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Controllability of impulsive neutral stochastic integro-differential systems driven by fractional Brownian motion with delay and Poisson jumps

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Abstract

In this paper the controllability of a class of impulsive neutral stochastic integro-differential systems driven by fractional Brownian motion and Poisson process in a separable Hilbert space with infinite delay is studied. The controllability result is obtained by using stochastic analysis and a fixed-point strategy. Finally, an illustrative example is given to demonstrate the effectiveness of the obtained result.

Keywords: *Controllability; impulsive neutral functional integro-differential equations; infinite delay; fractional Brownian motion; Poisson process.*

AMS Subject Classification: *35R10; 93B05; 60G22; 60H20.*

1. Introduction

One of the fundamental concepts in mathematical control theory is the controllability. It is a qualitative behaviour of dynamical systems and is of particular importance in control theory. There are various important relationships between controllability, stability and stabilizability of both deterministic and stochastic control systems. Any control system is said to be controllable if it is possible to steer the solution of the system from an arbitrary initial state to an arbitrary final state using the set of admissible controls, where the initial and final states may vary over the entire space. Different types of controllability have been defined, such as approximate, null, local null and local approximate null controllability. For more details on this matter, we refer to [3, 4, 10, 11, 12, 13] and references therein.

Many dynamical systems modeling real phenomena in different scientific areas (such as mechanics, control theory, chemistry, biology, medicine, economic, etc) have the property of after-effect, i.e, the future state depend not only on the present, but also on the past history. It is well-known that a lot of dynamical systems are subjective to sudden and abrupt changes such as shocks, harvesting and natural disasters. These short term perturbations often act instantaneously in the form of impulses. Impulsive neutral integro-differential equations with infinite delay have been widely developed in modeling such problems. Besides, practical systems are usually subjected to random abrupt perturbations, which may result from abrupt phenomena such as stochastic failures and repairs of the components, sudden environment changes, etc. These stochastic effects can lead to various complex dynamic performance. Therefore, it is interesting to add stochastic effect to the study of impulsive neutral differential equation. In recent years the stochastic functional differential equations driven by a fractional Brownian motion (fBm) have attracted the attention of many authors and many valuable results on existence, uniqueness and the controllability of the solution have been established. In addition, the study of neutral SFDEs driven by jumps process also have begun to gain attention and strong growth in recent years. To be more precise, we refer to [5, 6, 14, 15, 17, 18].

Furthermore, self-similar processes are of interest in physics, in the context of the renormalization group theory, and in hydrology, where they account for the so-called Hurst effect. The best known and widely used self-similar process is the fractional Brownian motion because it is gaussian, with stationary increments and exhibits long/short range dependance. It has been widely used to model a number of phenomena in diverse fields from biology to finance. This huge range of potential applications makes fBm an interesting object of study. On the other hand, in practice (for instance sudden price variation resulting from market crushes) where the path continuity supposition does not seem plausible for the model, one should consider stochastic processes with jumps in modeling such systems. Generally, these jump models are based on Poisson random measure. Recently, there has been an increasing interest in the study of stochastic differential equation with Poisson jumps. A sufficient condition for the existence of mild solution have been given by Taniguchi et al.[15]. Boufoussi & Hajji [17] have proved the existence and uniqueness result for a class of neutral functional stochastic differential equations driven both by the Brownian motion and by the Poisson point processes by using successive approximation. Very recently, Lakhel & Hajji [18] have studied the existence, uniqueness and asymptotic behavior of mild solutions for a class of neutral functional stochastic differential equations with Poisson jumps. For more result, we refer to [14, 19, 20].

This article mainly focusing on controllability result for impulsive neutral stochastic integro-differential equations of the form

$$\begin{aligned} d[x(t) \quad & -p(t,x_t,\textstyle\int_0^t \theta_1(t,s,x_s)ds)] = [Ax(t) + h(t,x_t,\textstyle\int_0^t \theta_2(t,s,x_s)ds) + Bu(t)]dt \\ & \qquad + \textstyle\int_{\mathcal{U}} g(t,x_t,\eta)\widetilde{N}(dt,d\eta) + \sigma(t)dB^H(t),\ t \in J = [0,T],\ t \neq t_k, \\ & \Delta x|_{t=t_i} = x(t_i^+) - x(t_i^-) = I_i(x(t_i^-)),\ \ i = 1,2,...,m,\ m \in \\ & x_0 = \varphi\ \in \mathcal{B} \end{aligned} \tag{1.1}$$

