in the form of $\frac{\beta_2 Z_2 V_1}{V}$. Since the growth rate α_1 of mosquitoes decreases as the population reaches carrying capacity K. Therefore, the mosquito population is considered in logistic growth, which is in the form of $\left(\alpha_1 - \frac{rV}{K}\right)V$. Thus, keeping the above discussion in mind, the proposed Japanese encephalitis model can be expressed mathematically using the following system of differential equations:

$$
\begin{cases}
\frac{dX_1}{dt} = \mu_h(1-v)N_1 + \delta X_2 - \frac{\beta_1 V_2 X_1}{N_1} - \mu_h X_1 \\
\frac{dX_2}{dt} = v\mu_h N_1 - (\mu_h + \delta)X_2 \\
\frac{dX_3}{dt} = \frac{\beta_1 V_2 X_1}{N_1} - (\mu_h + \epsilon)X_3 \\
\frac{dV_1}{dt} = \left(\alpha_1 - \frac{rV}{K}\right)V - \frac{\beta_2 Z_2 V_1}{V} - \alpha_2 V_1 \\
\frac{dV_2}{dt} = \frac{\beta_2 Z_2 V_1}{V} - \alpha_2 V_2 \\
\frac{dZ_1}{dt} = \mu_p N_2 - \frac{\beta_3 Z_1 V_2}{N_2} - \mu_p Z_1 \\
\frac{dZ_2}{dt} = \frac{\beta_3 Z_1 V_2}{N_2} - (\mu_p + \gamma_1)Z_2
\end{cases}
\tag{2.1}
$$

Parameter	Descriptions
α_1	Natural birth rate of mosquito population
α_2	The mortality rate of mosquito population
r	Intrinsic growth rate of mosquito population
K	Carrying capacity of mosquito population
μ_p	The mortality rate of pig population
β_1	The rate of disease transmission from infected mosquitoes to susceptible individuals in the human population
β_2	The rate of disease transmission from infected pigs to susceptible individuals in mosquito population
β_3	The rate of disease transmission from infected mosquitoes to susceptible individuals in pig population
γ_1	The rate at which infected pig population get partially recovered and goes to susceptible class
μ_h	The mortality rate of humans
v	Vaccination rate of humans
δ	The rate at which a vaccinated individual falls into the susceptible group of human
ϵ	Death rate of infected individuals in human population due to disease

Table 1: Biological description of the parameters used in the Japanese encephalitis framework (2.1)

In this case, we consider the model without vaccination. It is observed that if the vaccination rate v is zero, the second equation of the system (2.1) is meaningless. Also, if V be the total mosquito population then susceptible mosquitoes can be excluded, So these two equations can be excluded without loss of any all-inclusive statement. This permits us to assault the system (2.1) by examining the subsystem;

$$
\begin{cases}
\frac{dX_1}{dt} = \mu_h N_1 + \delta X_2 - \frac{\beta_1 V_2 X_1}{N_1} - \mu_h X_1 \\
\frac{dX_3}{dt} = \frac{\beta_1 V_2 X_1}{N_1} - (\mu_h + \epsilon) X_3 \\
\frac{dV}{dt} = r\left(1 - \frac{V}{K}\right) V \\
\frac{dV_2}{dt} = \frac{\beta_2 Z_2 (V - V_2)}{V} - \alpha_2 V_2 \\
\frac{dZ_1}{dt} = \mu_p N_2 - \frac{\beta_3 Z_1 V_2}{N_2} - \mu_p Z_1 \\
\frac{dZ_2}{dt} = \frac{\beta_3 Z_1 V_2}{N_2} - (\mu_p + \gamma_1) Z_2
\end{cases} \tag{2.2}
$$

When susceptible humans are bitten by infected mosquitoes, they are not answerable for the clinical side effects, during the intrinsic incubation time frame. To represent a class of people in this time period who are infected but do not show any symptoms. Let τ be the time delay to makes susceptible individuals infectious in the human population after interaction with infected mosquitoes. i.e. intrinsic incubation period. At that point, the quantity of recently happened infectious human per unit time t is

$$
\frac{\beta_1 V_2(t-\tau) X_1(t-\tau)}{N_1}
$$

Hence, the system (2.2), which is a representation of the system (2.1) is converted to a system of delay differential equations (DDEs):

$$
\begin{cases}
\frac{dX_1}{dt} = \mu_h N_1 - \frac{\beta_1 V_2(t-\tau) X_1(t-\tau)}{N_1} - \mu_h X_1 \\
\frac{dX_3}{dt} = \frac{\beta_1 V_2(t-\tau) X_1(t-\tau)}{N_1} - (\mu_h + \epsilon) X_3 \\
\frac{dV}{dt} = r\left(1 - \frac{V}{K}\right) V \\
\frac{dV_2}{dt} = \frac{\beta_2 Z_2 (V - V_2)}{V} - \alpha_2 V_2 \\
\frac{dZ_1}{dt} = \mu_p N_2 - \frac{\beta_3 Z_1 V_2}{N_2} - \mu_p Z_1 \\
\frac{dZ_2}{dt} = \frac{\beta_3 Z_1 V_2}{N_2} - (\mu_p + \gamma_1) Z_2
\end{cases} \tag{2.3}
$$

The framework (2.3) satisfies the initial conditions in the interval $[-\tau, 0]$ defined in the space

$$
\begin{aligned}
C_+ = \Big\{\phi \in C\Big([-\tau, 0], \mathbf{R}_+^6\Big) : X_1(\theta) = \phi_1(\theta),\ X_3(\theta) = \phi_2(\theta),\ V(\theta) = \phi_3(\theta),& \\
V_2(\theta) = \phi_4(\theta), Z_1(\theta) = \phi_5(\theta), Z_2(\theta) = \phi_6(\theta)\Big\}&
\end{aligned} \tag{2.4}
$$

where $\phi = \Big(\phi_1, \phi_2, \phi_3, \phi_4, \phi_5, \phi_6\Big) \in C\Big([-\tau, 0], \mathbf{R}_+^6\Big)$, the Banach space of all continuous mapping in the interval $[-\tau, 0]$ to $\mathbf{R}_+^6$ with $\phi_i(\theta) \geq 0$ $(\theta \in [-\tau, 0], i = 1, 2, 3, 4, 5, 6)$ and $\Big(\phi_1(0), \phi_2(0), \phi_3(0), \phi_4(0), \phi_5(0), \phi_6(0)\Big)$,$\mathbf{R}_+^6 = \Big\{(X_1, X_3, V, V_2, Z_1, Z_2) :$ $X_1 > 0, X_3 > 0, V > 0, V_2 > 0, Z_1 > 0, Z_2 > 0\Big\}$.

3. Positivity and boundedness of solutions

The positivity and boundedness of the system (2.3) as they represent populations are presented in this section. Positivity means that the population cannot be negative, that is, it must always be positive. Boundedness can be defined as a natural constraint to development as a result of limited resources. From the "fundamental theory of functional differential equations" [30], satisfying initial history (2.4), framework (2.3) has a exactly one solution. The boundedness and non-negativity of the solution with an initial history (2.4) are guaranteed by the following theorem:

Theorem 1. *All the solutions $(X_1(t), X_3(t), V(t), V_2(t), Z_1(t), Z_2(t))$ of framework (2.3) starting in $\mathbf{R}_+^6$ are bounded and enter the set* $\Omega = \Big\{(X_1, X_3, V, V_2, Z_1, Z_2) \in \mathbf{R}_+^6 : X_i(t) \geq 0, V(t) \geq 0, V_2(t) \geq 0, Z_j(t) \geq 0, i = 1, 3, j = 1, 2 : \ \Sigma X_i(t) \leq N_1, \ V(t) = K, V_2(t) \leq \frac{K\beta_2 Z_0}{\beta_2 Z_0 + \alpha_2 K}, \ \Sigma Z_j(t) \leq N_2\Big\}$

Remark 1. *In this region Ω, the elementary outcomes such as local existence, uniqueness and continuation of solutions are valid for framework (2.3). Hence, there exists a unique solution $(X_1(t), X_3(t), V(t), V_2(t), Z_1(t), Z_2(t))$ of framework (2.3) starting in the interior of Ω that exists on a maximal interval $[0, \infty)$, if solution remain bounded, [35].*

Theorem 2. *All state variables $(X_1(t), X_3(t), V(t), V_2(t), Z_1(t), Z_2(t))$ of framework (2.3) with initial conditions (2.4) exists in $[-\tau, 0)$ are positive for all t_0.*

Proof. Let us consider the framework (2.3) in vector form by setting

$$Y = (X_1, X_3, V, V_2, Z_1, Z_2)^T \in \mathbf{R}^6$$

and

$$f(Y)=\begin{pmatrix} f_1(Y)\\ f_2(Y)\\ f_3(Y)\\ f_4(Y)\\ f_5(Y)\\ f_6(Y)\end{pmatrix}=\begin{pmatrix} \mu_h N_1-\frac{\beta_1 V_2(t-\tau)X_1(t-\tau)}{N_1}-\mu_h X_1\\ \frac{\beta_1 V_2(t-\tau)X_1(t-\tau)}{N_1}-(\mu_h+\epsilon)X_3\\ r\left(1-\frac{V}{K}\right)V\\ \frac{\beta_2 Z_2(V-V_2)}{V}-\alpha_2 V_2\\ \mu_p N_2-\frac{\beta_3 Z_1 V_2}{N_2}-\mu_p Z_1\\ \frac{\beta_3 Z_1 V_2}{N_2}-(\mu_p+\gamma_1)Z_2 \end{pmatrix}$$

where the mapping $f : C_+ \to \mathbf{R}^6$ and $f \in C^\infty(\mathbf{R}^6)$, then the framework (2.3) reduces to

$$\frac{dY}{dt}=f(Y_t) \tag{3.1}$$

with initial history $Y_t(\phi) = Y(t+\phi), \phi \in [-\tau, 0)$. It is easy to verify that in Eq. (3.1), whenever we choose $Y(\phi) \in C_+$ in such a way that $Y_i = 0$, it is obtained that

$$f_i(Y)|_{Y_i(t)=0, Y_t\in C_+} = f_i(0) \geq 0,\ \forall\ i = 1, 2, 3, 4, 5, 6.$$

Due to lemma [36] and subsection (3.1) in [2], any solution of framework (2.3) with $Y_t(\phi) \in C_+$, say $Y(t) = Y(Y(0), t)$, is such that $Y(t) \in \mathbf{R}^6_{+0}$ $\forall\ t \geq 0$, i.e. it remains non-negative throughout the region $\mathbf{R}^6_+$. □

4. Dynamical behavior of the framework

4.1. Steady-states and basic reproductive number

The Japanese encephalitis disease transmission model formulated in delay framework (2.3) possess two types of equilibria:

1. Virus-free steady-state: The virus-free steady-state (disease-free equilibrium) is denoted by $E_0 = (\bar{X}_1, \bar{X}_3, \bar{V}, \bar{V}_2, \bar{Z}_1, \bar{Z}_2)$ and is given as

   $$\bar{X}_1 = N_1,\quad \bar{X}_3 = 0,\quad \bar{V} = K,\quad \bar{V}_2 = 0,\quad \bar{Z}_1 = N_2,\quad \bar{Z}_2 = 0.$$

   The existence of virus-free steady-state E_0 is obvious. This equilibrium state means that the JE virus cannot spread to the population.

To measure disease transmission potential, the basic reproduction number R_0 can be established by using the next generation matrix method, as described in, [2, 3, 5, 17].

$$R_0 = \frac{\beta_2\beta_3}{\alpha_2(\mu_p+\gamma_1)}$$

Biologically, this parameter R_0 is stated as: the average number of secondary infections generated by a typical single JE infected individual in the entirely susceptible individuals in which a few people have been immunized, [31].

