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Energy and Randić energy of special graphs

Jahfar T. K.
National Institute of Technology, India
and
Chithra A. V.
National Institute of Technology, India
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Abstract

In this paper, we determine the Randić energy of the m-splitting graph, the m-shadow graph and the m-duplicate graph of a given graph, m being an arbitrary integer. Our results allow the construction of an infinite sequence of graphs having the same Randić energy. Further, we determine some graph invariants like the degree Kirchhoff index, the Kemeny's constant and the number of spanning trees of some special graphs. From our results, we indicate how to obtain infinitely many pairs of equienergetic graphs, Randić equienergetic graphs and also, infinite families of integral graphs.

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1. Introduction

In this paper, we consider simple connected graphs. Let $G = (V, E)$ be a simple graph of order p and size q with vertex set $V(G) = \{v_1, v_2, ..., v_p\}$ and edge set $E(G) = \{e_1, e_2, ..., e_q\}$. The degree of a vertex v_i in G is the number of edges incident to it and is denoted by $d_i = d_G(v_i)$. The adjacency matrix $A(G) = [a_{ij}]$ of the graph G is a square symmetric matrix of order p whose $(i, j)^{th}$ entry is defined by

$$a_{i,j} = \begin{cases} 1, & \text{if } v_i \text{ and } v_j \text{ are adjacent,} \\ 0, & \text{otherwise.} \end{cases}$$

The eigenvalues $\lambda_1, \lambda_2, ..., \lambda_p$ of the graph G are defined as the eigenvalues of its adjacency matrix $A(G)$. If $\lambda_1, \lambda_2, ..., \lambda_t$ are the distinct eigenvalues of G, the spectrum of G can be written as

$$Spec(G) = \begin{pmatrix} \lambda_1 & \lambda_2 & ... & \lambda_t \\ m_1 & m_2 & ... & m_t \end{pmatrix},$$

where m_j indicates the algebraic multiplicity of the eigenvalue λ_j, $1 \leq j \leq t$ of G. The energy [11] of the graph G is defined as $\varepsilon(G) = \sum_{i=1}^{p} |\lambda_i|$. More results on graph energy are reported in [3,11]. The Randić matrix $R(G) = [R_{i,j}]$ of a graph G is a square matrix of order p whose $(i, j)^{th}$ entry is

$$R_{i,j} = \begin{cases} \frac{1}{\sqrt{d_i d_j}}, & \text{if } v_i \text{ and } v_j \text{ are adjacent,} \\ 0, & \text{otherwise.} \end{cases}.$$

The eigenvalues of $R(G)$ are called Randić eigenvalues of G and it is denoted by $\rho_i, 1 \leq i \leq p$. If $\rho_1, \rho_2, ..., \rho_s$ are the distinct Randić eigenvalues of G, then the Randić spectrum of G can be written as

$$RS(G) = \begin{pmatrix} \rho_1 & \rho_2 & ... & \rho_s \\ m_1 & m_2 & ... & m_s \end{pmatrix},$$

where m_j indicates the algebraic multiplicity of the eigenvalue ρ_j,$1 \leq j \leq s$ of G. If G has no isolated vertices, then $R(G) = D^{-1/2}A(G)D^{-1/2}$, where $D^{1/2}$ is the diagonal matrix with diagonal entries $\frac{1}{\sqrt{d_i}}$ for every i, $1 \leq i \leq p$ [4]. Randić energy of G is defined as $\varepsilon_R(G) = \sum_{i=1}^{p} |\rho_i|$ [4, 12]. More results on Randić energy are reported in [1, 10]. A graph is said to be integral if all the eigenvalues of its adjacency matrix are integers. of H, and by joining each vertex of the i^{th} copy of H to the i^{th} vertex of G, $1 \le i \le n$. The join $G + H$ of two graphs G and H is obtained by joining all the vertices of G to the vertices of H. Two non-isomorphic graphs G_1 and G_2 of the same order are said to be equienergetic if $\varepsilon(G_1) = \varepsilon(G_2)$ [18]. In analogy to this, two graphs G_1 and G_2 of same order are said to be Randić equienergetic if $\varepsilon_R(G_1) = \varepsilon_R(G_2)$ [2]. The normalized Laplacian matrix $\mathcal{L}(G) = (\mathcal{L}_{ij})$ is the square matrix of order p whose $(i, j)^{th}$ entry is defined as,

$$\mathcal{L}_{ij} = \begin{cases} 1, \text{if } v_i = v_j \text{ and } d_i \neq 0, \\ -\frac{1}{\sqrt{d_i d_j}}, \text{if } v_i \text{ and } v_j \text{ are adjacent in } G, \\ 0, \text{otherwise.} \end{cases} .$$

For a graph G without isolated vertices, the normalized Laplacian matrix can be written as

$$\mathcal{L}(G) = I_n - D^{-\frac{1}{2}}(G)A(G)D^{-\frac{1}{2}}(G).$$

The eigenvalues of the matrix $\mathcal{L}(G)$ are called the normalized Laplacian eigenvalues of G and it is denoted by $0 = \tilde{\mu}_1(G) \le \tilde{\mu}_2(G)... \le \tilde{\mu}_p(G)$. Let G be a graph without isolated vertices. Then its normalized Laplacian matrix $\mathcal{L}(G)$ and Randić matrix $R(G)$ are related by $\mathcal{L}(G) = I - R(G)$. The normalized Laplacian eigenvalues $\tilde{\mu}_i(G)$ and Randić eigenvalues $\rho_i(G)$ are related by $\tilde{\mu}_i(G) = 1 - \rho_i(G)$, for $i = 1, 2, ..., p$. The degree Kirchhoff index of connected graph G is defined in [6] as

$$Kf^*(G) = \sum_{i<j} d_i r^*_{i,j} d_j$$

where $r^*_{i,j}$ denotes the resistance distance [15] between vertices v_i and v_j in a graph G. In [6], the authors proved that

$$Kf^*(G) = 2q \sum_{i=2}^{p} \frac{1}{\tilde{\mu}_i(G)}.$$

In general, the computation of the degree Kirchhoff index of a graph is a difficult thing. Here we obtained the formula for finding the degree Kirchhoff index of some families of graphs.

The Kemeny's constant $K(G)$ of a connected graph G [5] is defined in terms of normalized Laplacian as

$$K(G) = \sum_{i=2}^{p} \frac{1}{\tilde{\mu}_i(G)}.$$

The various applications of the Kemeny's constant to perturbed Markov chains, random walks on directed graphs are studied in [13]. The number of spanning trees (distinct spanning subgraphs of G that are trees) of G [8] can be expressed in terms of the normalized Laplacian eigenvalues as

$$t(G) = \frac{\prod_{i=1}^{p} d_i \prod_{i=2}^{p} \tilde{\mu}_i(G)}{\sum_{i=1}^{p} d_i}.$$

We use the notations K_n, C_n and $K_{1,n-1}$ throughout this paper to denote the complete graph, the cycle and the star graph on n vertices respectively. Let J_m be the $m \times m$ matrix of all ones and I_m be the identity matrix of order m.