Where A is the infinitesimal generator of an analytic semigroup of bounded linear operators, $(S(t))_{t\geq 0}$, in a Hilbert space X; B^H is a fractional Brownian motion with Hurst parameter $H > \frac{1}{2}$ on a real and separable Hilbert space Y; and the control function $u(\cdot)$ takes values in $L^2([0,T],U)$, the Hilbert space of admissible control functions for a separable Hilbert space U; and B is a bounded linear operator from U into X. The history $x_t : (-\infty, 0] \to X$, $x_t(\theta) = x(t+\theta)$, belongs to an abstract phase space $\mathcal{B}$ defined axiomatically; $p, h : \ J \times \mathcal{B} \times X \to X$, $\theta_1, \theta_2 : \ D \times \mathcal{B} \to X$, $\sigma : \ J \to \mathcal{L}_2^0(Y,X)$ and $g : J \times \mathcal{B} \times \mathcal{U} \to X$ are appropriate functions and will be specified later, where $\mathcal{L}_2^0(Y,X)$ denotes the space of all Q-Hilbert-Schmidt operators from Y into X (see section 2 below) and $D = \{(t,s) \in J \times J : s < t\}$. Moreover, the fixed moments of time t_k satisfy $0 < t_1 < t_2 < ... < t_m < T$; $x(t_k^-)$ and $x(t_k^+)$ represent the left and right limits of $x(t)$ at time t_k respectively. $\Delta x(t_k)$ denotes the jump in the state x at time t_k with $I(.) : X \longrightarrow X$ determining the size of the jump.

The outline of this paper is as follows: Section 2 presents notation and preliminary results. Section 3, shows the controllability of impulsive neutral stochastic integro-differential systems driven by a fractional Brownian motion with infinit delay and Poisson process. Finally, Section 4, presents an example that illustrates our result.

2. Preliminaries

For details of the topics addressed in this section, we refer the reader to [1, 8, 9] and the references therein.

Let $(\mathcal{U}, B, \nu(du))$ be a σ-finite measurable space, given a Poisson point process $(q(t))_{t>0}$ wich is defined on a complete probabilty space $(\Omega, \mathcal{F}, \mathbf{P})$ with values in $\mathcal{U}$ and with characteristic measure ν (see [1]). $N(dt, du)$ denote the counting random measure associated to $q(\cdot)$,i.e $N(t, \Lambda) := N([0, t), \Lambda) = \sum_{s\in(0,t]} 1_\Lambda(q(s))$ such that $E(N(t, \Lambda)) = t\nu(\Lambda)$ for $\Lambda \in B$. Define $\widetilde{N}(dt, du) := N(dt, du) - dt\nu(du)$, the Poisson martingale measure generated by $q(t)$.

2.1. Fractional Brownian motion.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbf{P})$ be a complete probability space satisfying the usual conditions, meaning that the filtration is a right-continuous increasing family and $\mathcal{F}_0$ contains all P-null sets.

Consider a time interval $[0, T]$ with arbitrary fixed horizon T and let $\{\beta^H(t) : t \in [0, T]\}$ be a one-dimensional fractional Brownian motion with Hurst parameter $H \in (1/2, 1)$. By definition, β^H is a centered Gaussian process with covariance function

$$R_H(s, t) = \frac{1}{2}(t^{2H} + s^{2H} - |t - s|^{2H}).$$

Moreover, β^H has the following Wiener integral representation:

$$\beta^H(t) = \int_0^t K_H(t, s)d\beta(s), \tag{2.1}$$

where $\beta = \{\beta(t) : \ t \in [0, T]\}$ is a Wiener process and kernel $K_H(t, s)$ is the kernel given by

$$K_H(t, s) = c_H s^{\frac{1}{2}-H} \int_s^t (u - s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} du,$$

for $t > s$, where $c_H = \sqrt{\frac{H(2H-1)}{g(2-2H, H-\frac{1}{2})}}$ and $g(\cdot,\cdot)$ denotes the Beta function. We take $K_H(t,s) = 0$ if $t \leq s$.
We will denote by $\mathcal{H}$ the reproducing kernel Hilbert space of the fBm. Precisely, $\mathcal{H}$ is the closure of set of indicator functions $\{1_{[0;t]} : t \in [0,T]\}$ with respect to the scalar product

$$\langle 1_{[0,t]}, 1_{[0,s]}\rangle_{\mathcal{H}} = R_H(t,s).$$

The mapping $1_{[0,t]} \to \beta^H(t)$ can be extended to an isometry between $\mathcal{H}$ and the first Wiener chaos and we will denote by $\beta^H(\varphi)$ the image of φ by the previous isometry.