2. Endemic steady-state: The endemic steady-state (positive steady-state) is denoted by $E_1 = (X_1^*, X_3^*, V^*, V_2^*, Z_1^*, Z_2^*)$ and is given as

$$\begin{aligned}
X_1^* &= \frac{N_1^2\mu_h}{N_1\mu_h+\beta_3V_2^*}, \quad X_3^* = \frac{\beta_1V_2^*X_1^*}{N_1(\mu_h+\epsilon)}, \quad V^* = K, \\
V_2^* &= \frac{\beta_2Z_2^*K}{\alpha_2K+\beta_2Z_2^*} \\
Z_1^* &= \frac{N_1(\mu_p+\gamma_1)Z_2^*}{\beta_3V_2^*}, \quad Z_2^* = \frac{KN_2\alpha_2(\mu_p+\gamma_1)(R_0-1)}{N_2(\mu_p+\gamma_1)+\beta_3K}
\end{aligned}$$

It is observed that steady-state E_1 is feasible if $R_0 > 1$. This steady-state means that the infection would be spread to the susceptible population if infected humans, mosquitoes, and pigs remain in the system.

4.2. Stability analysis

In the analysis of dynamical systems, one approach to consider the local stability of steady states through linearization. This is built by characterizing $E = (\tilde{X}_1, \tilde{X}_3, \tilde{V}, \tilde{V}_2, \tilde{Z}_1, \tilde{Z}_2)$ as equilibrium of framework (2.3). Then the linearized framework of (2.3) at equilibrium is given by

$$(4.1)\quad \begin{cases} \frac{dX_1}{dt} = -\frac{\beta_1 \tilde{V_2}}{N_1} X_1(t-\tau) - \mu_h X_1(t) - \frac{\beta_1 V_2(t-\tau)\tilde{X_1}}{N_1} \\ \frac{dX_3}{dt} = \frac{\beta_1 \tilde{V_2} X_1(t-\tau)}{N_1} - (\mu_h + \epsilon) X_3(t) + \frac{\beta_1 V_2(t-\tau)\tilde{X_1}}{N_1} \\ \frac{dV}{dt} = r\left(1 - \frac{2\tilde{V}}{K}\right) V(t) \\ \frac{dV_2}{dt} = \frac{\beta_2 \tilde{Z_2} \tilde{V_2} V(t)}{\tilde{V}^2} - V_2(t)\left(\frac{\beta_2 \tilde{Z_2}}{\tilde{V}} + \alpha_2\right) + \beta_2\left(\frac{\tilde{V} - \tilde{V_2}}{\tilde{V}}\right) Z_2(t) \\ \frac{dZ_1}{dt} = -\frac{\beta_3 \tilde{Z_1} V_2(t)}{N_2} - \left(\frac{\beta_2 \tilde{V_2}}{N_2} + \mu_p\right) Z_1(t) \\ \frac{dZ_2}{dt} = \frac{\beta_3 \tilde{Z_1} V_2(t)}{N_2} + \frac{\beta_3 Z_1(t) \tilde{V_2}}{N_2} - (\mu_p + \gamma_1) Z_2(t) \end{cases}$$

The Jacobian matrix of linearized system (4.1) of system (2.3) is given by

$$(4.2)\qquad J(E) = \begin{pmatrix} J_{11} & J_{12} \\ O & J_{22} \end{pmatrix}$$

where,

$$J_{11} = \begin{pmatrix} -\frac{\beta_1 \tilde{V_2}}{N_1} e^{-\lambda\tau} - \mu_h & 0 \\ \frac{\beta_1 \tilde{V_2}}{N_1} e^{-\lambda\tau} & -(\mu_h + \epsilon) \end{pmatrix}, \qquad J_{12} = \begin{pmatrix} 0 & -\frac{\beta_1 \tilde{X_1}}{N_1} e^{-\lambda\tau} & 0 & 0 \\ 0 & \frac{\beta_1 \tilde{X_1}}{N_1} e^{-\lambda\tau} & 0 & 0 \end{pmatrix}$$

$$J_{22} = \begin{pmatrix} r(1 - \frac{2\tilde{V}}{K}) & 0 & 0 & 0 \\ \frac{\beta_2 \tilde{Z_2} \tilde{V_2}}{\tilde{V}^2} & -(\frac{\beta_2 \tilde{Z_2}}{\tilde{V}} + \alpha_2) & 0 & \frac{\beta_2(\tilde{V} - \tilde{V_2})}{\tilde{V}} \\ 0 & \frac{-\beta_3 \tilde{Z_1}}{N_2} & -(\frac{\beta_3 \tilde{V_2}}{N_2} + \mu_p) & 0 \\ 0 & \frac{\beta_3 \tilde{Z_1}}{N_2} & \frac{\beta_3 \tilde{V_2}}{N_2} & -(\mu_p + \gamma_1) \end{pmatrix}$$

4.2.1. Stability of virus-free steady-state

Theorem 3. The virus-free steady-state E_0 of framework (2.3) is absolutely stable for $\tau \geq 0$ if $R_0 < 1$.

Proof. The characteristic roots of the Jacobian matrix $J(E)$ at E_0 of Eq.(4.2) for linearized system of Eq.(4.1) are $-\mu_h,\ -(\mu_h+\epsilon),\ -r,\ -\mu_p$ $-\ \frac{1}{2}\left[(\alpha_2+\gamma_1+\mu_p)\pm\sqrt{(\alpha_2+\gamma_1+\mu_p)^2-4\alpha_2(\mu_p+\gamma_1)(1-R_0)}\right]$. Hence, all its roots are negative if $R_0<1$. Therefore, E_0 is absolutely stable for $\tau\geq 0$. □

Remark 2. *In order to get Hopf bifurcation around the equilibrium E_0, it is sufficient to show that $J(E_0)$ of Eq.(4.1) has purely imaginary roots. But at the equilibrium E_0, $J(E_0)$ has no purely imaginary roots. So, the system (4.1) does not undergoes Hopf bifurcation around E_0, [32].*

4.2.2. Stability of endemic steady-state

The Jacobian matrix $J(E)$ at E_1 of Eq.(4.1) for the linearized system of Eq.(2.3) have the characteristic roots are

$$-r,-\left[\alpha_2+\frac{\beta_2 Z_2^*}{K}\left\{1-\frac{N_1K}{R_0N_2V_2^*}\left(1-\frac{V_2^*}{K}\right)(1-K^*)\right\}\right],-\left(\frac{\beta_3V_2^*}{N_2}+\mu_p\right),\\ -(\mu_p+\gamma_1)$$

where, $K^*=\frac{\beta_3V_2^*}{\beta_3V_2^*+\mu_1N_2}<1$ and the roots of the polynomial equation of degree two

$$\lambda^2+(2\mu_h+\epsilon)\lambda+\mu_h(\mu_h+\epsilon)+\frac{\beta_1V_2^*}{N_1}(\lambda+\mu_h+\epsilon)e^{-\lambda\tau}=0 \tag{4.3}$$

If $\tau=0$, then Eq.(4.3) reduced to

$$\lambda^2+(2\mu_h+\epsilon)\lambda+\mu_h(\mu_h+\epsilon)+\frac{\beta_1V_2^*}{N_1}(\lambda+\mu_h+\epsilon)=0$$

From above, it is seen that the endemic steady-state E_1 is LAS if $R_0>1$, $\tau=0$.
If $\tau>0$, put $\lambda=iw, w>0$ in Eq.(4.3), we get.

$$\frac{\beta_1V_2^*}{N_1}w\sin(w\tau)+\frac{\beta_1V_2^*}{N_1}(\mu_h+\epsilon)\cos(\tau w)=w^2-\mu_h(\mu_h+\epsilon) \tag{4.4}$$

and

$$\frac{\beta_1 V_2^*}{N_1}(\mu_h+\epsilon)\sin(\tau w)-\frac{\beta_1 V_2^*}{N_1}w\cos(w\tau)=w(\mu_h+\epsilon) \tag{4.5}$$

Figuring out and including Eqs. (4.4) and (4.5), we get

$$w^4+\left\{2\mu_h^2+\epsilon^2+2\mu_h\epsilon-\frac{\beta_1^2 V_2^{*2}}{N_1^2}\right\}w^2+\left(\mu_h^2-\frac{\beta_1^2 V_2^* 2}{N_1^2}\right)(\mu_h+\epsilon)^2=0$$

or

$$y_1^2+y_1c_1+c_2=0 \tag{4.6}$$

where,

$$y_1=w^2, c_1=\left\{2\mu_h^2+\epsilon^2+2\mu_h\epsilon-\frac{\beta_1^2 V_2^{*2}}{N_1^2}\right\}, c_2=\left(\mu_h^2-\frac{\beta_1^2 V_2^* 2}{N_1^2}\right)(\mu_h+\epsilon)^2$$

From above, we see that if $c_2>0$ then obviously $c_1>0$, so we can consider two case as-

Case 1. *If $R_0>1$ and $c_2>0$, then by Descarte's rule of signs all the solutions of Eq.(4.6) have negative real parts $\forall\ \tau\geq 0$, along these lines, by Definition 3.1, [2] the accompanying theorem is acquired.*

Theorem 4. If $R_0>1$ and $c_2>0$, then endemic steady-state E_1 of the framework (2.3) is absolutely stable $\forall\ \tau\geq 0$.

Lemma 1. *[29] For the equation of a quadratic transcendental polynomial*

$$\lambda^2+p_1\lambda+p_2+(p_3\lambda+p_4)e^{-\lambda\tau}=0 \tag{4.7}$$

If,

1. $p_1+p_3>0$;

2. $p_2 + p_4 > 0$;
3. *either* $p_3^2 - p_1^2 + 2p_2 < 0$ *and* $p_2^2 - p_4^2 > 0$ *or* $(p_3^2 - p_1^2 + 2p_2)^2 < 4(p_2^2 - p_4^2)$;
4. *either* $p_2^2 - p_4^2 < 0$ *or* $p_3^2 - p_1^2 + 2p_2 > 0$ *and* $(p_3^2 - p_1^2 + 2p_2)^2 = 4(p_2^2 - p_4^2)$;

Then there are the following outcomes,

1. *All the solutions of (4.7) have negative real parts* $\forall \tau \geq 0$, *if (a)-(c) holds.*
2. *If (a), (b), and (d) holds and* $\tau = \tau_0$ *then Eq. (4.7) has purely imaginary roots* $\pm iw$ *and all others solutions of Eq. (4.7) have negative real parts.*
3. *If* $\lambda = 0$, $p_1 + p_3 > 0$ *and* $p_2 + p_4 > 0$ *then, all the solutions of (4.7) have negative real parts.*

Case 2. *If* $R_0 > 1$ *and* $c_2 < 0$, *then by Descarte's rule of signs Eq.(4.6) has positive solutions. Thus, the Eq.(4.3) has pair of purely imaginary roots (say)* $\lambda = \pm iw_0$. *Put* $w = w_0$ *in Eq.(4.7) and solve it for* τ, *we get the corresponding* $\tau_j > 0, j = 0, 1, 2, ...$ *such that*

$$\tau_0 = \frac{1}{w_0} arc\cos\left(\frac{Ap_1(\mu_h + \epsilon) + Aw_0\sqrt{A^2((\mu_h + \epsilon)^2 + w_0^2) - p_1^2}}{A^2 N_1(w_0^2 + (\mu_h + \epsilon)^2)}\right) \tag{4.8}$$

where $A = \frac{\beta_1 V_2^*}{N_1}$, $p_1 = 1 - \mu_h(\mu_h + \epsilon)$. *Using the Lemma (1) all the solutions of Eq.(11) are have negative real parts for* $\tau \in [0, \tau_0)$. *Consequently, by Lemma 3.5 (ii)[2] the accompanying theorem is acquired.*

Theorem 5. If $R_0 > 1$ and $c_2 < 0$ then positive steady-state E_1 of the framework (2.3) is conditionally steady $\forall\ \tau \in [0, \tau_0)$.