The rest of the paper is organized as follows. In Section 2, we give a list of some previously known results which are useful for further reference in this paper. In Section 3, Randić energy of the m-splitting graph, the m-shadow graph and the m-duplicate graphs are obtained. In Section 4, our results allow the construction of an infinitely many integral and Randić integral graphs. Also, our results show how to construct equienergetic and Randić equienergetic graphs. In Section 5, we discuss the graph invariants like the degree Kirchhoff index, the Kemeny's constant and the number of spanning trees of resulting graphs from various graph operations.

2. Preliminaries

In this section, we recall the concepts of the m-splitting graph, the m-shadow graph and the m-duplicate graph of a graph and list some results that will be used in the subsequent sections.

Definition 2.1. *The Kronecker product of two graphs G_1 and G_2 is the graph $G_1 \times G_2$ with vertex set $V(G_1) \times V(G_2)$ and the vertices (x_1, x_2) and (y_1, y_2) are adjacent if and only if (x_1, y_1) and (x_2, y_2) are edges in G_1 and G_2 respectively.*

Definition 2.2. *Let $A \in R^{m\times n}$,$B \in R^{p\times q}$. Then the Kronecker product of A and B is defined as follows*

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & a_{13}B & \dots & a_{1n}B \\ a_{21}B & a_{22}B & a_{23}B & \dots & a_{2n}B \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & a_{m3}B & \dots & a_{mn}B \end{bmatrix}.$$

Proposition 2.1. *Let $A, B \in R^{n\times n}$. Let λ be an eigenvalue of matrix A with corresponding eigenvector x and μ be an eigenvalue of matrix B with corresponding eigenvector y, then $\lambda\mu$ is an eigenvalue of $A \otimes B$ with corresponding eigenvector $x \otimes y$.*

Definition 2.3. *Let G be a simple (p, q) graph. Then the m-splitting graph of a graph G, $Spl_m(G)$ is obtained by adding to each vertex v of G new m vertices say, $v_1, v_2, ..., v_m$ such that v_i ,$1 \leq i \leq m$ is adjacent to each vertex that is adjacent to v in G. The adjacency matrix of the m-splitting graph of the graph G is*

$$A(Spl_m(G)) = \begin{bmatrix} A(G) & A(G) & A(G) & \dots & A(G) \\ A(G) & O & O & \dots & O \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A(G) & O & O & \dots & O \end{bmatrix}_{(m+1)p}.$$

If $m = 1$, the m-splitting graph of the graph G is known as splitting graph of G[21], denoted by $Spl(G)$. The number of vertices and the number of edges in $Spl_m(G)$ are $(m + 1)p$ and $(m + 1)q$ respectively.

Proposition 2.2. *The energy of the m-splitting graph of G is $\varepsilon(Spl_m(G)) = \sqrt{1 + 4m}\varepsilon(G)$.*

Definition 2.4. *Let G be a simple (p, q) graph. Then the m-shadow graph $D_m(G)$ of a connected graph G is constructed by taking m copies of G say, $G_1, G_2, ..., G_m$ then join each vertex u in G_i to the neighbors of the corresponding vertex v in $G_j, 1 \leq i \leq m, 1 \leq j \leq m$. The adjacency matrix of the m-shadow graph of G is*

$$A(D_m(G)) = \begin{bmatrix} A(G) & A(G) & A(G) & \dots & A(G) \\ A(G) & A(G) & A(G) & \dots & A(G) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A(G) & A(G) & A(G) & \dots & A(G) \end{bmatrix}_{mp}.$$

If $m = 2$, the m-shadow graph of G is known as shadow graph of G[16]. The number of vertices and the number of edges in $D_m(G)$ are pm and m^2q respectively.

Proposition 2.3. *The energy of the m-shadow graph of G is $\varepsilon(D_m(G)) = m\varepsilon(G)$.*

Definition 2.5. *Let $G = (V, E)$ be a simple (p, q) graph with vertex set V and edge set E. Let V' be a set such that $V \bigcap V' = \emptyset$, $|V| = |V'|$ and $f : V \to V'$ be bijective (for $a \in V$ we write $f(a)$ as a' for convenience). A duplicate graph of G is $D(G) = (V_1, E_1)$, where the vertex set $V_1 = V \cup V'$ and the edge set E_1 of $D(G)$ is defined as, the edge ab is in E if and only if both ab' and $a'b$ are in E_1.*

In general the m-duplicate graph of the graph G, $D^m(G)$ is defined as $D^m(G) = D^{m-1}(D(G))$.

The number of vertices and the number of edges in the m-duplicate graph of the graph are $2^m p$ and $2^m q$ respectively. With suitable labeling of the vertices, the adjacency matrix of $D(G)$ is

$$A(D(G)) = \begin{bmatrix} O_{p\times p} & A(G) \\ A(G) & O_{p\times p} \end{bmatrix}.$$

Proposition 2.4. *The energy of the duplicate graph of G is $\varepsilon(D(G)) = 2\varepsilon(G)$.*

In [17], authors remarked that, the m-duplicate graph of G, $D^m(G) = G \times K_2 \times K_2 ... \times K_2$ (K_2 repeats m-times). The energy of the m-duplicate graph of G is $\varepsilon(D^m(G)) = \varepsilon(G).\varepsilon(K_2)...\varepsilon(K_2) = 2^m\varepsilon(G)$[3].

3. Randić energy of the m-splitting, the m-shadow and the m-duplicate graphs

In this section, we present the Randić energy of the m-splitting graph, the m-shadow graph and the m-duplicate graphs of G. Also, we obtain some new families of Randić equienergetic graphs. In addition, our results show how to construct infinitely many families of integral graphs.

Theorem 3.1. *Let G be a simple (p, q) graph without isolated vertices. Then the Randić energy of the m-splitting graph of G is $\varepsilon_R(Spl_m(G)) = \frac{2m+1}{m+1}\varepsilon_R(G)$.*