Recall that for $\psi, \varphi \in \mathcal{H}$, the scalar product in $\mathcal{H}$ is given by

$$\langle \psi, \varphi\rangle_{\mathcal{H}} = H(2H-1)\int_0^T\int_0^T \psi(s)\varphi(t)|t-s|^{2H-2}dsdt.$$

Consider the operator K_H^* from $\mathcal{H}$ to $L^2([0,T])$ defined by

$$(K_H^*\varphi)(s) = \int_s^T \varphi(r)\frac{\partial K_H}{\partial r}(r,s)dr.$$

The proof of the fact that K_H^* is an isometry between $\mathcal{H}$ and $L^2([0,T])$ can be found in [8]. Moreover, for any $\varphi \in \mathcal{H}$, we have

$$\beta^H(\varphi) = \int_0^T (K_H^*\varphi)(t)d\beta(t).$$

It follows from [8] that the elements of $\mathcal{H}$ may be not functions but rather distributions of negative order. In order to obtain a space of functions contained in $\mathcal{H}$, we consider the linear space $|\mathcal{H}|$ generated by the measurable functions ψ such that

$$\|\psi\|^2_{|\mathcal{H}|} := \alpha_H\int_0^T\int_0^T |\psi(s)||\psi(t)||s-t|^{2H-2}dsdt < \infty,$$

where $\alpha_H = H(2H-1)$. We have the following lemma (see [8]).

Lemma 2.1. *The space $|\mathcal{H}|$ is a Banach space with the norm $\|\psi\|_{|\mathcal{H}|}$; the following inclusions hold*

$$\mathbf{L}^2([0,T]) \subseteq \mathbf{L}^{1/H}([0,T]) \subseteq |\mathcal{H}| \subseteq \mathcal{H};$$

and for any $\varphi \in \mathbf{L}^2([0,T])$,

$$\|\psi\|^2_{|\mathcal{H}|} \leq 2HT^{2H-1}\int_0^T |\psi(s)|^2 ds.$$

Let X and Y be two real, separable Hilbert spaces and let $\mathcal{L}(Y,X)$ be the space of bounded linear operator from Y to X. For convenience, we shall use the same notation to denote the norms in X, Y and $\mathcal{L}(Y,X)$. Let $Q \in \mathcal{L}(Y,Y)$ be an operator defined by $Qe_n = \lambda_n e_n$ with finite trace $trQ = \sum_{n=1}^{\infty} \lambda_n < \infty$, where $\lambda_n \geq 0$ $(n = 1, 2...)$ are nonnegative real numbers and $\{e_n : n = 1, 2...\}$ is a complete orthonormal basis in Y. Let $B^H = (B^H(t))$ be a $Y-$ valued fbm on $(\Omega, \mathcal{F}, \mathbf{P})$ with covariance Q given by

$$B^H(t) = B^H_Q(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n} e_n \beta^H_n(t),$$

where β^H_n are real, independent fBm's. This process is Gaussian, it starts from 0, has zero mean, and covariance:

$$\mathbf{E}\langle B^H(t), x\rangle\langle B^H(s), y\rangle = R(s,t)\langle Q(x), y\rangle \text{ for all } x, y \in Y \text{ and } t, s \in [0,T].$$

In order to define Wiener integrals with respect to the Q-fBm, we introduce the space $\mathcal{L}_2^0 := \mathcal{L}_2^0(Y,X)$ of all Q-Hilbert-Schmidt operators $\psi : Y \to X$. Recall that $\psi \in \mathcal{L}(Y,X)$ is called a Q-Hilbert-Schmidt operator if

$$\|\psi\|^2_{\mathcal{L}_2^0} := \sum_{n=1}^{\infty} \|\sqrt{\lambda_n}\psi e_n\|^2 < \infty,$$

and that the space $\mathcal{L}_2^0$ equipped with the inner product $\langle \varphi, \psi\rangle_{\mathcal{L}_2^0} = \sum_{n=1}^{\infty}\langle \varphi e_n, \psi e_n\rangle$ is a separable Hilbert space.