5. Hopf bifurcation analysis

Here, we have taken the time delay τ as bifurcation parameter. Let $\lambda(\tau) = \alpha(\tau) + iw(\tau), w > 0$ be the root of Eq.(4.3) to such an extent that $\alpha(\tau_0) = 0$ and $w(\tau_0) = w_0$ for some underlying estimation of the bifurcation parameter

τ_0. To build up Hopf bifurcation at $\tau = \tau_0$, it have to check $\lambda(\tau_0) = iw_0$ is a simple and $\left(\frac{d(Re(\lambda))}{d\tau}\right)_{\tau=\tau_0} > 0$. Diff. Eq.(4.3) w.r.t. τ, we get

$$\left(2\lambda + 2\mu_h + \epsilon + A((1 - \tau(\lambda + \mu_h + \epsilon)e^{-\lambda\tau})\right)\frac{d\lambda}{d\tau} = A\lambda\left(\lambda + \mu_h + \epsilon\right)e^{-\lambda\tau} \tag{5.1}$$

Suppose $\lambda(\tau) = iw_0$ is not simple, so $\left(\frac{d(Re(\lambda))}{d\tau}\right)_{\tau=\tau_0} = 0$. At that point, Eq.(4.3) can be composed as-

$$\lambda A(\lambda + \mu_h + \epsilon)e^{-\lambda\tau} = 0 \tag{5.2}$$

Put $\lambda = iw_0$ in Eq.(4.4), we get

$$\begin{aligned} &Aw_0(\mu_h + \epsilon)\sin(w_0\tau_0) - Aw_0^2\cos(\tau_0 w_0) = 0 \\ &Aw_0^2\sin(\tau_0 w_0) + Aw_0(\mu_h + \epsilon)\cos(w_0\tau_0) = 0 \end{aligned} \tag{5.3}$$

Putting $w = w_0$ and $\tau = \tau_0$ in Eqs.(4.4) and (4.5), we get

$$\begin{aligned} Aw_0^2\sin(\tau_0 w_0) + Aw_0(\mu_h + \epsilon)\cos(w_0\tau_0) &= w_0^3 - w\mu_h(\mu_h + \epsilon) \\ Aw_0(\mu_h + \epsilon)\sin(w_0\tau_0) - Aw_0^2\cos(\tau_0 w_0) &= w_0^2(\mu_h + \epsilon) \end{aligned} \tag{5.4}$$

Using Eq.(5.3) in Eq.(5.4), we get $w_0 = 0$, this is not possible. This contradict our supposition that $\lambda(\tau) = iw_0$ is not simple. Hence $\lambda(\tau) = iw_0$ is a simple root of Eq.(4.3).
Now, we will show that $\left(\frac{d(Re(\lambda))}{d\tau}\right)_{\tau=\tau_0} > 0$.
Eq.(5.1) can be composed as-

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{(2\lambda + 2\mu_h + \epsilon)}{A\lambda(\lambda + \mu_h + \epsilon)e^{-\lambda\tau}} + \frac{1}{\lambda(\lambda + \mu_h + \epsilon)} - \frac{\tau}{\lambda}$$

or,

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{-(2\lambda + 2\mu_h + \epsilon)}{\lambda(\lambda^2 + (2\mu_h + \epsilon)\lambda + \mu_h(\mu_h + \epsilon))} + \frac{1}{\lambda(\lambda + \mu_h + \epsilon)} - \frac{\tau}{\lambda}$$

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = \left[\frac{-(2\lambda + 2\mu_h + \epsilon)}{\lambda(\lambda^2 + (2\mu_h + \epsilon)\lambda + \mu_h(\mu_h + \epsilon))} + \frac{1}{\lambda(\lambda + \mu_h + \epsilon)} - \frac{\tau}{\lambda}\right]_{\lambda=iw_0}$$

$$= \left[\frac{-(\lambda^2-\mu_h(\mu_h+\epsilon))}{\lambda^2(\lambda^2+(2\mu_h+\epsilon)\lambda+\mu_h(\mu_h+\epsilon))} - \frac{\mu_h+\epsilon}{\lambda^2(\lambda+\mu_h+\epsilon)} - \frac{\tau}{\lambda}\right]_{\lambda=iw_0}$$

After separating real and imaginary parts, we obtain the following simplified real part equation

$$Re\left[\left(\frac{d\lambda}{d\tau}\right)^{-1}\right] = \frac{1}{w_0^2}\left[\frac{(\mu_h+\epsilon)^2}{w_0^2+(\mu_h+\epsilon)^2} + \frac{\mu_h^2(\mu_h+\epsilon)^2-w_0^4}{(\mu_h^2+w_0^2)((\mu_h+\epsilon)^2+w_0^2)}\right]$$

Thus,

$$\text{sign}\left[\left(\frac{d\lambda}{d\tau}\right)^{-1}\right]_{\tau=\tau_0} = \frac{1}{w_0^2}\text{sign}\left[\frac{(\mu_h+\epsilon)^2}{w_0^2+(\mu_h+\epsilon)^2} + \frac{\mu_h^2(\mu_h+\epsilon)^2-w_0^4}{(\mu_h^2+w_0^2)((\mu_h+\epsilon)^2+w_0^2)}\right] \quad (5.5)$$

The expression on right hand side of Eq.(5.5) is positive if $\mu_h(\mu_h+\epsilon) > w_0^2$ (since $\mu_h+\epsilon$ is mortality rate of mosquitoes, it never surpass unity).

Hence, the transversality condition for example $\left(\frac{d(Re(\lambda))}{d\tau}\right)_{\tau=\tau_0} > 0$ is fulfilled. As the above outcomes, the solution of characteristic Eq.(4.3) goes through imaginary axis from left to right as τ changes from a number less than τ_0. Hence, the condition for Hopf bifurcation are fulfilled at $\tau = \tau_0$. From the above investigation, we have obtained the Theorem (6) as:

Theorem 6. If $R_0 > 1$, the positive steady-state E_1 of the framework (2.3) is

1. If $c_2 > 0$, absolutely stable $\forall\ \tau \geq 0$.
2. If $c_2 < 0$, conditionally stable $\forall\ \tau \in [0, \tau_0)$. Moreover, the system experiences Hopf bifurcation at E_1 when $\tau = \tau_0$.
3. The largest positive simple root of the Eq.(4.3) is w_0, then E_1 of the delay induced framework (2.3) is asymptotically steady when $\tau < \tau_0$ and unstable when $\tau > \tau_0$, where,

$$\tau_0 = \frac{1}{w_0} arc\cos\left(\frac{Ap_1(\mu_h+\epsilon) + Aw_0\sqrt{A^2((\mu_h+\epsilon)^2+w_0^2)-p_1^2}}{A^2N_1(w_0^2+(\mu_h+\epsilon)^2)}\right)$$

At τ goes through the critical point $\tau = \tau_0$, a group of periodic solutions bifurcates from E_1.

6. Direction and stability of the Hopf bifurcation

In this part, we have acquired the condition for Hopf bifurcation at the critical value $\tau = \tau_0$ and prevailing with regards to getting unequivocally the articulation for τ_0 by utilizing the normal structure strategy and manifold theory presented by Hassard et al. [7]. In this investigation, our underlying supposition that will be that the system (4.3) shows Hopf bifurcation and $\pm i w_0$ is corresponding purely imaginary roots of the characteristic equation at positive steady-state $E_1(X_1^*, X_3^*, V^*, V_2^*, Z_1^*, Z_2^*)$. Let us think about the transformation

$$\begin{aligned} v_1(t) &= X_1(t) - X_1^*, v_2(t) = X_3(t) - X_3^*, v_3(t) = V(t) - V^*, v_4(t) = V_2(t) - V_2^* \\ v_5(t) &= Z_1(t) - Z_1^*, v_6(t) = Z_2(t) - Z_2^*, \bar{v_i}(t) = v_i(t\tau), (i = 1, 2, 3, 4, 5, 6) \text{ and} \\ \tau &= \tau_0 + \mu \end{aligned}$$

the system (4.3) is converted to a functional differential equation (FDE) in $C = C([-1, 0], \mathbf{R}^6)$.

$$\frac{dy}{dt} = L_\mu(y_t) + f(\mu, y_t) \tag{6.1}$$

where,

$$(y_1(t), y_2(t), y_3(t), y_4(t), y_5(t), y_6(t)) \in \mathbf{R}^6$$

and $L_\mu : C \to \mathbf{R}^6$, $f : \mathbf{R} \times C \to \mathbf{R}^6$ are respectively given by

$L_\mu(\phi) =$

$$(\tau_0+\mu)\begin{pmatrix} -\mu_h & 0 & 0 & 0 & 0 & 0 \\ 0 & -(\mu_h+\epsilon) & 0 & 0 & 0 & 0 \\ 0 & 0 & \left(r-\frac{2V^*}{K}\right) & 0 & 0 & 0 \\ 0 & 0 & \frac{\beta_2 Z_2^* V_2^*}{V^{*2}} & -\left(\frac{\beta_2 Z_2^*}{V^*}+\alpha_2\right) & 0 & \frac{\beta_2(V^*-V_2^*)}{V^*} \\ 0 & 0 & 0 & -\frac{\beta_3 Z_1^*}{N_2} & -\left(\frac{\beta_3 V_2^*}{N_2}+\mu_p\right) & 0 \\ 0 & 0 & 0 & \frac{\beta_3 Z_1^*}{N_2} & \frac{\beta_3 V_2^*}{N_2} & -(\mu_p+\gamma_1) \end{pmatrix}$$

(6.2)

$$\begin{pmatrix} \phi_1(0) \\ \phi_2(0) \\ \phi_3(0) \\ \phi_4(0) \\ \phi_5(0) \\ \phi_6(0) \end{pmatrix} + (\tau_0+\mu)\begin{pmatrix} -\frac{\beta_1 V_2^*}{N_1} & 0 & 0 & -\frac{\beta_1 X_1^*}{N_1} & 0 & 0 \\ \frac{\beta_1 V_2^*}{N_1} & 0 & 0 & \frac{\beta_1 X_1^*}{N_1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \phi_1(-1) \\ \phi_2(-1) \\ \phi_3(-1) \\ \phi_4(-1) \\ \phi_5(-1) \\ \phi_6(-1) \end{pmatrix}$$

and

$$f(\mu,\phi) = (\tau_0+\mu)\begin{pmatrix} -\frac{\beta_1\phi_1(-1)\phi_4(-1)}{N_1} \\ \frac{\beta_1\phi_1(-1)\phi_4(-1)}{N_1} \\ -\frac{r\phi_3^2(0)}{K} \\ -\frac{\beta_2\phi_6(0)\phi_4(0)}{\phi_3(0)} \\ -\frac{\beta_3\phi_5(0)\phi_4(0)}{N_2} \\ \frac{\beta_3\phi_5(0)\phi_4(0)}{N_2} \end{pmatrix} \tag{6.3}$$

By the Riesz representation theorem, we can locate a bounded variation function $\eta(\theta,\mu)$ for $\theta \in [-1,0]$ with the end goal that

$$\text{For } \phi \in C, L_\mu(\phi) = \int_{-1}^{0} \phi(\theta)d\eta(\theta,\mu) \tag{6.4}$$

We can choose,

$\eta(\theta, \mu) =$

$$(\tau_0+\mu)\begin{pmatrix} -\mu_h & 0 & 0 & 0 & 0 & 0 \\ 0 & -(\mu_h+\epsilon) & 0 & 0 & 0 & 0 \\ 0 & 0 & \left(r-\frac{2V^*}{K}\right) & 0 & 0 & 0 \\ 0 & 0 & \frac{\beta_2 Z_2^* V_2^*}{V^{*2}} & -\left(\frac{\beta_2 Z_2^*}{V^*}+\alpha_2\right) & 0 & \frac{\beta_2(V^*-V_2^*)}{V^*} \\ 0 & 0 & 0 & -\frac{\beta_3 Z_1^*}{N_2} & -\left(\frac{\beta_3 V_2^*}{N_2}+\mu_p\right) & 0 \\ 0 & 0 & 0 & \frac{\beta_3 Z_1^*}{N_2} & \frac{\beta_3 V_2^*}{N_2} & -(\mu_p+\gamma_1) \end{pmatrix}\delta(\theta) \tag{6.5}$$

(6.6)

$$+(\tau_0+\mu)\begin{pmatrix} -\frac{\beta_1 V_2^*}{N_1} & 0 & 0 & -\frac{\beta_1 X_1^*}{N_1} & 0 & 0 \\ \frac{\beta_1 V_2^*}{N_1} & 0 & 0 & \frac{\beta_1 X_1^*}{N_1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}\delta(\theta+1) \tag{6.7}$$

where δ is the Dirac delta function, let

$$M(\mu)(\phi) = \begin{cases} \frac{d\phi(\theta)}{d\theta}, & \theta \in [-1, 0) \\ \int_{-1}^{0} d\eta(\mu, t_1)\phi(t_1), & \theta = 0 \end{cases} \tag{6.8}$$

and

$$S(\mu)(\phi) = \begin{cases} 0, & \theta \in [-1, 0) \\ f(\mu, \phi), & \theta = 0 \end{cases} \tag{6.9}$$

Then, the Eq.(6.1) can be composed as-

$$\frac{dy_t}{dt} = M(\mu)y_t + S(\mu)y_t \tag{6.10}$$

where, $y_t(\theta) = y(t+\theta)$ for $\theta \in [-1, 0]$. Also let,

$$(6.11) \qquad M^*\psi(t) = \begin{cases} -\frac{d\psi(t)}{dt}, & t \in (0,1] \\ \int_{-1}^{0} d\eta^T(s,0)\psi(-s), & t = 0 \end{cases}$$

for $\psi \in C^1\left([0,1], (\mathbf{R}^6)^*\right)^{-1}$
We define bilinear inner product (here, $\eta(\theta) = \eta(\theta, 0)$)

$$(6.12) \; < \psi(t), \phi(\theta) >= \bar{\psi(0)}\phi(0) - \int_{-1}^{0}\int_{\xi=0}^{\theta} \bar{\psi}(\xi-\theta)d\eta(\theta)\phi(\xi)d\xi$$

At that point M(0) and M^* are called adjoint operators. Presently it very well may be effectively demonstrated that $\pm iw_0\tau_0$ are eigenvalues of M(0). Consequently, $\pm iw_0\tau_0$ are likewise eigenvalues of M^*.