Proof. The Randić matrix of m- splitting graph of G is $R(Spl_m(G))$

$$= \begin{bmatrix} ((m+1)\,D)^{-\frac{1}{2}} & O & \cdots & O \\ O & D^{-\frac{1}{2}} & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & D^{-\frac{1}{2}} \end{bmatrix}_{p(m+1)}$$

$$\cdot \begin{bmatrix} A(G) & A(G) & A(G) & \cdots & A(G) \\ A(G) & O & O & \cdots & O \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A(G) & O & O & \cdots & O \end{bmatrix}_{p(m+1)}$$

$$\begin{bmatrix} ((m+1)D)^{-\frac{1}{2}} & O & \cdots & O \\ O & D^{-\frac{1}{2}} & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & D^{-\frac{1}{2}} \end{bmatrix}_{p(m+1)}$$

$$= \begin{bmatrix} ((m+1)D)^{-\frac{1}{2}} \mathrm{A}(G)((m+1)D)^{-\frac{1}{2}} & ((m+1)D)^{-\frac{1}{2}} \mathrm{A}(G)\mathrm{D}^{-\frac{1}{2}} & \cdot \cdots & ((m+1)D)^{-\frac{1}{2}} \mathrm{A}(G)\mathrm{D}^{-\frac{1}{2}} \\ ((m+1)D)^{-\frac{1}{2}} \mathrm{A}(G)\mathrm{D}^{-\frac{1}{2}} & 0 & 0\cdots & 0 \\ \vdots & \vdots & \vdots\ddots & \vdots \\ ((m+1)D)^{-\frac{1}{2}} \mathrm{A}(G)\mathrm{D}^{-\frac{1}{2}} & 0 & 0\cdots & 0 \end{bmatrix}_{\mathrm{p(m+1)}}$$

$$= \begin{bmatrix} \frac{1}{\sqrt{m+1}} & 1 & 1 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix}_{m+1} \otimes \frac{1}{\sqrt{m+1}} D^{\frac{-1}{2}} A(G) D^{\frac{-1}{2}}$$

$$= B \otimes \frac{1}{\sqrt{m+1}} D^{\frac{-1}{2}} A(G) D^{\frac{-1}{2}}, \text{ where } B = \begin{bmatrix} \frac{1}{\sqrt{m+1}} & 1 & 1 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix}_{(m+1)}$$

The eigenvalues of B are $\sqrt{m+1}$, $\frac{-m}{\sqrt{m+1}}$ and 0, and 0 has multiplicity $m-1$. So spectrum of B is

$$Spec(B) = \begin{pmatrix} 0 & \sqrt{m+1} & \frac{-m}{\sqrt{m+1}} \\ m-1 & 1 & 1 \end{pmatrix}.$$

Thus the Randić spectrum of the m-splitting graph is,

$$RS(Spl_m(G)) = \begin{pmatrix} 0 & \rho_1 & \rho_2 & \cdots & \rho_p & \frac{-m}{m+1}\rho_1 & \frac{-m}{m+1}\rho_2 & \cdots & \frac{-m}{m+1}\rho_p \\ p(m-1) & 1 & 1 & \ldots & 1 & 1 & 1 & \ldots & 1 \end{pmatrix}.$$

Hence the Randić energy of the m-splitting graph of G is $\varepsilon_R(Spl_m(G)) = \frac{2m+1}{m+1}\varepsilon_R(G)$. □

If $m = 1$ in Theorem 3.1, we get the Randić energy of splitting graph of G is $\varepsilon_R(Spl(G)) = \frac{3}{2}\varepsilon_R(G)$[7].

Corollary 3.2. *Let G_1 and G_2 be Randić equienergetic graphs. Then $Spl_m(G_1)$ and $Spl_m(G_2)$ are Randić equienergetic.*

In [19], Rojo et al. have obtained the construction of bipartite graphs having the same Randić energy. We indicate how to obtain infinitely many pairs of graphs (other than bipartite graphs) having the same Randić energy. The following theorem gives some information how to construct a new family of graphs having the same Randić energy as that of G.

Theorem 3.3. *Let G be a simple (p,q) graph without isolated vertices. Then Randić energy of the m-shadow graph of G, $m>1$ is $\varepsilon_R(D_m(G)) = \varepsilon_R(G)$.*

Proof. The Randić matrix of the m-shadow graph of G is

$$= \begin{bmatrix} (mD)^{-\frac{1}{2}} & O & O & \cdots & O \\ O & (mD)^{-\frac{1}{2}} & O & \cdots & O \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ O & O & O & \cdots & (mD)^{-\frac{1}{2}} \end{bmatrix}_{pm} \begin{bmatrix} A(G) & A(G) & A(G) & \cdots & A(G) \\ A(G) & A(G) & A(G) & \cdots & A(G) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A(G) & A(G) & A(G) & \cdots & A(G) \end{bmatrix}_{pm} \begin{bmatrix} (mD)^{-\frac{1}{2}} & O & O & \cdots & O \\ O & (mD)^{-\frac{1}{2}} & O & \cdots & O \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ O & O & O & \cdots & (mD)^{-\frac{1}{2}} \end{bmatrix}_{pm}$$

$$= \begin{bmatrix} (mD)^{-\frac{1}{2}} \mathrm{A}(G) (mD)^{-\frac{1}{2}} & (mD)^{-\frac{1}{2}} \mathrm{A}(G) (mD)^{-\frac{1}{2}} & \cdots & (mD)^{-\frac{1}{2}} \mathrm{A}(G) (mD)^{-\frac{1}{2}} \\ (mD)^{-\frac{1}{2}} \mathrm{A}(G) (mD)^{-\frac{1}{2}} & (mD)^{-\frac{1}{2}} \mathrm{A}(G) (mD)^{-\frac{1}{2}} & \cdots & (mD)^{-\frac{1}{2}} \mathrm{A}(G) (mD)^{-\frac{1}{2}} \\ \vdots & \vdots & \ddots & \vdots \\ (mD)^{-\frac{1}{2}} \mathrm{A}(G) (mD)^{-\frac{1}{2}} & (mD)^{-\frac{1}{2}} \mathrm{A}(G) (mD)^{-\frac{1}{2}} & \cdots & (mD)^{-\frac{1}{2}} \mathrm{A}(G) (mD)^{-\frac{1}{2}} \end{bmatrix}_{\mathrm{pm}}$$

$$= \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix}_{m} \otimes \frac{1}{m} D^{\frac{-1}{2}} A(G) D^{\frac{-1}{2}}$$

The eigenvalues of J_m are m and 0, and 0 has multiplicity $m-1$. Therefore, the Randić spectrum of the m-shadow graph is

$$RS(D_m(G)) = \begin{pmatrix} 0 & \rho_1 & \rho_2 & \cdots & \rho_p \\ p(m-1) & 1 & 1 & \ldots & 1 \end{pmatrix}.$$

Hence $\varepsilon_R(D_m(G)) = \varepsilon_R(G)$. □

The following Proposition helps us to construct infinite sequence of Randić integral graphs.

Proposition 3.1. *Let G be a simple (p,q) graph and $m \geq 2$ an integer. Then G is Randić integral if and only if the m-shadow graph of G is Randić integral.*

For example, $D_m(C_4)$ and $D_m(K_{1,4})$ are Randić integral for every m.

The following theorem gives a relation between the Randić energy of the m-duplicate graph of the graph and Randić energy of the original graph.