Let $\phi : [0,T] \to \mathcal{L}_2^0(Y,X)$ be a given function. The Wiener integral of ϕ with respect to B^H is defined by

$$(2.2) \int_0^t \phi(s) dB^H(s) = \sum_{n=1}^{\infty} \int_0^t \sqrt{\lambda_n}\phi(s) e_n d\beta^H_n(s) = \sum_{n=1}^{\infty} \int_0^t \sqrt{\lambda_n}(K_H^*(\phi e_n)(s) d\beta_n(s),$$

where β_n is the standard Brownian motion used to define β^H_n as in (2.1). We conclude this subsection by stating the following result which is critical in the proof of our result. It can be proved using arguments similar to those used to prove Lemma 2 in [7].

Lemma 2.2. *If $\psi : [0,T] \to \mathcal{L}_2^0(Y,X)$ satisfies $\int_0^T \|\psi(s)\|^2_{\mathcal{L}_2^0} ds < \infty$, then* (2.2) *is well-defined as an X-valued random variable and*

$$\mathbf{E}\|\int_0^t \psi(s) dB^H(s)\|^2 \leq 2Ht^{2H-1}\int_0^t \|\psi(s)\|^2_{\mathcal{L}_2^0} ds.$$

Proof. For the reader's convenience, we give here a simple proof. By Lemma 2.1, we have

$$\begin{aligned}
\mathbf{E}\|\int_0^t \psi(s)dB^H(s)\|^2 &= \sum_{n=1}^{\infty} \mathbf{E}\|\int_0^t \sqrt{\lambda_n}(K_H^*(\psi e_n)(s)d\beta_n(s)\|^2 \\
&\leq \sum_{n=1}^{\infty} 2Ht^{2H-1}\int_0^t \lambda_n\|\psi(s)e_n\|^2 ds \\
&= 2Ht^{2H-1}\int_0^t \|\psi(s)\|^2_{\mathcal{L}_2^0} ds.
\end{aligned}$$

□

It is known that the study of theory of differential equation with infinite delays depends on a choice of the abstract phase space. We will employ an axiomatic definition of the phase space $\mathcal{B}$ introduced by Hale and Kato [21]. To establish the axiom of the phase space $\mathcal{B}$, we follow the terminology used in Hino et al.[22]. The axioms of the space $\mathcal{B}$ are established for functions mapping $(-\infty, 0]$ into X, endowed with a seminorm $\|\cdot\|_{\mathcal{B}}$, which satisfies the following axiom.

Axiom 2.3. (A_1) *If* $x:]-\infty, T] \longrightarrow X$, $T > 0$ *is such that* $x_0 \in \mathcal{B}$, *then, for every* $t \in J$, *the following properties hold*

1. $x_t \in \mathcal{B}$;
2. $\|x(t)\| \leq H\|x_t\|_{\mathcal{B}}$;
3. $\|x_t\|_{\mathcal{B}} \leq K(t)\sup_{0\leq s\leq t}\|x(s)\| + N(t)\|x_0\|_{\mathcal{B}}$,

   where $L > 0$ *is a constant;* $K, N : [0, +\infty[\longrightarrow [1, +\infty[$, K *is continuous,* N *is locally bounded, and* L, K, N *are independent of* $x(\cdot)$.

(A_2) *the space* $\mathcal{B}$ *is complete.*

The next result is a consequence of the phase space axiom. see for example [5].

Lemma 2.4. *Let* $x:]-\infty, T] \to X$ *be a measurable process such that* $x_0 = \varphi \in L^2(\Omega, \mathcal{B})$; *then*

$$\mathbf{E}\|x_s\|_{\mathcal{B}} \leq K_T \sup_{0\leq s\leq T}\mathbf{E}\|x(s)\| + N_T\mathbf{E}\|\varphi\|_{\mathcal{B}},$$

where $N_T = \sup_{t\in J} N(t)$ *and* $K_T = \sup_{t\in J} K(t)$.