Let $P(\theta) = (1, e_1, e_2, e_3, e_4, e_5)^T e^{iw_0\tau_0\theta}$ be the eigenvector of $M(0)$ corresponding to $iw_0\tau_0$, then $M(0)P(\theta) = iw_0\tau_0 P(\theta)$. Then, from Eqs.(6.2), (6.4), (6.5) and definition of $M(0)$, we obtain, $P(0) = (1, e_1, 0, e_3, e_4, e_5)^T$ where,

$$\begin{aligned} e_1 &= \frac{\beta_1 e^{-iw_0\tau_0}(V_2^* + X_1^* e_3)}{N_1(iw_0 + \mu_h + \epsilon)}, \quad e_3 = -\left(\frac{N_1(iw_0+\mu_h)}{\beta_1 X_1 e^{-iw_0\tau_0}} - \frac{V_2^*}{X_1^*}\right) \\ e_4 &= \frac{N_2 e_5(iw_0+\mu_p+\gamma_1) - e_3\beta_3 Z_1^*}{\beta_3 V_2^*}, \quad e_5 = \frac{e_3(iw_0V^* + \beta_2 Z_2^* + \alpha_2 V^*)}{\beta_2(V_2^* - V^*)} \end{aligned}$$

Similarly, let $P^*(\theta^*) = D(1, e_1^*, e_2^*, e_3^*, e_4^*, e_5^*)e^{-iw_0\tau_0\theta^*}$ be the eigenvector of M^* corresponding to the eigenvalue $-iw_0\tau_0$. In the same manner we have obtain,

$$\begin{aligned} e_1^* &= 0, \; e_2^* = \frac{e_5^*(iw_0+\mu_p+\gamma_1)K\beta_2 Z_2^* V_2^*}{\beta_2 V^*(V^* - V_2^*)(iw_0K - rK + 2V^*)}, \\ e_3^* &= \frac{e_5^* V^*(iw_0+\mu_p+\gamma_1)}{\beta_2(V_2^* - V^*)}, \; e_4^* = \frac{e_5^*\beta_2 V_2^*}{iw_0 N_2 + \beta_3 V_2^* + \mu_p N_2}, \\ e_5^* &= \frac{\beta_1 X_1^* e^{-iw_0\tau_0}}{N_1\left(\frac{\beta_3 Z_1^*(\mu_p+iw_0)}{\beta_3 V_2^* + N_2(\mu_p+iw_0)} + \frac{(iw_0+\mu_p+\gamma_1)(\alpha_2 V^* + iw_0 V^* + \beta_2 Z_2^*)}{\beta_2(V^* - V_2^*)}\right)} \end{aligned}$$

To assure $\langle P(\theta), P^*(\theta^*)\rangle = 1$, we use Eq.(6.12) to calculate D as follows:

$$(6.13) \qquad \langle P^*(\theta^*), P(\theta)\rangle = \bar{P^*}(0)P(0) - \int_{\theta=-1}^{0}\int_{\xi=0}^{\theta} \bar{P}^*(\xi-\theta)d\eta(\theta)P(\xi)d\xi$$

$$= \bar{D}(1,0,e_2^*,e_3^*,e_4^*,e_5^*)(1,e_1,0,e_3,e_4,e_5) -$$

$$\int_{\theta=-1}^{0}\int_{\xi=0}^{\theta}\bar{D}(1,0,e_2^*,e_3^*,e_4^*,e_5^*)e^{-iw_0\tau_0(\xi-\theta)}d\eta(\theta)(1,e_1,0,e_3,e_4,e_5)e^{iw_0\tau_0\xi}d\xi$$

$$= \bar{D}(1+\bar{e_3^*}e_3+\bar{e_4^*}e_4+\bar{e_5^*}e_5)\left(1-\int_{\theta=-1}^{0}\int_{\xi=0}^{\theta}e^{iw_0\tau_0\theta}d\eta(\theta)d\xi\right)$$

$$= \bar{D}(1+\bar{e_3^*}e_3+\bar{e_4^*}e_4+\bar{e_5^*}e_5)\left(\frac{w_0^2\tau_0^2-1+e^{-iw_0\tau_0}+iw_0\tau_0e^{-iw_0\tau_0}}{w_0^2\tau_0^2}\right)$$

Choosing,

$$D = \frac{w_0^2\tau_0^2}{(1+\bar{e_3^*}e_3+\bar{e_4^*}e_4+\bar{e_5^*}e_5)(w_0^2\tau_0^2-1+e^{-iw_0\tau_0}+iw_0\tau_0e^{-iw_0\tau_0})}$$

Thus, we achieve the property $\langle P(\theta), P^*(\theta^*)\rangle = 1$. To figure the directions portraying the center manifold C_0 at $\mu = 0$, we utilize the possibility of Hassard et al. [7]. Let y_t be the solution of Eq.(6.10) at $\mu = 0$, we define a function as follows:

(6.14) $Z(t) = \langle P^*(\theta^*), y_t\rangle$ and $W(t,\theta) = y_t(\theta) - 2Re\{Z(t), P(\theta)\}$

On center manifold C_0, we have

$$W(t,\theta) = W(Z(t),\bar{Z(t)},\theta)$$

where,

$$\begin{aligned} W(Z(t),\bar{Z(t)},\theta) &= W_{20}(\theta)\frac{Z^2}{2}+W_{02}(\theta)\frac{\bar{Z}^2}{2}+W_{11}(\theta)Z\bar{Z}+W_{30}(\theta)\frac{Z^3}{6} \\ &+ W_{03}(\theta)\frac{\bar{Z}^3}{6}+W_{12}(\theta)\frac{Z\bar{Z}^2}{2}+... \end{aligned} \tag{6.15}$$

Here, Z and $\bar{Z}$ are local coordinates for center manifold C_0 toward P^* and $\bar{P}^*$ respectively. The expression $W(Z,\bar{Z},\theta)$ will be real, if y_t is real and we are keen on real solution as it were. For real solution $y_t \in C_0$ of Eq.(6.10), (since $\mu = 0$) we have,

$$\begin{aligned}
\frac{dZ(t)}{dt} &= iw_0\tau_0 Z(t) + \bar{P^*}(0) f(0, W(Z,\bar{Z},0) + 2Re\{ZP(\theta)\}) \\
&= iw_0\tau_0 Z(t) + \bar{P^*}(0) f_0(Z,\bar{Z}) \\
&= iw_0\tau_0 Z(t) + h(Z,\bar{Z}) \qquad (6.16)
\end{aligned}$$

where,

$$h(Z,\bar{Z}) = \bar{P^*}(0) f_0(Z,\bar{Z}) = h_{20}\frac{Z^2}{2} + h_{02}\frac{\bar{Z}^2}{2} + h_{11}Z\bar{Z} + h_{21}\frac{Z^2\bar{Z}}{2} + ... \qquad (6.17)$$

From Eq.(6.14),

$$\begin{aligned}
y_t(\theta) &= W(t,\theta) + 2Re(Z(t), P(\theta)) \\
&= W_{20}(\theta)\frac{Z^2}{2} + W_{02}(\theta)\frac{\bar{Z}^2}{2} + W_{11}(\theta)Z\bar{Z} + ZP(\theta) + \bar{Z}\bar{P}(\theta) + ...
\end{aligned}$$

where,

$P(\theta) = (1, e_1, 0, e_3, e_4, e_5)^T e^{iw_0\tau_0\theta}$ and $\bar{P}(\theta) = (1, \bar{e_1}, 0, \bar{e_3}, \bar{e_4}, \bar{e_5})^T e^{-iw_0\tau_0\theta}$

Eq.(6.16) can also be written as

$$\begin{aligned}
h(Z,\bar{Z}) &= \bar{P^*}(0) f_0(Z,\bar{Z}) \\
&= \bar{P^*}(0) f(0, x_t) \\
&= \tau_0 \bar{D}(1, 0, \bar{e_2^*}, \bar{e_3^*}, \bar{e_4^*}, \bar{e_5^*}) \begin{pmatrix} -\frac{\beta_1 y_{1t}(-1) y_{4t}(-1)}{N_1} \\ \frac{\beta_1 y_{1t}(-1) y_{4t}(-1)}{N_1} \\ -\frac{r y_{3t}^2(0)}{K} \\ -\frac{\beta_2 y_{6t}(0) y_{4t}(0)}{y_{3t}(0)} \\ -\frac{\beta_3 y_{5t}(0) y_{4t}(0)}{N_2} \\ \frac{\beta_3 y_{5t}(0) y_{4t}(0)}{N_2} \end{pmatrix}
\end{aligned}$$

$$= -\tau_0 \bar{D}\left[\frac{\beta_1 y_{1t}(-1) y_{4t}(-1)}{N_1} + \bar{e_2^*}\frac{r y_{3t}^2(0)}{K} + \bar{e_3^*}\frac{\beta_2 y_{6t}(0) y_{4t}(0)}{y_{3t}(0)} + \bar{e_4^*}\frac{\beta_3 y_{5t}(0) y_{4t}(0)}{N_2} - \bar{e_5^*}\frac{\beta_3 y_{5t}(0) y_{4t}(0)}{N_2}\right]$$

$$
\begin{aligned}
= \quad & -\tau_0 \bar{D}\Bigg\{\frac{\beta_1}{N_1}\Bigg[W_{20}^{(1)}(-1)\frac{Z^2}{2} + W_{02}^{(1)}(-1)\frac{\bar{Z}^2}{2} + W_{11}^{(1)}(-1)Z\bar{Z} + Ze^{iw_0\tau_0} + \bar{Z}e^{-iw_0\tau_0} \\
+ \quad & O(|(Z,\bar{Z})|^3)\Bigg]\Bigg[W_{20}^{(4)}(-1)\frac{Z^2}{2} + W_{02}^{(4)}(-1)\frac{\bar{Z}^2}{2} + W_{11}^{(4)}(-1)Z\bar{Z} + e_3Ze^{iw_0\tau_0} + \bar{e_3}\bar{Z}e^{-iw_0\tau_0} \\
+ \quad & O(|(Z,\bar{Z})|^3)\Bigg] + \frac{\bar{e_2^*}r}{K}\Bigg[W_{20}^{(3)}(-1)\frac{Z^2}{2} + W_{02}^{(3)}(-1)\frac{\bar{Z}^2}{2} + W_{11}^{(3)}(-1)Z\bar{Z} + O(|(Z,\bar{Z})|^3)\Bigg]^2 \\
+ \quad & \bar{e_3^*}\beta_2\Bigg[W_{20}^{(6)}(-1)\frac{Z^2}{2} + W_{02}^{(6)}(-1)\frac{\bar{Z}^2}{2} + W_{11}^{(6)}(-1)Z\bar{Z} + e_5Z + \bar{e_5}\bar{Z} + O(|(Z,\bar{Z})|^3)\Bigg] \\
& \Bigg[W_{20}^{(4)}(-1)\frac{Z^2}{2} + W_{02}^{(4)}(-1)\frac{\bar{Z}^2}{2} + W_{11}^{(4)}(-1)Z\bar{Z} + e_3Z + \bar{e_3}\bar{Z} + O(|(Z,\bar{Z})|^3)\Bigg] \\
& \Bigg[W_{20}^{(3)}(-1)\frac{Z^2}{2} + W_{02}^{(3)}(-1)\frac{\bar{Z}^2}{2} + W_{11}^{(3)}(-1)Z\bar{Z} + O(|(Z,\bar{Z})|^3)\Bigg]^{-1} - \frac{\beta_3}{N_2}(\bar{e_3^*} - \bar{e_5^*}) \\
& \Bigg[W_{20}^{(4)}(-1)\frac{Z^2}{2} + W_{02}^{(4)}(-1)\frac{\bar{Z}^2}{2} + W_{11}^{(4)}(0)Z\bar{Z} + e_3Z + \bar{e_3}\bar{Z} + O(|(Z,\bar{Z})|^3)\Bigg] \\
& \Bigg[W_{20}^{(5)}(0)\frac{Z^2}{2} + W_{02}^{(5)}(0)\frac{\bar{Z}^2}{2} + W_{11}^{(5)}(0)Z\bar{Z} \\
& + e_4Z + \bar{e_4}\bar{Z} + O(|(Z,\bar{Z})|^3)\Bigg]\Bigg\}
\end{aligned}
$$