Theorem 3.4. *Let G be a simple (p,q) graph and $D^m(G)$ be the m-duplicate of graph G. Then $\varepsilon_R(D^m(G)) = 2^m \varepsilon_R(G)$.*

Proof. The Randić matrix of $D^m(G)$ is

$$R(D^m(G)) = \begin{bmatrix} O & O & O & \ldots & O & R(G) \\ O & O & O & \ldots & R(G) & O \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ R(G) & O & O & \ldots & O & O \end{bmatrix}_{2^m p}$$

$$= \begin{bmatrix} 0 & 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & 0 & \ldots & 1 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & 0 & \ldots & 0 & 0 \end{bmatrix}_{2^m} \otimes R(G).$$

Then the Randić spectrum of $D^m(G)$ is

$$RS(D^m(G)) = \begin{pmatrix} -\rho_1 & -\rho_2 & \cdots & -\rho_p & \rho_1 & \rho_2 & \cdots & \rho_p \\ 2^{m-1} & 2^{m-1} & \ldots & 2^{m-1} & 2^{m-1} & 2^{m-1} & \ldots & 2^{m-1} \end{pmatrix}.$$

Hence Randić energy of $D^m(G)$ is $\varepsilon_R(D^m(G)) = 2^m \varepsilon_R(G)$. □

Remark 3.1. *The graphs $D^m(G)$ and $D_{2^m}(G)$ are non-cospectral equienergetic but not Randić equienergetic.*

Proposition 3.2. *Let G be a simple (p,q) graph and $m \geq 1$. Then G is Randić integral if and only if the m-duplicate graph of G is Randić integral.*

4. Energy and Randić energy of some non-regular graphs

In this section, we define some new operations on a graph G and calculate the energy and Randić energy of the resultant graphs. Moreover, our results allow the construction of new pairs of equienergetic and Randić equienergetic graphs.

Operation 4.1. *Let G be a simple (p,q) graph and $D_m(G), m > 3$ be the m-shadow graph of G and $G_1, G_2, ..., G_m$ are the m copies of G in $D_m(G)$. The graph $H_1^m(G)$ is defined by $H_1^m(G) = D_m(G) - E(G_i) - E(G_j)$, for a pair $i \neq j, 1 \leq i, j \leq m$.*

The number of vertices and the number of edges in $H_1^m(G)$ are pm and $(m^2-2)q$ respectively.
We can easily compute the energy of $H_1^m(G)$ in terms of energy of G.

Theorem 4.1. *The energy of the graph $H_1^m(G)$ is*

$$\varepsilon(H_1^m(G)) = \left[1 + \sqrt{m^2+2m-7}\right]\varepsilon(G).$$

Proof. With the suitable labeling of the vertices, the adjacency matrix of $H_1^m(G)$ is

$$A(H_1^m(G)) = \begin{bmatrix} O & A(G) & A(G) & \dots & A(G) & A(G) \\ A(G) & A(G) & A(G) & \dots & A(G) & A(G) \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ A(G) & A(G) & A(G) & \dots & A(G) & A(G) \\ A(G) & A(G) & A(G) & \dots & A(G) & O \end{bmatrix}_{pm}$$

$$= \begin{bmatrix} 0 & 1 & 1 & \dots & 1 & 1 \\ 1 & 1 & 1 & \dots & 1 & 1 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 1 & 1 & 1 & \dots & 1 & 1 \\ 1 & 1 & 1 & \dots & 1 & 0 \end{bmatrix}_{m} \otimes A(G) = V_1 \otimes A(G),$$

$$\text{where } V_1 = \begin{bmatrix} 0 & 1 & 1 & \dots & 1 & 1 \\ 1 & 1 & 1 & \dots & 1 & 1 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 1 & 1 & 1 & \dots & 1 & 1 \\ 1 & 1 & 1 & \dots & 1 & 0 \end{bmatrix}_{m}.$$

The simple eigenvalues of V_1 are $\frac{m-1+\sqrt{m^2+2m-7}}{2}$, $\frac{m-1-\sqrt{m^2+2m-7}}{2}$,$-1$, and 0 has multiplicity $m-3$. Thus spectrum of $H_1^m(G)$ is

$$Spec(H_1^m(G)) = \begin{pmatrix} (\frac{m-1+\sqrt{m^2+2m-7}}{2})\lambda_i & (\frac{m-1-\sqrt{m^2+2m-7}}{2})\lambda_i & -\lambda_i & 0 \\ 1 & 1 & 1 & m-3 \end{pmatrix},$$

$1 \leq i \leq p$.

Hence $\varepsilon(H_1^m(G)) = \left[1 + \sqrt{m^2+2m-7}\right]\varepsilon(G)$. □

Corollary 4.2. *Let G_1 and G_2 be equienergetic graphs. Then $H_1^m(G_1)$ and $H_1^m(G_2)$ are equienergetic for all $m > 3$.*

Theorem 4.3. *The Randić energy of the graph $H_1^m(G), m > 3$ is $\varepsilon_R(H_1^m(G)) = \varepsilon_R(G) + \frac{2\varepsilon_R(G)}{m}$.*

Proof. The Randić matrix of $H_1^m(G)$ is $R(H_1^m(G))$

$$= \begin{bmatrix} ((m-1)D)^{-\frac{1}{2}} & O & \dots & O \\ O & (mD)^{-\frac{1}{2}} & \dots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \dots & ((m-1)D)^{-\frac{1}{2}} \end{bmatrix}_{\text{pm}} \begin{bmatrix} O & A(G) & \dots & A(G) \\ A(G) & A(G) & \dots & A(G) \\ \vdots & \vdots & \ddots & \vdots \\ A(G) & A(G) & \dots & O \end{bmatrix}_{\text{pm}}$$

$$\begin{bmatrix} ((m-1)D)^{-\frac{1}{2}} & O & \dots & O \\ O & (mD)^{-\frac{1}{2}} & \dots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \dots & ((m-1)D)^{-\frac{1}{2}} \end{bmatrix}_{pm}$$

$$= \begin{bmatrix} 0 & \frac{1}{\sqrt{m(m-1)}} & \frac{1}{\sqrt{m(m-1)}} & \dots & \frac{1}{m-1} \\ \frac{1}{\sqrt{m(m-1)}} & \frac{1}{m} & \frac{1}{m} & \dots & \frac{1}{\sqrt{m(m-1)}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{m-1} & \frac{1}{\sqrt{m(m-1)}} & \frac{1}{\sqrt{m(m-1)}} & \dots & 0 \end{bmatrix}_m \otimes D^{-\frac{1}{2}}A(G)D^{-\frac{1}{2}}$$

$= V_2 \otimes D^{-\frac{1}{2}}A(G)D^{-\frac{1}{2}}$, where

$$V_2 = \begin{bmatrix} 0 & \frac{1}{\sqrt{m(m-1)}} & \frac{1}{\sqrt{m(m-1)}} & \cdots & \frac{1}{m-1} \\ \frac{1}{\sqrt{m(m-1)}} & \frac{1}{m} & \frac{1}{m} & \cdots & \frac{1}{\sqrt{m(m-1)}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{m-1} & \frac{1}{\sqrt{m(m-1)}} & \frac{1}{\sqrt{m(m-1)}} & \cdots & 0 \end{bmatrix}_m .$$