We now consider the space $\mathcal{B}_T$ given by

$$\mathcal{B}_T = \{x : (-\infty, T] \to X : x(\cdot) \text{ is càdlàg such that} x_0 = \varphi \in \mathcal{B} \text{ and } \sup_{0\le t\le T} \mathbf{E}(\|x(t)\|^2) < \infty\},$$

Set $\|.\|_{\mathcal{B}_T}$ to be a semi-norm in $\mathcal{B}_T$, defined by

$$\|x\|_{\mathcal{B}_T} = \mathbf{E}\|x_0\|_{\mathcal{B}} + \sup_{0\le t\le T}(\mathbf{E}(\|x(t)\|^2))^{\frac{1}{2}}.$$

Next, we introduce some notations and basic facts about the theory of semigroups and fractional power operators. Let $A : D(A) \to X$ be the infinitesimal generator of an analytic semigroup, $(S(t))_{t\ge 0}$, of bounded linear operators on X. The theory of strongly continuous is thoroughly discussed in [9] and [2]. It is well-known that there exist $M \ge 1$ and $\lambda \in \mathbf{R}$ such that $\|S(t)\| \le Me^{\lambda t}$ for every $t \ge 0$. If $(S(t))_{t\ge 0}$ is a uniformly bounded, analytic semigroup such that $0 \in \rho(A)$, where $\rho(A)$ is the resolvent set of A, then it is possible to define the fractional power $(-A)^\alpha$ for $0 < \alpha \le 1$, as a closed linear operator on its domain $D(-A)^\alpha$. Furthermore, the subspace $D(-A)^\alpha$ is dense in X, and the expression

$$\|z\|_\alpha = \|(-A)^\alpha z\|$$

defines a norm in $D(-A)^\alpha$. If X_α represents the space $D(-A)^\alpha$ endowed with the norm $\|.\|_\alpha$, then the following properties hold (see [9], p. 74).

Lemma 2.5. *Suppose that A, X_α, and $(-A)^\alpha$ are as described above.*

(i) For $0 < \alpha \le 1$, X_α is a Banach space.

(ii) If $0 < \beta \le \alpha$, then the injection $X_\alpha \hookrightarrow X_\beta$ is continuous.

(iii) For every $0 < \alpha \le 1$, there exists $M_\alpha > 0$ such that

$$\|(-A)^\alpha S(t)\| \le M_\alpha t^{-\alpha} e^{-\lambda t}, \quad t > 0, \;\; \lambda > 0.$$

3. Controllability Result

In this section we prove the main result. First we give the definition of mild solutions for equation (1.1).

Definition 3.1. *A stochastic process $x(\cdot) : (-\infty, T] \longrightarrow X$ is a mild solution of (1.1) if*

1. $x(\cdot)$ *has càdlàg path on* $[0,T]-\{t_1,t_2,...,t_m\}$, *and* $\int_0^T \|x(t)\|^2dt<\infty$ *almost surely;*
2. $x(t)=\varphi(t)$ *on* $(-\infty,0]$;
3. *for every* $0\leq s<t$ *the function* $AS(t-s)p(s,x_s,\int_0^s\theta_1(s,\tau,x_\tau)d\tau)$ *is integrable such that the following integral equation is satisfied*

$$\begin{aligned} x(t) &= S(t)(\varphi(0)-g(0,\varphi,0))+p(t,x_t,\textstyle\int_0^t\theta_1(t,s,x_s)ds) \\ &+\textstyle\int_0^t AS(t-s)p(s,x_s,\int_0^s\theta_1(s,\tau,x_\tau)d\tau)ds+\int_0^t S(t-s)Bu(s)ds \\ &+\textstyle\int_0^t S(t-s)h(s,x_s,\int_0^s\theta_2(s,\tau,x_\tau)d\tau)ds+\int_0^t S(t-s)\sigma(s)dB^H(s) \\ &+\textstyle\int_0^t\int_{\mathcal{U}} S(t-s)g(s,x_s,\eta)\widetilde{N}(dt,d\eta) \\ &+\textstyle\sum_{0<t_k<t}S(t-t_k)I_k(x(t_k^-)),\quad \mathbf{P}-a.s. \end{aligned} \tag{3.1}$$

Definition 3.2. *The impulsive neutral stochastic functional integro-differential equation (1.1) is said to be controllable on the interval* $(-\infty,T]$ *if for every initial stochastic process* φ *defined on* $(-\infty,0]$, *there exists a stochastic control* $u\in L^2([0,T],U)$ *such that the mild solution* $x(\cdot)$ *of (1.1) satisfies* $x(T)=x_1$, *where* x_1 *and* T *are the preassigned terminal state and time, respectively.*

In order to establish the controllability of (1.1), we impose the following assumptions.