Comparing the coefficient with (6.17), we get

$$
\begin{aligned}
h_{20} &= -2\tau_0\bar{D}\left[\frac{\beta_1}{N_1}e_3e^{2iw_0\tau_0} + \frac{\beta_3}{N_2}(\bar{e_4^*} - \bar{e_5^*})e_3e_4\right], \\
h_{02} &= -2\tau_0\bar{D}\left[\frac{\beta_1}{N_1}\bar{e_3}e^{-2iw_0\tau_0} + \frac{\beta_3}{N_2}(\bar{e_4^*} - \bar{e_5^*})\bar{e_3}\bar{e_4}\right]
\end{aligned}
$$

$$
\begin{aligned}
h_{11} &= -\tau_0\bar{D}\left[\tfrac{\beta_1}{N_1}(\bar{e_3} + e_3) + \tfrac{\beta_3}{N_2}(\bar{e_4^*}e_3 - \bar{e_3^*}e_4)\right], \\
h_{21} &= -2\tau_0\bar{D}\Bigg[\tfrac{\beta_1}{N_1}\Bigg(e_3e^{iw_0\tau_0}W_{11}^{(1)}(-1) + e^{iw_0\tau_0}W_{11}^{(4)}(-1) \\
&+ \tfrac{1}{2}\bar{e_3}e^{-iw_0\tau_0}W_{20}^{(1)}(-1) + \tfrac{1}{2}e^{-iw_0\tau_0}W_{20}^{(4)}(-1)\Bigg) \\
&+ \tfrac{\beta_3}{N_2}\Bigg(e_4W_{11}^{(4)}(0) + \tfrac{\bar{e_4}}{2}W_{20}^{(4)}(0) + e_3W_{11}^{(5)}(0) + \tfrac{\bar{e_3}}{2}W_{20}^{(5)}(0)\Bigg)\Bigg]
\end{aligned} \tag{6.18}
$$

Next, we center around the calculation of $W_{20}(\theta)$ and $W_{11}(\theta)$, since they are in the outflow of h_{21}. Putting (6.10) and (6.16) into (6.14), we get

$$\begin{aligned} W^{'}(t) &= y_t^{'}(\theta) - Z^{'}(t)P(\theta) - Z^{\bar{'}}(t)P(\bar{\theta}) \\ &= \begin{cases} M(0)W - 2Re\Big\{P^{*\bar{}}(0)f_0(Z,\bar{Z})P(\theta)\Big\}, & \theta \in [-1,0) \\ M(0)W - 2Re\Big\{P^{*\bar{}}(0)f_0(Z,\bar{Z})P(\theta)\Big\} + f_0(Z,\bar{Z}), & \theta = 0 \end{cases} \\ &\cong M(0)W + L(Z,\bar{Z},\theta) \end{aligned} \tag{6.19}$$

where,

$$L(Z,\bar{Z},\theta) = L_{20}(\theta)\frac{Z^2}{2} + L_{02}(\theta)\frac{\bar{Z}^2}{2} + L_{11}(\theta)(Z\bar{Z}) + \dots \tag{6.20}$$

Using the value of Eqs.(6.15) and (6.20) in Eq.(6.19) and comparing the coefficient, we have

$$\begin{aligned} \Big(M(0) - 2iw_0\tau_0 I\Big)W_{20}(\theta) &= -L_{20}(\theta) \\ M(0)W_{11}(\theta) &= -L_{11}(\theta) \end{aligned} \tag{6.21}$$

From Eq.(6.18), it is found that for $\theta \in [-1,0)$

$$\begin{aligned} L(Z,\bar{Z},\theta) &= -2Re\Big\{P^{*\bar{}}(0)f_0(Z,\bar{Z})P(\theta)\Big\} \\ &= -P^{*\bar{}}(0)f_0(Z,\bar{Z})P(\theta) - P^{*}(0)\bar{f_0}(Z,\bar{Z})P(\bar{\theta}) \\ &= -h(Z,\bar{Z})P(\theta) - \bar{h}(Z,\bar{Z})P(\bar{\theta}) \\ &= -\Big(h_{20}\tfrac{Z^2}{2} + h_{02}\tfrac{\bar{Z}^2}{2} + h_{11}Z\bar{Z} + \dots\Big) \\ &\quad P(\theta) - \Big(\bar{h_{20}}\tfrac{\bar{Z}^2}{2} + \bar{h_{02}}\tfrac{Z^2}{2} + \bar{h_{11}}Z\bar{Z} + \dots\Big)P(\bar{\theta}) \end{aligned} \tag{6.22}$$

Comparing the coefficient of Eqs.(6.20) and (**??**), we obtain

$$L_{20}(\theta) = -h_{20}P(\theta) - \bar{h_{02}}P(\bar{\theta}) \tag{6.23}$$

$$L_{11}(\theta) = -h_{11}P(\theta) - \bar{h_{11}}P(\bar{\theta}) \tag{6.24}$$

From definition of $M(0)$, Eqs.(6.21), (6.23) and (6.24), we get

$$M(0)W_{20}(\theta) = W_{20}^{'}(\theta) = 2iw_0\tau_0 W_{20}(\theta) + h_{20}P(\theta) + \bar{h_{02}}P(\bar{\theta}) \tag{6.25}$$

$$M(0)W_{11}(\theta) = W_{11}^{'}(\theta) = h_{11}P(\theta) + \bar{h_{11}}\bar{P(\theta)} \tag{6.26}$$

Putting $P(\theta) = (1, e_1, 0, e_3, e_4, e_5)^T e^{iw_0\tau_0\theta}$ in the last equation, we obtain the solution

$$W_{20}(\theta) = \frac{ih_{20}}{w_0\tau_0}P(0)e^{iw_0\tau_0\theta} + \frac{i\bar{h_{02}}}{3w_0\tau_0}\bar{P(0)}e^{-iw_0\tau_0\theta} + J_1 e^{2iw_0\tau_0\theta} \tag{6.27}$$

and similarly,

$$W_{11}(\theta) = \frac{-ih_{11}}{w_0\tau_0}P(0)e^{iw_0\tau_0\theta} + \frac{i\bar{h_{11}}}{3w_0\tau_0}\bar{P(0)}e^{-iw_0\tau_0\theta} + J_2 \tag{6.28}$$

where,

$$J_1 = (J_1^{(1)}, J_1^{(2)}, J_1^{(3)}, J_1^{(4)}, J_1^{(5)}, J_1^{(6)})^T,\ J_2 = (J_2^{(1)}, J_2^{(2)}, J_2^{(3)}, J_2^{(4)}, J_2^{(5)}, J_2^{(6)})^T$$

are constant vectors. Presently, we need to locate a fitting consistent vector J_1 and J_2 which fulfills the above conditions. From the definition of M and Eq.(6.26), we get

$$\int_{-1}^{0} d\eta(\theta)W_{20}(\theta) = M(0)W_{20}(\theta) = 2iw_0\tau_0 W_{20}(0) - L_{20}(0) \tag{6.29}$$

and

$$\int_{-1}^{0} d\eta(\theta)W_{11}(\theta) = M(0)W_{11}(\theta) = -L_{11}(0) \tag{6.30}$$

From Eq.(6.19),

$$\begin{aligned} L(Z,\bar{Z},0) &= -h(Z,\bar{Z})P(0) - \bar{h}(Z,\bar{Z})\bar{P(0)} + f_0(Z,\bar{Z}) \\ &= -\left(h_{20}\frac{Z^2}{2} + h_{02}\frac{\bar{Z}^2}{2} + h_{11}Z\bar{Z} + \dots\right) \\ &\quad P(0) - \left(h_{20}\frac{Z^2}{2} + h_{02}\frac{\bar{Z}^2}{2} + h_{11}Z\bar{Z} + \dots\right) \\ &\quad P(\theta) + 2\tau_0 M_1\frac{Z^2}{2} + M_2\tau_0 Z\bar{Z} + \dots \end{aligned} \tag{6.31}$$

where,

$$M_1 = \begin{pmatrix} -\frac{\beta_1}{N_1} e_3 e^{2iw_0\tau_0} \\ \frac{\beta_1}{N_1} e_3 e^{2iw_0\tau_0} \\ 0 \\ 0 \\ -\frac{\beta_3}{N_2} e_3 e_4 \\ \frac{\beta_3}{N_2} e_3 e_4 \end{pmatrix} \quad \text{and} \quad M_2 = \begin{pmatrix} -\frac{\beta_1}{N_1}(\bar{e_3} + e_3) \\ \frac{\beta_1}{N_1}(\bar{e_3} + e_3) \\ 0 \\ 0 \\ -\frac{\beta_3}{N_2}(e_3\bar{e_4} + \bar{e_3}e_4) \\ \frac{\beta_3}{N_2}(e_3\bar{e_4} + \bar{e_3}e_4) \end{pmatrix}$$

Comparing the coefficients of the Eq.(6.31), we get

$$L_{20}(0) = -h_{20}P(0) - \bar{h_{02}}\bar{P(0)} + 2\tau_0 M_1 \tag{6.32}$$

$$L_{11}(0) = -h_{11}P(0) - \bar{h_{11}}\bar{P(0)} + \tau_0 M_2 \tag{6.33}$$

Since $iw_0\tau_0$ is the eigenvalue of $M(0)$ and $P(0)$ is the corresponding eigenvector, then

$$\Big(iw_0\tau_0 I - \int_{-1}^{0} e^{iw_0\tau_0} d\eta(\theta)\Big)P(0) = 0 \text{ and } \Big(-iw_0\tau_0 I - \int_{-1}^{0} e^{-iw_0\tau_0} d\eta(\theta)\Big)\bar{P(0)} = 0$$

Using the Eqs.(6.27) and (6.32) in Eq.(6.29), we get

$$\left(2iw_0\tau_0 I - \int_{-1}^{0} e^{2iw_0\tau_0\theta} d\eta(\theta)\right) J_1 = 2\tau_0 M_1$$