The simple eigenvalues of V_2 are 1, $\frac{-1}{m-1}$, $\frac{-(m-2)}{m(m-1)}$, and 0 has multiplicity $m-3$. Thus Randić spectrum of $H_1^m(G)$ is

$$RS(H_1^m(G)) = \begin{pmatrix} \rho_i & \frac{-1}{m-1}\rho_i & \frac{-(m-2)}{m(m-1)}\rho_i & 0 \\ 1 & 1 & 1 & m-3 \end{pmatrix}, 1 \leq i \leq p.$$

Thus $\varepsilon_R(H_1^m(G)) = \varepsilon_R(G) + \frac{2\varepsilon_R(G)}{m}$. □

Corollary 4.4. *Let G_1 and G_2 be Randić equienergetic graphs. Then $H_1^m(G_1)$ and $H_1^m(G_2)$ are Randić equienergetic for all $m > 3$.*

Operation 4.2. *Let G be a simple (p,q) graph with vertex set $V(G) = \{v_1, v_2, ..., v_p\}$ and $G_1, G_2, ..., G_{m-1}$ are the $m-1$ copies of $G, m \geq 2$. Define a graph $H_2^m(G)$ with vertex set $V(H_2^m(G)) = V(G) \cup \{\cup_{j=1}^{m-1} V(G_j)\}$ and edge set $E(H_2^m(G))$ consisting edges of G and $G_j, 1 \leq j \leq m-1$ together with those edges joining i^{th} vertex of G_j's, $1 \leq j \leq m-1$, to the neighbors of v_i in G, $1 \leq i \leq p$.*

The number of vertices and the number of edges in $H_2^m(G)$ are pm and $(3m-2)q$ respectively.
The following theorem gives a relation between the energy of $H_2^m(G)$ and energy of the original graph.

Theorem 4.5. *The energy of the graph $H_2^m(G)$ is $\varepsilon(H_2^m(G)) = \left[m - 2 + 2\sqrt{m-1}\right]\varepsilon(G),\ m \geq 2$.*

Proof. With the suitable labeling of the vertices, the adjacency matrix of $H_2^m(G)$ is

$$A(H_2^m(G)) = \begin{bmatrix} A(G) & A(G) & A(G) & \dots & A(G) & A(G) \\ A(G) & A(G) & O & \dots & O & O \\ A(G) & O & A(G) & \dots & O & O \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ A(G) & O & O & \dots & A(G) & O \\ A(G) & O & O & \dots & O & A(G) \end{bmatrix}_{pm}$$

$$= \begin{bmatrix} 1 & 1 & 1 & \dots & 1 & 1 \\ 1 & 1 & 0 & \dots & 0 & 0 \\ 1 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 1 & 0 & 0 & \dots & 1 & 0 \\ 1 & 0 & 0 & \dots & 0 & 1 \end{bmatrix}_m \otimes A(G) = W_1 \otimes A(G),$$

$$\text{where} \quad W_1 = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 & 1 \\ 1 & 1 & 0 & \dots & 0 & 0 \\ 1 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 1 & 0 & 0 & \dots & 1 & 0 \\ 1 & 0 & 0 & \dots & 0 & 1 \end{bmatrix}_m .$$

Let $X = \begin{bmatrix} \sqrt{m-1} \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}_{m\times 1}$, then $W_1X = (1+\sqrt{m-1})X$ and let $Y = \begin{bmatrix} -\sqrt{m-1} \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}_{m\times 1}$, then $W_1Y = (1-\sqrt{m-1})Y$. Let $E_j = \begin{bmatrix} 0 \\ -1 \\ f_j \end{bmatrix}_{m\times 1}$, $1 \le j \le m-2$, where f_j is the column vector having j^{th} entry one, all other entries zeros. Then $W_1E_j = E_j$.

So the simple eigenvalues of W_1 are $1+\sqrt{m-1}, 1-\sqrt{m-1}$, and 1 has multiplicity $m-2$. Thus spectrum of $H_2^m(G)$ is

$$Spec(H_2^m(G)) = \begin{pmatrix} (1+\sqrt{m-1})\lambda_i & (1-\sqrt{m-1})\lambda_i & \lambda_i \\ 1 & 1 & m-2 \end{pmatrix}, 1 \leq i \leq p.$$

Hence we get, $\varepsilon(H_2^m(G)) = \left[m-2+2\sqrt{m-1}\right]\varepsilon(G)$. □

Corollary 4.6. *Let G be an integral graph and $m-1$ a perfect square. Then $H_2^m(G)$ is integral.*

Corollary 4.7. *Let G_1 and G_2 be equienergetic graphs. Then $H_2^m(G_1)$ and $H_2^m(G_2)$ are equienergetic for all $m > 1$.*

Remark 4.1. *If $m = 2$, the graph $H_2^m(G)$ coincide with the shadow graph $D_2(G)$.*

Theorem 4.8. *The Randić energy of the graph $H_2^m(G), m > 3$ is $\varepsilon_R(H_2^m(G)) = \varepsilon_R(G) + \frac{(m+1)(m-2)\varepsilon_R(G)}{2m}$.*

Proof. The Randić matrix of $H_2^m(G)$ is $R(H_2^m(G))$

$$= \begin{bmatrix} (mD)^{-\frac{1}{2}} & O & O & \dots & O \\ O & (2D)^{-\frac{1}{2}} & O & \dots & O \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ O & O & O & \dots & (2D)^{-\frac{1}{2}} \end{bmatrix}_{\mathrm{pm}} \begin{bmatrix} A(G) & A(G) & A(G) & \dots & A(G) \\ A(G) & A(G) & O & \dots & O \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A(G) & O & O & \dots & A(G) \end{bmatrix}_{\mathrm{pm}}$$

$$\begin{bmatrix} (mD)^{-\frac{1}{2}} & O & O & \dots & O \\ O & (2D)^{-\frac{1}{2}} & O & \dots & O \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ O & O & O & \dots & (2D)^{-\frac{1}{2}} \end{bmatrix}_{pm}$$

$$= \begin{bmatrix} \frac{1}{m} & \frac{1}{\sqrt{2m}} & \frac{1}{\sqrt{2m}} & \dots & \frac{1}{\sqrt{2m}} \\ \frac{1}{\sqrt{2m}} & \frac{1}{2} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\sqrt{2m}} & 0 & 0 & \dots & \frac{1}{2} \end{bmatrix}_m \otimes D^{-\frac{1}{2}}A(G)D^{-\frac{1}{2}}$$

$$= W_2 \otimes D^{-\frac{1}{2}} A(G) D^{-\frac{1}{2}}, \text{ where } \quad W_2 = \begin{bmatrix} \frac{1}{m} & \frac{1}{\sqrt{2m}} & \frac{1}{\sqrt{2m}} & \cdots & \frac{1}{\sqrt{2m}} \\ \frac{1}{\sqrt{2m}} & \frac{1}{2} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\sqrt{2m}} & 0 & 0 & \cdots & \frac{1}{2} \end{bmatrix}_m .$$