($\mathcal{H}$.1) A is the infinitesimal generator of an analytic semigroup, $(S(t))_{t\geq0}$, of bounded linear operators on X. Further, $0\in\rho(A)$, and there exist constants M, $M_{1-\beta}$ such that

$$\|S(t)\|^2\leq M \quad \text{and} \quad \|(-A)^{1-\beta}S(t)\|\leq\frac{M_{1-\beta}}{t^{1-\beta}},\ \text{for all } t\in[0,T]$$

(see Lemma 2.5).

($\mathcal{H}$.2) The mapping $p:J\times\mathcal{B}\times X\to X$ satisfies the following conditions

(i) The function $\theta_1:D\times\mathcal{B}\to X$ satisfies the following condition. There exists a constant $k_1>0$, for $x_1,x_2\in\mathcal{B}$ such that

$$\mathbf{E}\|\int_0^t[\theta_1(t,s,x_1)-\theta_1(t,s,x_2)]ds\|^2\leq k_1\{\mathbf{E}\|\mathbf{x_1}-\mathbf{x_2}\|^{\mathbf{2}}_{\mathcal{B}},\quad (\mathbf{t},\mathbf{s})\in\mathbf{D},$$

and $\overline{k}_1 = \sup_{(t,s)\in D} \| \int_0^t \theta_1(t,s,0)ds\|^2$.

(ii) There exist constants $0 < \beta < 1$, $\mathrm{k}_2 > 0$ such that the function p is X_β-valued and for $x_1, x_2 \in \mathcal{B}$, $y_1, y_2 \in X$ and satisfies for all $t \in J$

$$\begin{aligned}&\mathbf{E}\|(-A)^\beta p(t,x_1,y_1) - (-A)^\beta p(t,x_2,y_2)\|^2\\ &\leq k_2[\mathbf{E}\|x_1 - x_2\|^2_{\mathcal{B}} + \mathbf{E}\|y_1 - y_2\|^2],\end{aligned}$$

and $\overline{k}_2 = \sup_{t\in[0,T]} \|(-A)^{-\beta} p(t,0,0)\|^2$.

(iii) the function $(-A)^\beta p$ is continuous in the quadratic mean sense, i.e for $x_1, x_2 \in \mathcal{B}$, $y_1, y_2 \in X$

$$\lim_{t\longrightarrow s} \mathbf{E}\|(-A)^\beta p(t,x_1,y_1) - (-A)^\beta p(s,x_2,y_2)\|^2 = 0,$$

($\mathcal{H}$.3) The mapping $h : J \times \mathcal{B} \times X \to X$ satisfies the following Lipschitz conditions

(i) There exist positive constants $k_3, \overline{k}_3$ for $t \in [0,T]$, $x_1, x_2 \in \mathcal{B}$, $y_1 y_2 \in X$ such that

$$\mathbf{E}\|h(t,x_1,y_1) - h(t,x_2,y_2)\|^2 \leq k_3[\mathbf{E}\|x_1 - x_2\|^2_{\mathcal{B}} + \mathbf{E}\|y_1 - y_2\|^2], \text{ and}$$

$$\overline{k}_3 = \sup_{t\in[0,T]} \|h(t,0,0)\|^2.$$

(ii) The function $\theta_2 : D \times \mathcal{B} \to X$ satisfies the following condition. There exists a constant $k_4 > 0$, for $x_1, x_2 \in \mathcal{B}$ such that

$$\mathbf{E}\| \int_0^t [\theta_2(t,s,x_1) - \theta_2(t,s,x_2)]ds\|^2 \leq k_4\{\mathbf{E}\|\mathbf{x_1} - \mathbf{x_2}\|^{\mathbf{2}}_{\mathcal{B}}, \quad (\mathbf{t},\mathbf{s}) \in \mathbf{D},$$

and $\overline{k}_4 = \sup_{(t,s)\in D} \| \int_0^t \theta_2(t,s,0)ds\|^2$.