That is

$$\begin{pmatrix} a_{11} & 0 & 0 & \frac{\beta_1}{N_1}X_1^* e^{-2iw_0\tau_0} & 0 & 0 \\ b_{11} & b_{12} & 0 & -\frac{\beta_1}{N_1}X_1^* e^{-2iw_0\tau_0} & 0 & 0 \\ 0 & 0 & 2iw_0 - \left(r - \frac{2V^*}{K}\right) & 0 & 0 & 0 \\ 0 & 0 & -\frac{\beta_2 Z_2^* V_2^*}{V^{*2}} & 2iw_0 + \left(\frac{\beta_2 Z_2^*}{V^*} + \alpha_2\right) & 0 & -\frac{\beta_2(V^* - V_2^*)}{V^*} \\ 0 & 0 & 0 & \frac{\beta_3 Z_1^*}{N_2} & c_{11} & 0 \\ 0 & 0 & 0 & -\frac{\beta_3 Z_1^*}{N_2} & -\frac{\beta_3 V_2^*}{N_2} & (2iw_0 + \mu_p + \gamma_1) \end{pmatrix}$$

$$\begin{pmatrix} J_1^{(1)} \\ J_1^{(2)} \\ J_1^{(3)} \\ J_1^{(4)} \\ J_1^{(5)} \\ J_1^{(6)} \end{pmatrix} = 2 \begin{pmatrix} -\frac{\beta_1}{N_1} e_3 e^{2iw_0\tau_0} \\ \frac{\beta_1}{N_1} e_3 e^{2iw_0\tau_0} \\ 0 \\ 0 \\ -\frac{\beta_3}{N_2} e_3 e_4 \\ \frac{\beta_3}{N_2} e_3 e_4 \end{pmatrix}$$

where,

$$\begin{aligned} a_{11} &= 2iw_0 + \mu_h + \frac{\beta_1}{N_1} V_2^* e^{-2iw_0\tau_0},\ b_{11} = -\frac{\beta_1}{N_1} V_2^* e^{-2iw_0\tau_0},\ b_{12} = 2iw_0 + (\mu_h + \epsilon) \\ c_{11} &= 2iw_0 + \left(\frac{\beta_3 V_2^*}{N_2} + \mu_p\right) \end{aligned}$$

From above we can undoubtedly figure a constant vector $J_1 = \left(J_1^{(1)}, J_1^{(2)}, J_1^{(3)}, J_1^{(4)}, J_1^{(5)}, J_1^{(6)}\right)^T \in \mathbf{R}^6$. Similarly, using Eqs.(6.28) and (6.33) into (6.30), we get

$$\begin{pmatrix} -\left(\mu_h + \frac{\beta_1}{N_1} V_2^*\right) & 0 & 0 & -\frac{\beta_1}{N_1} X_1^* & 0 & 0 \\ \frac{\beta_1}{N_1} V_2^* & -(\mu_h + \epsilon) & 0 & \frac{\beta_1}{N_1} X_1^* & 0 & 0 \\ 0 & 0 & \left(r - \frac{2V^*}{K}\right) & 0 & 0 & 0 \\ 0 & 0 & \frac{\beta_2 Z_2^* V_2^*}{V^{*2}} & -\left(\frac{\beta_2 Z_2^*}{V^*} + \alpha_2\right) & 0 & \frac{\beta_2 (V^* - V_2^*)}{V^*} \\ 0 & 0 & 0 & -\frac{\beta_3 Z_1^*}{N_2} & -\left(\frac{\beta_3 V_2^*}{N_2} + \mu_p\right) & 0 \\ 0 & 0 & 0 & \frac{\beta_3 Z_1^*}{N_2} & \frac{\beta_3 V_2^*}{N_2} & -(\mu_p + \gamma_1) \end{pmatrix}$$

$$\begin{pmatrix} J_2^{(1)} \\ J_2^{(2)} \\ J_2^{(3)} \\ J_2^{(4)} \\ J_2^{(5)} \\ J_2^{(6)} \end{pmatrix} = 2 \begin{pmatrix} -\frac{\beta_1}{N_1} (\bar{e_3} + e_3) \\ \frac{\beta_1}{N_1} (\bar{e_3} + e_3) \\ 0 \\ 0 \\ -\frac{\beta_3}{N_2} (e_3 \bar{e_4} + \bar{e_3} e_4) \\ \frac{\beta_3}{N_2} (e_3 \bar{e_4} + \bar{e_3} e_4) \end{pmatrix}$$

In a similar way we can compute the constant vector

$J_2 = \left(J_2^{(1)}, J_2^{(2)}, J_2^{(3)}, J_2^{(4)}, J_2^{(5)}, J_2^{(6)}\right)^T \in \mathbf{R}^6$. Thus, finally we can determine $W_{20}(0)$ and $W_{11}(0)$ from Eq.(6.27) and (6.28). Hence, at last we can decide $W_{20}(0)$ and $W_{11}(0)$ from Eq.(6.27) and (6.28). Also, h_{ij} from Eq.(6.18) can be dictated by the parameters of the model. Therefore, we can register the accompanying qualities:

$$
\begin{aligned}
C_{11}(0) &= \frac{i}{2w_0\tau_0}\left(h_{20}h_{11} - 2|h_{11}|^2 - \frac{|h_{02}|^2}{3}\right) + \frac{h_{21}}{2} \\
\mu_{22} &= -\frac{Re\{C_{11}(0)\}}{Re\{\lambda'(\tau_0)\}} \\
\beta_{22} &= 2Re\{C_{11}(0)\} \\
T_{22} &= -\frac{Im\{C_{11}(0) + \mu_{22}Im\{\lambda'(\tau_0)\}\}}{w_0\tau_0}
\end{aligned}
\tag{6.34}
$$

where, $\lambda'(\tau_0) = \frac{d\lambda(\tau)}{d\tau}|_{\tau=\tau_0}$. Therefore, we have the following results:

Theorem 7. The expressions in (6.34) give the evaluation of the bifurcating periodic solutions in the center manifold at the critical value τ_0, for the delayed model (2.3). Then,

1. The indication of μ_{22} decides the direction of Hopf bifurcation. The Hopf bifurcation is forward or in backward proportionately as $\mu_{22} > 0$ or μ_{22} and the bifurcating periodic solution exists for $\tau > \tau_0$ and $\tau < \tau_0$ respectively.

2. The indication of β_{22} evaluates the stability of the bifurcating periodic solutions; if $\beta_{22} << 0$, bifurcating periodic solutions are stable and if $\beta_{22} > 0$, bifurcating periodic solutions are unstable.

3. The indication of T_{22} quantifies the period of the bifurcating periodic solutions; if $T_{22} < 0$, period decreases and if $T_{22} > 0$, period increases.

7. Numerical simulations and its biological involvement

To validate our theoretical results of the delayed model (2.3), we used MATLAB (R2014a) and MATHEMATICA 11 with the following set of parameters to integrate numerically, [1, 17, 27]:

$$\begin{aligned}\alpha_1 &= 1.51, \alpha_2 = 0.06, r = 1.45, \mu_h = 0.001538,\ \mu_p = 0.01, \beta_1 = 0.08, \beta_3 = 0.1\\ \gamma_1 &= 0.25, \delta = 0.001, \epsilon = 0.0222,\ K = 1000,\ N_1 = 2500, N_2 = 50\end{aligned} \tag{7.1}$$

where β_2 and τ are varied and $R_0 = 6.41\beta_2$. Utilizing the parameter values (7.1), virus-free and positive steady-state of the delay model (2.3) are given by $E_0 = (2500, 0, 1000, 0, 50, 0)$ and $E_1 = (X_1^*, X_3^*, V^*, V_2^*, Z_1^*, Z_2^*)$ where,

$$\begin{aligned}X_1^* &= \frac{125.525 - 14.4408\beta_2 + 92.5658\beta_2^2}{0.0502101 - 0.156006\beta_2 + 1.25\beta_2^2},\ X_3^* = \frac{124.783\beta_2(1.25\beta_2^2 - 0.156006)}{0.05021 - 0.156006\beta_2 + 1.25\beta_2^2}\\ V^* &= K = 1000,\ V_2^* = \frac{6902.65\beta_2(6.14\beta_2 - 1)}{60 + 6.90265\beta_2(6.41\beta_2 - 1)},\ Z_2^* = 6.9026(6.41\beta_2 - 1)\\ Z_1^* &= 287.597\beta_2 + \frac{390}{\beta_2} - 44.8669\end{aligned}$$

Putting $R_0 = 1$ and solving it for β_2, gives

$$\beta_2 = \beta_2^* = \frac{\alpha_2(\mu_p + \gamma_1)}{\beta_3} = 0.156$$

where β_2^* has taken as a bifurcation parameter. In proposed framework the impact of virus transmission rate β_2 on the stabilities of the framework (2.3) is explored as β_2 goes through β_2^*. The acquired outcomes are arranged in Table (7.1).

β_2	$\beta_2 < \beta_2^*$ 0.05	$\beta_2 < \beta_2^*$ 0.08	$\beta_2 < \beta_2^*$ 0.10	$\beta_2 = \beta_2^*$ 0.156	$\beta_2 < \beta_2^*$ 0.20	$\beta_2 < \beta_2^*$ 0.40	$\beta_2 < \beta_2^*$ 0.60
R_0	0.3201	0.5128	0.6510	1	1.2820	2.5641	3.8461
X_1	2500	2500	2500	2500	2141.03	910.36	474.35
X_3	0.0	0.0	0.0	0.0	18.60	82.39	104.99
V	1000	1000	1000	1000	1000	1000	1000
V_2	0.0	0.0	0.0	0.0	6.44	67.13	164.19
Z_1	50	50	50	50	1962.62	1024.17	777.69
Z_2	0.0	0.0	0.0	0.0	1.940	10.790	19.640
Stable state	E_0	E_0	E_0	E_0	E_1	E_1	E_1

Table 7.1: The effect of β_2 on $R_0, X_1, X_3, V, V_2, Z_1andZ_2$ at equilibrium points of delayed model (2.3):

According to the table (7.1), increasing β_2 causes an increase in the basic reproduction number, R_0, which leads to an increase in the quantity of disease and a decrease in the quantity of susceptible.

Example 1. *In this example, the stability of virus-free steady-state E_0 is explored by reproducing the delayed model (2.3) as follows. Utilizing the parametric values in (7.1) and $\beta_2 = 0.05$, gives $R_0 = 0.3201 < 1$. It is seen that the model (2.3) has the virus-free steady-state $E_0 = (2500, 0, 1000, 0, 50, 0)$. Fig. (1) shows that E_0 is absolutely stable for all $\tau \geq 0$ as in accordance with Theorem (3). By looking at the estimation of τ, the profiles of susceptible individuals initially decrease, but as time goes on, these are continuously increasing and then go on saturated, see Fig. 1(a). Similarly, the profiles of the mosquito population continuously increase and then saturated, see Fig.1(c), and the susceptible pigs are initially decreasing, but as time goes on, these are continuous increases and then go on saturated, see Fig. 1(e). while the profiles of infected individuals in 1(b) and 1(d) are given off an impression of being pandemic at first and are at the end terminated but in Fig. 1(f) infected individuals are continuous decreases. It has been determined that the virus is eradicated from the population. As a consequence of the above discussion, we have concluded that the number of susceptible individuals will increase while the number of infected individuals will remain constant. It is predicted that the population will be free of viruses.*