Let $X^* = \begin{bmatrix} \sqrt{\frac{m}{2}} \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}_{m\times 1}$, then $W_2 X^* = 1.X^*$ and let $Y^* = \begin{bmatrix} -(m-1)\sqrt{\frac{2}{m}} \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}_{m\times 1}$,

then $W_2 Y^* = \frac{-(m-2)}{2m} Y^*$. Let E_j be as in Theorem 4.5, then $W_2 E_j = \frac{1}{2} E_j$. So the simple eigenvalues of W_2 are 1, $\frac{-(m-2)}{2m}$, and $\frac{1}{2}$ has multiplicity $m-2$. Thus Randić spectrum of $H_2^m(G)$ is

$$RS(H_2^m(G)) = \begin{pmatrix} \rho_i & \frac{-(m-2)}{2m}\rho_i & \frac{1}{2}\rho_i \\ 1 & 1 & m-2 \end{pmatrix}, 1 \leq i \leq p.$$

Hence $\varepsilon_R(H_2^m(G)) = \varepsilon_R(G) + \frac{(m+1)(m-2)\varepsilon_R(G)}{2m}$. □

Corollary 4.9. *Let G_1 and G_2 be Randić equienergetic graphs. Then $H_2^m(G_1)$ and $H_2^m(G_2)$ are Randić equienergetic for all $m > 3$.*

Operation 4.3. *Let G be a simple (p, q) graph with vertex set $V(G) = \{v_1, v_2, ..., v_p\}$ and $G_1, G_2, ..., G_{m-1}$ are the $m-1$ copies of G. Define a graph $H_3^m(G), m > 1$ with vertex set $V(H_3^m(G)) = V(G) \cup \{\cup_{i=1}^{m-1} V(G_i)\}$ and edge set $E(H_3^m(G))$ consisting only of those edges joining i^{th} vertex of G_j $1 \leq j \leq m-1$, to the neighbors of v_i in G, $1 \leq i \leq p$ and then removing edges of G. $G_1, G_2, ..., G_{m-1}$ are the $m-1$ copies of G. Let $H_3^m(G)$ be the graph obtained by making i^{th} vertex of $G_i, 1 \leq i \leq m-1$ adjacent to the vertices in $N(v_i)$, where $N(v_i)$ is the neighborhood of vertex v_i for every i and remove edges of G.*

The number of vertices and the number of edges in $H_3^m(G)$ are pm and $3(m-1)q$ respectively.

Theorem 4.10. *The energy of the graph $H_3^m(G)$ is*

$$\varepsilon(H_3^m(G)) = \left[m - 2 + \sqrt{4m-3} \right] \varepsilon(G).$$

Proof. The adjacency matrix of $H_3^m(G)$ is

$$A(H_3^m(G)) = \begin{bmatrix} O & A(G) & A(G) & \dots & A(G) & A(G) \\ A(G) & A(G) & O & \dots & O & O \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ A(G) & O & O & \dots & O & A(G) \end{bmatrix}_{pm}$$

$$= \begin{bmatrix} 0 & 1 & 1 & \dots & 1 & 1 \\ 1 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 1 & 0 & 0 & \dots & 0 & 1 \end{bmatrix}_{m} \otimes A(G)$$

$$= Z_1 \otimes A(G), \text{ where } Z_1 = \begin{bmatrix} 0 & 1 & 1 & \dots & 1 & 1 \\ 1 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 1 & 0 & 0 & \dots & 0 & 1 \end{bmatrix}.$$

Let $P = \begin{bmatrix} \frac{-1+\sqrt{4m-3}}{2} \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}_{m\times 1}$, then $Z_1 P = \left(\frac{1+\sqrt{4m-3}}{2}\right)P$ and let $Q = \begin{bmatrix} \frac{-1-\sqrt{4m-3}}{2} \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}_{m\times 1}$, then $Z_1 Q = \left(\frac{1-\sqrt{4m-3}}{2}\right)Q$. Let E_j be as in Theorem 4.5, then $Z_1 E_j = E_j$. So the simple eigenvalues of Z_1 are $\frac{1+\sqrt{4m-3}}{2}$, $\frac{1-\sqrt{4m-3}}{2}$, and 1 has multiplicity $m-2$. Thus spectrum of $H_3^m(G)$ is

$$Spec(H_3^m(G)) = \begin{pmatrix} (\frac{1+\sqrt{4m-3}}{2})\lambda_i & (\frac{1-\sqrt{4m-3}}{2})\lambda_i & \lambda_i \\ 1 & 1 & m-2 \end{pmatrix}, 1 \le i \le p.$$

Hence we have

$$\varepsilon(H_3^m(G)) = \left[m - 2 + \sqrt{4m-3}\right]\varepsilon(G).$$

□

Corollary 4.11. *Let G be an integral graph. Then $H_3^m(G)$ is an integral graph if $4m-3$ is a perfect square.*

For example, $H_3^3(K_2)$, $H_3^7(K_2)$, $H_3^{13}(K_2)$, $H_3^{21}(K_2)$ etc.

Corollary 4.12. *Let G_1 and G_2 be equienergetic graphs. Then $H_3^m(G_1)$ and $H_3^m(G_2)$ are equienergetic for all $m > 1$.*

Theorem 4.13. *The Randić energy of graph $H_3^m(G), m > 2$ is $\varepsilon_R(H_3^m(G)) = \varepsilon_R(G) + \frac{(m-1)\varepsilon_R(G)}{2}$.*

Proof. The Randić matrix of $H_3^m(G)$ is $R(H_3^m(G))$

$$= \begin{bmatrix} ((m-1)D)^{-\frac{1}{2}} & O & O & \dots & O \\ O & (2D)^{-\frac{1}{2}} & O & \dots & O \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ O & O & O & \dots & (2D)^{-\frac{1}{2}} \end{bmatrix}_{\text{pm}} \begin{bmatrix} O & A(G) & A(G) & \dots & A(G) \\ A(G) & A(G) & O & \dots & O \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A(G) & O & O & \dots & A(G) \end{bmatrix}_{\text{pm}}$$

$$\begin{bmatrix} ((m-1)D)^{-\frac{1}{2}} & O & O & \dots & O \\ O & (2D)^{-\frac{1}{2}} & O & \dots & O \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ O & O & O & \dots & (2D)^{-\frac{1}{2}} \end{bmatrix}_{pm}$$

$$= \begin{bmatrix} 0 & \frac{1}{\sqrt{2(m-1)}} & \frac{1}{\sqrt{2(m-1)}} & \dots & \frac{1}{\sqrt{2(m-1)}} \\ \frac{1}{\sqrt{2(m-1)}} & \frac{1}{2} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\sqrt{2(m-1)}} & 0 & 0 & \dots & \frac{1}{2} \end{bmatrix}_m$$

$$\otimes D^{\frac{-1}{2}} A(G) D^{\frac{-1}{2}} = Z_2 \otimes D^{\frac{-1}{2}} A(G) D^{\frac{-1}{2}},$$

$$\text{where } Z_2 = \begin{bmatrix} 0 & \frac{1}{\sqrt{2(m-1)}} & \frac{1}{\sqrt{2(m-1)}} & \dots & \frac{1}{\sqrt{2(m-1)}} \\ \frac{1}{\sqrt{2(m-1)}} & \frac{1}{2} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\sqrt{2(m-1)}} & 0 & 0 & \dots & \frac{1}{2} \end{bmatrix}_m.$$

Let $P^* = \begin{bmatrix} \sqrt{\frac{m-1}{2}} \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}_{m\times 1}$

then $Z_2P^* = 1.P^*$ and let $Q^* = \begin{bmatrix} -(m-1)\sqrt{\frac{2}{m-1}} \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}_{m\times 1}$

then $Z_2Q^* = -\frac{1}{2}Q^*$. Let E_j be as in Theorem 4.5, then $Z_2E_j = \frac{1}{2}E_j$. So the simple eigenvalues of Z_2 are $1, -\frac{1}{2}$, and $\frac{1}{2}$ has multiplicity $m-2$. Thus Randić spectrum of $H_3^m(G)$ is

$$RS(H_3^m(G)) = \begin{pmatrix} \rho_i & -\frac{1}{2}\rho_i & \frac{1}{2}\rho_i \\ 1 & 1 & m-2 \end{pmatrix}, 1 \le i \le p.$$

Hence $\varepsilon_R(H_3^m(G)) = \varepsilon_R(G) + \frac{(m-1)\varepsilon_R(G)}{2}$. □

Corollary 4.14. *Let G_1 and G_2 be Randić equienergetic graphs. Then $H_3^m(G_1)$ and $H_3^m(G_2)$ are Randić equienergetic for all $m > 1$.*

5. Applications

In this section, we compute the degree Kirchhoff index, the Kemeny's constant and the number of spanning trees of $Spl_m(G)$ in terms of original graph. Analogous to this, results for $D_m(G)$, $D^m(G)$,$H_1^m(G)$, $H_2^m(G)$ and $H_3^m(G)$ are included in Appendix.

Theorem 5.1. *Let G be a simple connected (p, q) graph with Randić spectrum $\{\rho_1, \rho_2, ..., \rho_p\}$. Then*

$$K(Spl_m(G)) = p(m-1) + K(G) + \sum_{i=2}^{p} \frac{m+1}{1+m(1+\rho_i(G))}.$$

Theorem 5.2. *Let G be a simple connected (p,q) graph with Randić spectrum $\{\rho_1, \rho_2, ..., \rho_p\}$. Then*

$$f^*(Spl_m(G)) = 2(m+1)q\left[p(m-1) + \sum_{i=2}^{p} \frac{m+1}{1+m(1+\rho_i(G))}\right] + (m+1)Kf^*(G).$$

From the following theorem, we obtain the number of spanning trees of graphs in terms of Randić eigenvalues.

Theorem 5.3. *Let G be a connected simple (p,q) graph with Randić spectrum $\{\rho_1, \rho_2, ..., \rho_p\}$. Then*

$$t(Spl_m(G)) = \frac{(m+1)^p(\prod_{i=1}^{p} d_i)^m t(G) \prod_{i=1}^{p}\left(1 + \frac{m\rho_i(G)}{m+1}\right)}{2m+1}.$$

6. Conclusion

In this paper, we compute the energy and Randić energy of some specific graphs which are obtained by some graph operations on G. Also, our results show how to construct some new class of graphs having the same Randić energy as that of G. In addition, some new family of equienergetic, Randić equienergetic, integral and Randić integral graphs are obtained. Moreover, we discuss some graph invariants like the degree Kirchhoff index, the Kemeny's constant and the number of spanning trees of graph $Spl_m(G)$.

7. Appendix

Let G be a connected graph, then $D_m(G)$, $H_1^m(G)$, $H_2^m(G)$ and $H_3^m(G)$ are connected. Here we discuss the degree Kirchhoff index, the Kemeny's constant and the number of spanning trees of graphs $D_m(G)$, $H_1^m(G)$, $H_2^m(G)$ and $H_3^m(G)$.

Theorem 7.1. *Let G be a simple connected (p,q) graph with Randić spectrum $\{\rho_1, \rho_2, ..., \rho_p\}$. Then*

1. $K(D_m(G)) = p(m-1) + K(G)$.

2. $K(H_1^m(G)) = m-3+K(G)+\sum_{i=2}^{p} \frac{m-1}{m-1+\rho_i(G)}+\sum_{i=2}^{p} \frac{m(m-1)}{m^2-m+(m-2)\rho_i(G)}$.

3. $K(H_2^m(G)) = m - 3 + K(G) + \sum_{i=2}^{p} \frac{2m}{2m+(m-2)\rho_i(G)} + \sum_{i=2}^{p} \frac{2}{2-\rho_i(G)}$.

4. $K(H_3^m(G)) = 6(m-1)q \left[K(G) + \sum_{i=2}^{p} \frac{2}{2+\rho_i(G)} + \sum_{i=2}^{p} \frac{2}{2-\rho_i(G)}\right]$.

Theorem 7.2. *Let G be a simple connected (p, q) graph with Randić spectrum $\{\rho_1, \rho_2, ..., \rho_p\}$. Then*

1. $Kf^*(D_m(G)) = 2m^2 q\,(p(m-1)) + m^2 Kf^*(G)$.

2. $Kf^*(H_1^m(G)) = 2(m^2-2)q \left[m-3+\sum_{i=2}^{p} \frac{m-1}{m-1+\rho_i(G)} + \sum_{i=2}^{p} \frac{m(m-1)}{m^2-m+(m-2)\rho_i(G)}\right] + (m^2-2)Kf^*(G)$.

3. $Kf^*(H_2^m(G)) = 2(3m-2)q \left[m - 3 + \sum_{i=2}^{p} \frac{2m}{2m+(m-2)\rho_i(G)} + \sum_{i=2}^{p} \frac{2}{2-\rho_i(G)}\right] + (3m-2)Kf^*(G)$.

4. $Kf^*(H_3^m(G)) = 6(m-1)q \left[\sum_{i=2}^{p} \frac{2}{2+\rho_i(G)} + \sum_{i=2}^{p} \frac{2}{2-\rho_i(G)}\right] + 3(m-1)Kf^*(G)$.

From the following theorem, we obtain the number of spanning trees of graphs in terms of Randić eigenvalues.