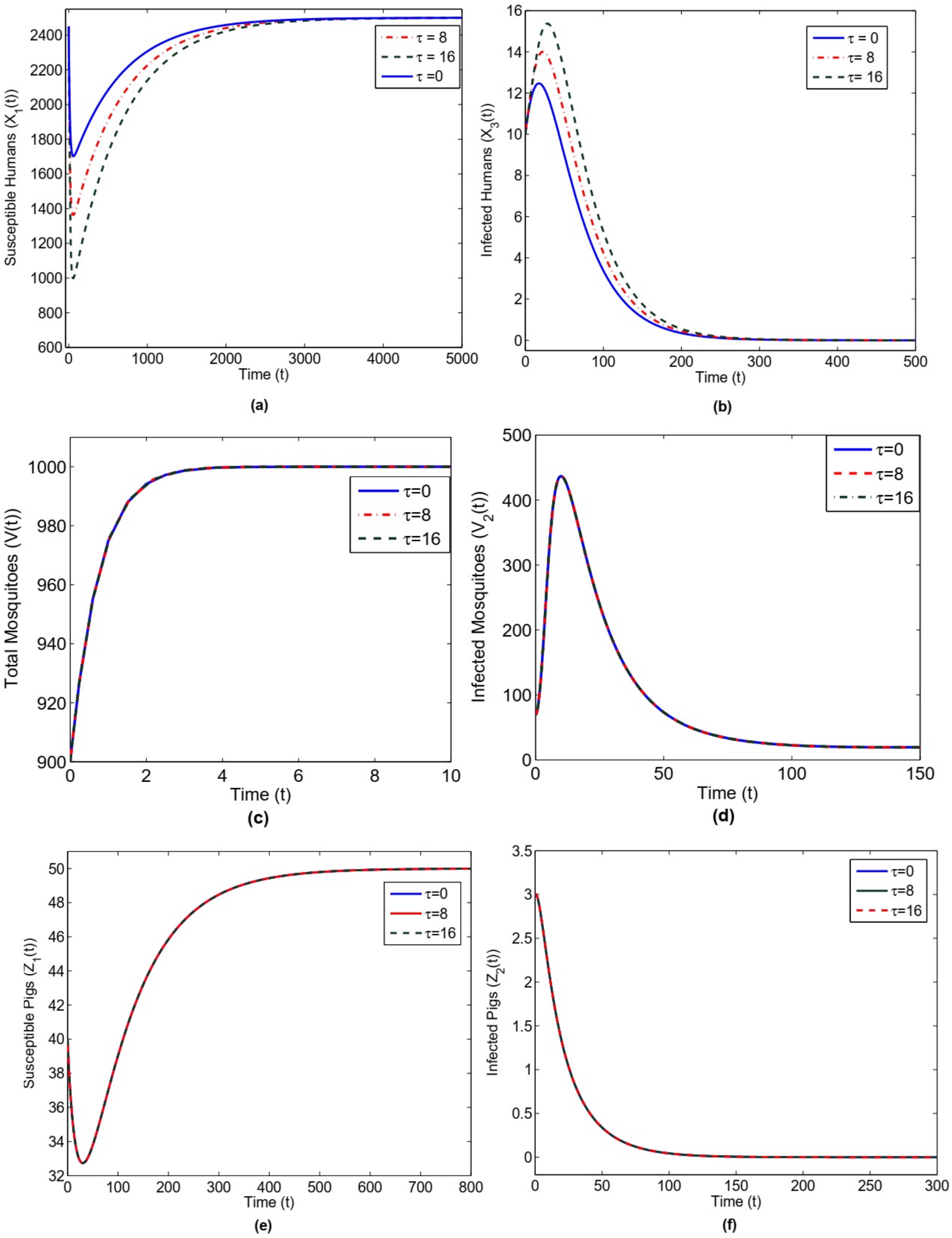

Fig. 1: The stability behavior of delayed model (2.1) at virus-free steady-state E_0 for $\tau = 0, 8, 16$ and the parametric values are taken from (7.1), when the basic reproduction number $R_0 < 1$.

Example 2. *Here, the stability of positive steady-state E_1 is examined by simulating the framework (2.3) as follows. Utilizing the parameters in (7.1) and $\beta_2 = 0.4$, gives $R_0 = 2.5641$ and the condition $c_2 > 0$ holds. Hence, the delayed model (2.3) have an equilibrium*
$E_1(910.360, 82.390, 1000, 67.130, 1024.17, 10.79)$. It is shown in Fig. (2), this signifies that E_1 is absolutely stable $\forall\ \tau \geq 0$ as guaranteed by Theorem (4). This result shows that the influence of time delay in the transmission of Japanese encephalitis. From Figs. 2(a) -(b) it is more clear that a higher value of τ increases the infection in the human population. This is because of increasing the value of τ, susceptible humans decrease whereas infected humans increase monotonically before the convergence to the steady-state E_1. While Figs. 2(c)-(f) shows that the profiles of individuals without time delay. These figures only show the stability of a positive steady-state as shown in Theorem (4).

Example 3. *In this example, we are going to analyze the stability and Hopf bifurcation of the delayed model, when $R_0 > 1$ and $c_1 < 0$. If the parametric values are $\beta_1 = 0.6, \beta_2 = 0.7, \beta_3 = 0.7, \gamma_1 = 0.025, \mu_h = 0.01538, \epsilon = 0.222$ and the remaining values are in (7.1), then $R_0 = 85$ and the delayed model passes a unique endemic equilibrium point $(133, 37, 1000, 227, 37, 29)$. The critical-time delay value τ_0 is $\tau = \tau_0 = 7.43922$. Consequently, the behaviors of the model for different values of τ are manifested in Fig. 3(a)-(d) and Fig. 4(a)-(b). It is found that as τ passes through τ_0, the steady-state E_1 losses its stability and a Hopf bifurcation happened as guaranteed by Theorem 6(2). Furthermore, using the Theorem (7), we can determine the value of $C_{11}(0)$, μ_{22}, β_{22} and T_{22} as $C_{11}(0) = -5.38742 \times 10^{-7} + 1.15966 \times 10^{-7}i$, $\mu_{22} = 6.37095 \times 10^{-5}$, $\beta_{22} = -1.07748 \times 10^{-6}$, $T_{22} = 1.60986 \times 10^{-7} > 0$. Since $\mu_{22} > 0$ and $\beta_{22} < 0$, therefore by Theorem (7), the Hopf bifurcation is supercritical and the periodic solution bifurcating from E_1 is steady, which can be further demonstrated by Fig. 5(a)-(b). From Fig. (6), we also observe that on increasing the value of τ, the periodic solutions have a smaller period and they disappear for large enough. Thus, the numerical estimation shows that if the τ is sufficiently large then nonexistence periodic solution and for a small value of τ, the periodic solution exists. Since we realize that the steady-state E_1 is asymptotically steady if $\tau < \tau_0$. When τ goes through τ_0, the state E_1 losses its stability and a Hopf bifurcation happens. The solution plots for $\tau > \tau_0$ ("unstable positive equilibrium and stable periodic solution") have appeared in Fig. 5(a)- (b). Fig. (6) shows that the period and amplitude of the periodic solutions of the delayed (2.3) turns out to be large as the* *delay increases. From the top line to the base the delays are* $\tau = 60, \tau = 61$ *and* $\tau = 62$*, respectively. These plots show that the instability strong and stronger with an increasing value of* τ.

Example 4. *The effect of* β_1 *and* μ_h *on the occurrence of Hopf bifurcation has been discussed in this example. The influence of disease transmission rate from an infected mosquito to susceptible individuals and the natural death rate of humans is investigated of the delayed model (2.3) with various values of* β_1 *and* μ_h*. For* $\beta_2 = 0.6$*, the critical value of* β_1 *and* μ_h *of the model (2.3) is determined by the value of* c_1 *from Theorem (4) as* $\beta_1 > 0.3$ *and* $\mu_h > 0.001$*. For* $\beta_1 > 0.3$ *and* $\mu_h > 0.001$*, the stable and unstable regions of the model (2.3) have appeared in Fig. (7). This figure gives the base estimation of time delay for various estimations of* β_1 *and* μ_h *so the model gets stable inside the endorsed them. Figures show that as* β_1 *and* μ_h *increase, the critical time delay* τ_0 *increases. These outcomes can be deciphered that the event of Hopf bifurcation in the delayed model is delayed as a bigger infers a more drawn out intermingling time.*

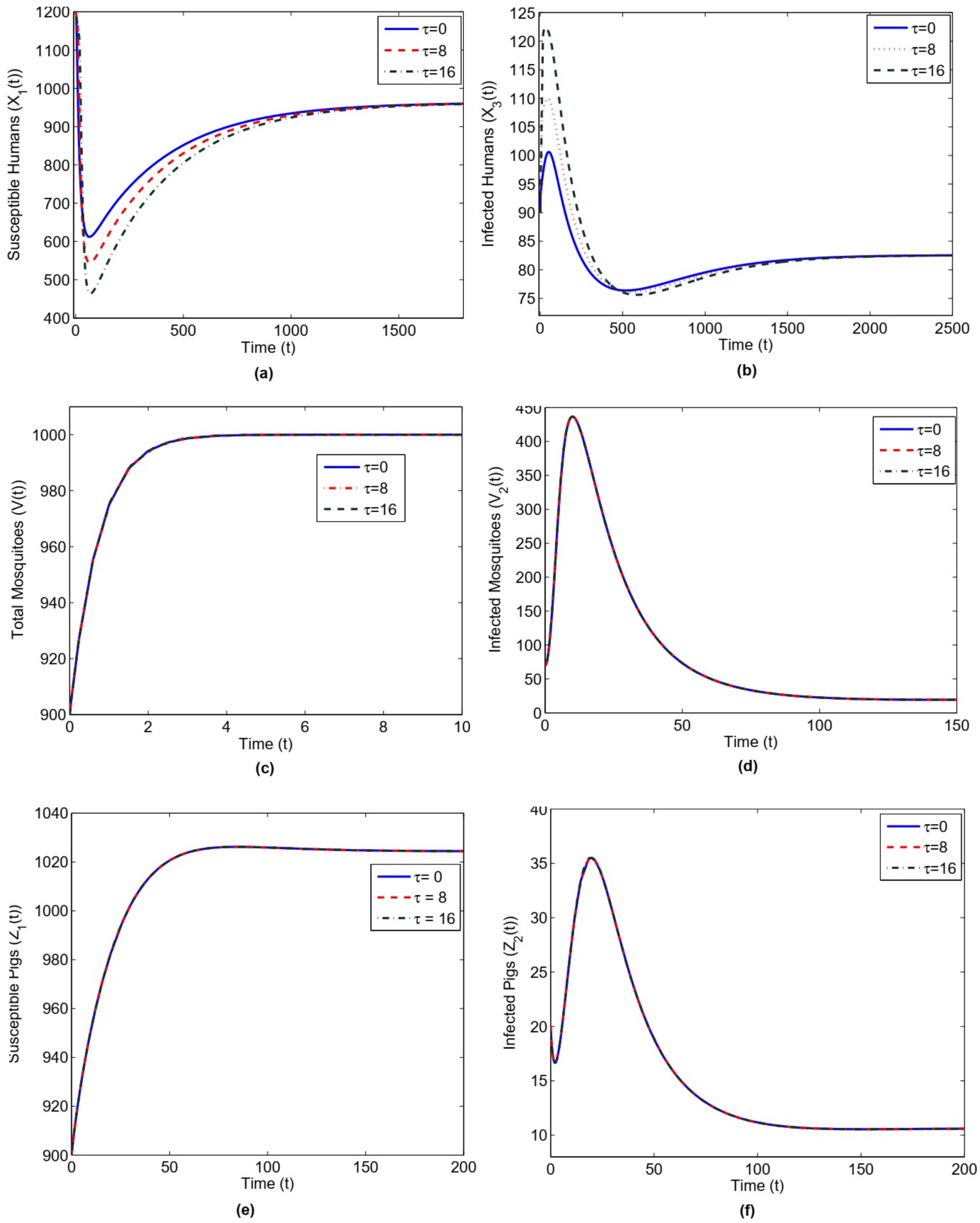

Fig. 2: The stability behavior of delayed model (2.3) at positive steady-state E_1 for $\tau = 0, 8, 16$ and the parametric values are taken from (7.1), when the basic reproduction number $R_0 > 1$.