Theorem 7.3. *Let G be a connected simple (p, q) graph with Randić spectrum $\{\rho_1, \rho_2, ..., \rho_p\}$. Then*

1. $t(D_m(G)) = \frac{m^{mp}(\prod_{i=1}^{p} d_i)^{m-1} t(G)}{m^2}$.

2. $t(H_1^m(G)) = \frac{m^{(m-2)p}(m-1)^2(\prod_{i=1}^{p} d_i)^{m-1} t(G) \prod_{i=1}^{p}\left(1+\frac{\rho_i(G)}{m-1}\right) \prod_{i=1}^{p}\left(1+\frac{(m-2)\rho_i(G)}{m(m-1)}\right)}{m^2-2}$.

3. $t(H_2^m(G)) = \frac{2^{(m-1)p} m^p(\prod_{i=1}^{p} d_i)^{m-1} t(G) \prod_{i=1}^{p}\left(1+\frac{(m-2)\rho_i(G)}{2m}\right) \prod_{i=1}^{p}\left(1+\frac{\rho_i(G)}{2}\right)}{3m-2}$.

4. $t(H_3^m(G)) = \frac{2^{(m-1)p}(m-1)^p(\prod_{i=1}^{p} d_i)^{m-1} t(G) \prod_{i=1}^{p}\left(1-\frac{\rho_i(G)}{2}\right) \prod_{i=1}^{p}\left(1+\frac{\rho_i(G)}{2}\right)}{3m-3}$.

References

[1] S. Alikhani and N. Ghanbari, "Randić energy of specific graphs", *Applied Mathematics and Computation,* vol. 269, pp. 722-730, 2015. doi: 10.1016/j.amc.2015.07.112

[2] I. Altindag, "Some statistical results on Randić energy of graphs", *Match (Mülheim),* vol. 79, pp. 331-339, 2018. [On line]. Available: https://bit.ly/3PB0Xwr

[3] R. Balakrishnan, "The energy of a graph", *Linear Algebra and its Applications,* vol. 387, pp. 287-295, 2004. doi: 10.1016/j.laa.2004.02.038

[4] S. Burcu Bozkurt, A Dilek Gungor, I. Guntman, and A. Sinan Cevik, "Randić matrix and Randić energy", *Match (Mülheim),* vol. 64, no. 1, pp. 239-250, 2010. [On line]. Available: https://bit.ly/3cy2WTn

[5] S. Butler, "Algebraic aspects of the normalized Laplacian," in *Recent* Trends in Combinatorics, A. Beveridge, J. Griggs, L. Hogben, G. Musiker, and P. Tetali, Eds. Springer, 2016, pp. 295–315. doi: 10.1007/978-3-319-24298-9_13

[6] H. Chen and F. Zhang, "Resistance distance and the normalized Laplacian spectrum", *Discrete Applied Mathematics*, vol. 155, no. 5, pp. 654-661, 2007. doi: 10.1016/j.dam.2006.09.008

[7] Z.-Q. Chu, S. Nazeer, T. J. Zia, I. Ahmed, and S. Shahid, "Some new results on various graph energies of the splitting graph", *Journal of Chemistry*, vol. 2019, Art. ID. 7214047, 2019. doi: 10.1155/2019/7214047

[8] F. R. K. Chung, *Spectral graph theory,* vol. 92. Providence, RI: American Mathematical Society, 1997. doi: 10.1090/cbms/092

[9] D. M. Cvetković, M. Doob and H. Sachs, *Spectral of graphs: Theory and applications,* vol. 10. New York, NY: Academic Press, 1980.

[10] K. C. Das, S. Sun and I. Gutman, "Normalized Laplacian eigenvalues and Randić energy of graphs", *Match (Mülheim),* vol. 77, no. 1, pp. 45-59, 2017. [On line]. Available: https://bit.ly/3RYURrd

[11] I. Gutman, "The energy of a graph", *Berichte der Mathematisch-Statistischen Sektion in der Forschungsgesellschaft Joanneum,* no. 103, pp. 100-105, 1978.

[12] I. Gutman, B. Furtula and S. Burcu Bozkurt, "On Randić energy", *Linear Algebra and its Applications,* vol. 442, pp. 50-57, 2014. doi: 10.1016/j.laa.2013.06.010

[13] J. J. Hunter, “The role of Kemeny’s constant in properties of Markov chains”, *Communications in Statistics- Theory and Methods,* vol. 43, no. 7, pp. 1309-1321, 2014. doi: 10.1080/03610926.2012.741742

[14] G. Indulal and A. Vijayakumar, “On a pair of equienergetic graphs”, *Match (Mülheim),* vol. 55, no. 1, pp. 83-90, 2006. [On line]. Available: https://bit.ly/3PB0J8n

[15] D. J. Klein and M. Randić, “Resistance distance”, *Journal of mathematical chemistry,* vol. 12, no. 1, pp. 81-95, 1993. doi: 10.1007/BF01164627

[16] E. Munarini, C. Perelli Cippo, A. Scagliola and N. Zagaglia Salvi, “Double graphs”, *Discrete mathematics*, vol. 38, no. 2-3, pp. 242-254, 2008. doi: 10.1016/j.disc.2006.11.038

[17] H. P. Patil and V. Raja, “On tensor product of graphs, girth and triangles”, *Iranian Journal of Mathematical Sciences and Informatics*, vol. 10, no. 1, pp. 139-147, 2015. doi: 10.7508/ijmsi.2015.01.011

[18] H. S. Ramane, H. B. Walikar, S. Bhaskara Rao, B. D. Acharya, P. R Hampiholi. S. R. Jog, and I. Gutman, “Equienergetic graphs”, *Kragujevac journal of mathematics*, vol. 26, pp. 5-13, 2004. [On line]. Available: https://bit.ly/3Pzn1Ye

[19] O. Rojo and L. Medina, “Construction of bipartite graphs having the same Randić energy”, *Match (Mülheim),* vol. 68, no. 3, pp. 805-814, 2012. [On line]. Available: https://bit.ly/3J7HHEb

[20] E. Sampathkumar, “On duplicate graphs”, *Journal of the Indian Mathematical Society,* vol. 37, pp. 285-293, 1973. [On line]. Available: https://bit.ly/3PPrjKP

[21] E. Sampathkumar and H. B. Walikar, “On splitting graph of a graph”, *Karnatak University journal of science*, vol. 25, no. 13, pp. 13-16, 1981. [On line]. Available: https://bit.ly/3vdA5dK

[22] S. K. Vaidya and K. M. Popat, “Energy of m-splitting and m-shadow graphs”, *Far East Journal of Mathematical Sciences,* vol. 102, pp. 1571-1578, 2017. doi: 10.17654/MS102081571

Jahfar T. K.
Department of Mathematics,
National Institute of Technology,
Calicut, Kerala,
India 673601
India
e-mail: jahfartk@gmail.com
Corresponding author

and

Chithra A. V.
Department of Mathematics,
National Institute of Technology,
Calicut, Kerala,
India 673601
India
e-mail: chithra@nitc.ac.in