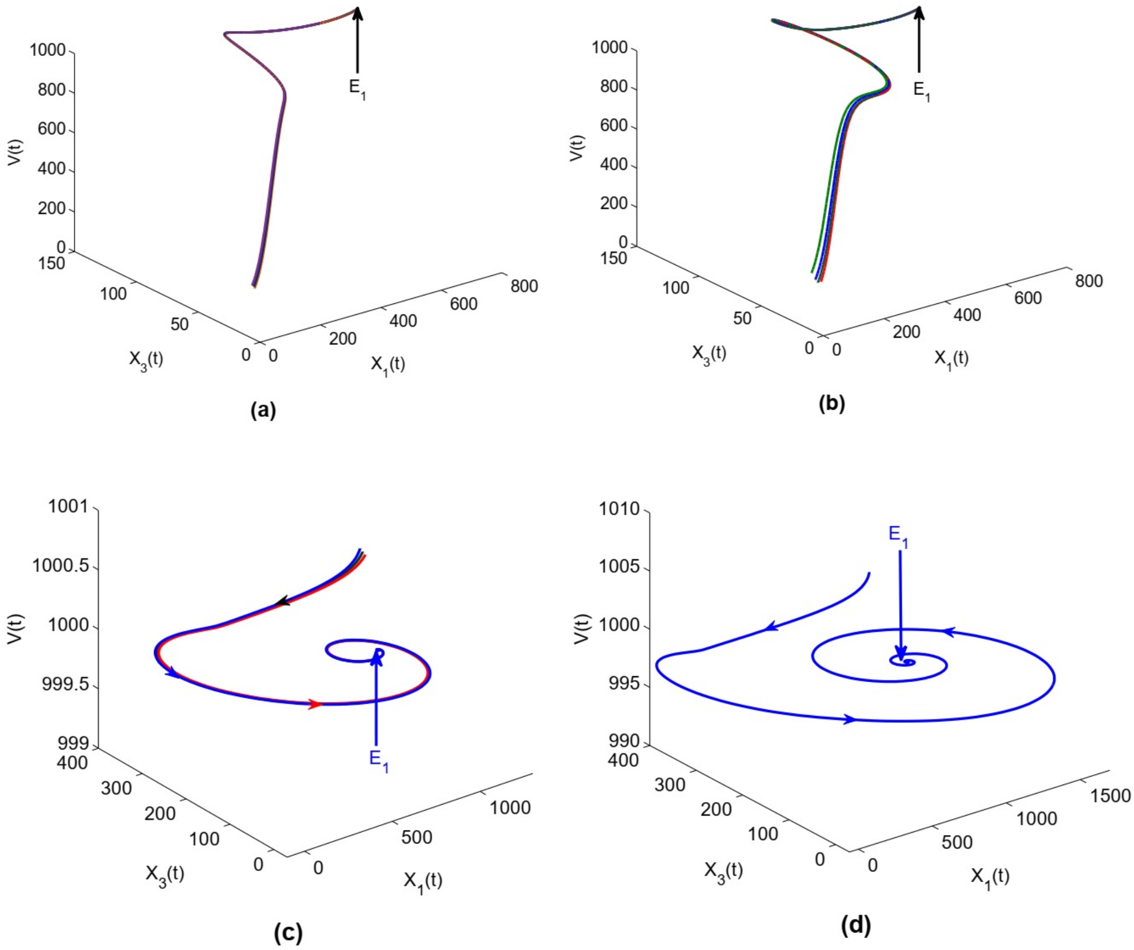

Fig. 3: The stability of delayed model (2.3) at positive state E_1 for $R_0 > 1$, when $\beta_1 = 0.6$, $\beta_2 = 0.7$, $\beta_3 = 0.7$, $\epsilon = 0.222$, $\gamma_1 = 0.025$, $\mu_h = 0.01538$ and different time delay (a): $\tau = 0$, (b): $\tau = 5.8$, (c): $\tau = 20$, (d): $\tau = 30$.

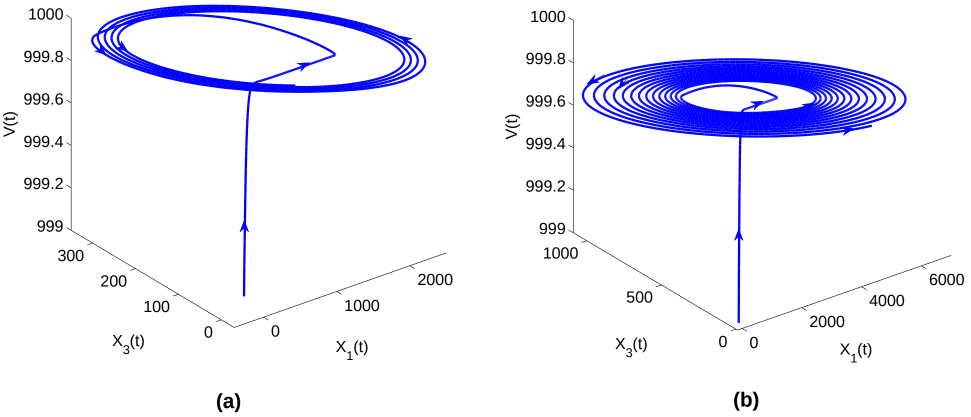

Fig. 4: The delayed model (2.3) is unstable and Hopf bifurcation happens for $R_0 > 1$, when $\beta_1 = 0.6$, $\beta_2 = 0.7$, $\beta_3 = 0.7$, $\epsilon = 0.222$, $\gamma_1 = 0.025$, $\mu_h = 0.01538$ and different time delay (a): $\tau = 65$, (b): $\tau = 70$

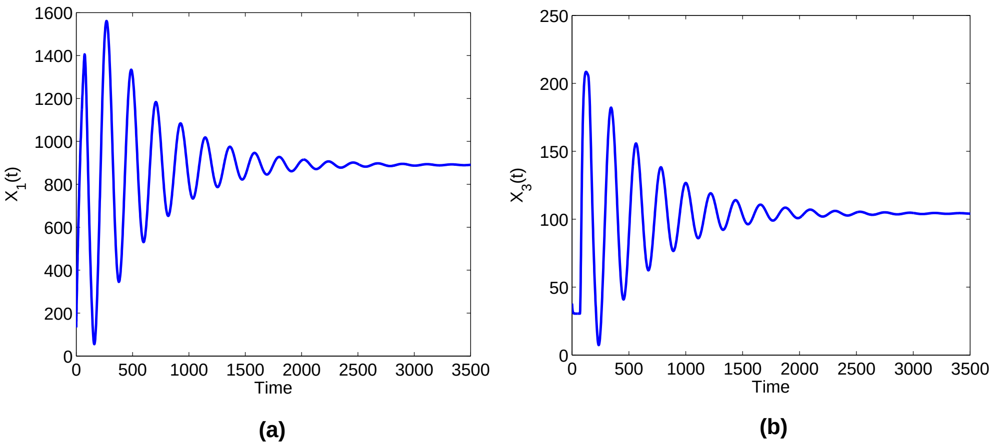

Fig. 5: The state E_1 is asymptotically stable when $\tau < \tau_0$, where, $\beta_1 = 0.6$, $\beta_2 = 0.65$, $\beta_3 = 0.65$, $\epsilon = 0.222$, $\gamma_1 = 0.027$, $\mu_h = 0.01538$ and different time delay (a): $\tau = 45$, (b): $\tau = 50$

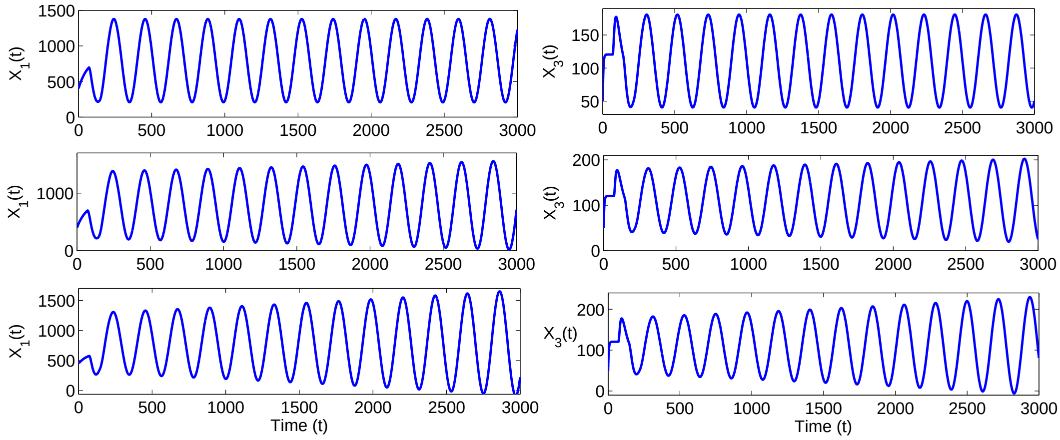

Fig. 6: Periods and amplitudes of periodic solutions of delayed model (2.3), when $\tau = 60, 61, 62$ respectively, where $\beta_1 = 0.7$, $\beta_2 = 0.6$, $\beta_3 = 0.6$, $\epsilon = 0.222$, $\gamma_1 = 0.027$, $\mu_h = 0.01538$.

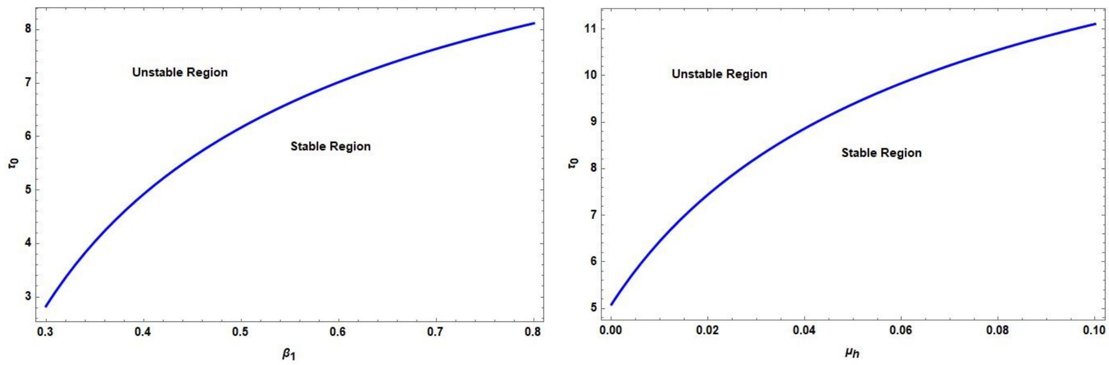

Fig. 7: The graphs showing that variation of time delay w.r.t. β_1 and μ_h, which indicates that the stable and unstable regions of the delayed model (2.3). where $\beta_1 = 0.3$, $\beta_2 = 0.6$, $\beta_3 = 0.6$, $\epsilon = 0.222$, $\gamma_1 = 0.027$, $\mu_h = 0.01538$.

8. Conclusions

We investigated the impact of time delay on the dynamics of the Japanese encephalitis system in this study. In particular, we considered a six-dimensional delay differential model of Japanese encephalitis after removing the vaccination strategies from the human population. Our theoretical and numerical simulation outcomes show that the time delay incredibly influences the dynamics of the framework. Beginning from the more reasonable actuality that when people get an infection from JEV, they do not turn out to be sick or infected with the disease in a split second. It implies that the infective class requires time to have the option of transferring the disease to a defenseless level. As a result, it considers a delayed effect on the dynamics of the Japanese encephalitis model. The models are thoroughly broke down to pick up bits of knowledge into the impact of time delay and their subjective dynamics. The existence and stability of steady-states, the Hopf bifurcation and the stability sharing marvels, and the direction and stability of Hopf bifurcation are all considered as dynamical properties of the system. The main finding of the work is summarized below.

1. The proposed delayed model confirms R_0, called the basic reproduction number and it have two steady-state, namely virus-free E_0 and positive E_1. The existence of E_0 is obvious while E_1 exists if $R_0 > 1$.

2. If $R_0 < 1$, E_0 is absolutely stable $\forall\ \tau \geq 0$ and it is unstable if $R_0 > 1$. This signifies that if $R_0 < 1$ then virus dies out from the population and if $R_0 > 1$ then virus is present in the population.

3. If $R_0 > 1$, there exist different form of stability and Hopf bifurcation incidence at E_1, including (1) If $c_2 > 0$, E_1 is absolutely stable $\forall$ $\tau \geq 0$, (2) there exist a unique critical-delay $\tau_0 > (0)$ such that E_1 is conditionally stable $\forall\ \tau_0 \in [0, \tau)$ and E_1 is unstable and Hopf bifurcation happens when $\tau = \tau_0$. Moreover, τ goes through the critical point τ_0, a groups of periodic solutions bifurcates from E_1.

4. If $R_0 > 1$, the direction and stability of Hopf bifurcation have talked about, see Theorem (7). Furthermore, it is viewed as supercritical and steady for the considered data.

5. By numerical simulations, it is shown that the larger values of virus transmission rate from an V_2 to X_1 and the natural death rate of humans of a delayed model (2.3) affect the existence of Hopf bifurcation (it can be verified by Fig. (7)).

Finally, the introduction of a time delay in the JEV transmission term can balance out the structure as the DDEs framework is consistent $\forall\ \tau \geq 0$. It can moreover destabilize the system and occasional arrangements can arise through Hopf bifurcation while using the time delay as a bifurcation boundary. It has been pointed out that just R_0 accepts a huge occupation in the elements of the delayed model. That is, R_0 changes the stability of steady-state when it experiences unity. The τ changes the strength of the positive state employing Hopf bifurcation when it goes through τ_0. This infers the framework will be unsteady and the virus can not be viably controlled. Truly, different components can influence the infection episode. The paper centers around breaking down the impact of a time delay with different parameters. In our future work, we can contemplate the effect of a period delay with vaccination on the DDEs of encephalitis framework with time-varying parameters.

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