($\mathcal{H}$.4) There exist a positive constant $k_5 > 0$ such that, for all $t \in J$ and $x, y \in \mathcal{B}$

$$\int_{\mathcal{U}} \mathbf{E}\|g(t,x,\eta) - g(t,y,\eta)\|^2 \nu(d\eta) \leq k_5\{\mathbf{E}\|\mathbf{x} - \mathbf{y}\|^{\mathbf{2}}_{\mathcal{B}}$$

and $\overline{k}_5 = \sup_{t\in J} \int_{\mathcal{U}} \|g(t,0,\eta)\|^2 \nu(d\eta)$.

($\mathcal{H}$.5) The impulses functions I_k for $k = 1, 2, ..., m$, satisfies the following condition. There exist positive constants M_k, $\widetilde{M}_k$ such that $\|I_k(x) - I_k(y)\|^2 \leq M_k\{\mathbf{E}\|\mathbf{x} - \mathbf{y}\|^{\mathbf{2}}_{\mathcal{B}}$ and $\|I_k(x)\|^2 \leq \widehat{M}_k$ for all $x, y \in \mathcal{B}$.

$(\mathcal{H}.6)$ The function $\sigma : [0, \infty) \to \mathcal{L}_2^0(Y, X)$ satisfies

$$\int_0^T \|\sigma(s)\|^2_{\mathcal{L}_2^0} ds < \infty, \ \ \forall T > 0.$$

$(\mathcal{H}.7)$ The linear operator W from U into X defined by

$$Wu = \int_0^T S(T-s)Bu(s)ds$$

has an inverse operator W^{-1} that takes values in $L^2([0,T], U)\backslash kerW$, where

$$kerW = \{x \in L^2([0,T], U) : \ Wx = 0\}$$

(see [3]), and there exists finite positive constants M_b, M_w such that $\|B\|^2 \le M_b$ and $\|W^{-1}\|^2 \le M_w$.

$(\mathcal{H}.8)$ There exists a constant $\omega > 0$ such that

$$\begin{aligned} \omega = \ \ & 12K_T^2(1 + 5MM_bM_wT^2)[(c_1^2 + \tfrac{(M_{1-\beta}T^\beta)^2}{2\beta-1})k_2(1+2k_1) \\ & + MT^2k_3(1+k_4) + MTk_5 + mM\textstyle\sum_{k=1}^m M_k] < 1, \end{aligned}$$

and $c_1 = \|(-A)^{-\beta}\|$.

We prove the following theorem by using the Banach fixed point theorem.

Theorem 3.3. *Suppose that* $(\mathcal{H}.1) - (\mathcal{H}.8)$ *hold. Then, the system (1.1) is controllable on* $(-\infty, T]$ *provided that*

$$8\mathrm{K_T^2}(1 + 9\mathrm{MM_bM_wT^2})\{8(\mathrm{c_1^2} + \frac{(\mathrm{M}_{1-\beta}\mathrm{T}^\beta)^2}{2\beta - 1})\mathrm{k_2}(1+2\mathrm{k_1}) + 8\mathrm{MT}[\mathrm{k_3}(1+2\mathrm{k_4})\mathrm{T} + \mathrm{k_5}]\} < 1. \tag{3.2}$$

Proof. Using the hypothesis $(\mathcal{H}.7)$ for an arbitrary function $x(\cdot)$, Define the control u_x by

$$\begin{aligned} u_x(t) \ &= \ W^{-1}\{x_1 - S(T)(\varphi(0) - p(0, x_0, 0)) - p(T, x_T, \textstyle\int_0^T \theta_1(T, s, x_s)ds)) \\ &- \ \textstyle\int_0^T AS(T-s)p(s, x_s, \int_0^s \theta_1(s, \eta, x_\eta)d\eta)ds \\ &- \ \textstyle\int_0^T S(T-s)h(s, x_s, \int_0^s \theta_2(s, \eta, x_\eta)d\eta)ds - \int_0^T S(T-s)\sigma(s)dB^H(s) \\ &- \ \textstyle\int_0^T \int_{\mathcal{U}} S(T-s)g(s, x_s, \eta)\widetilde{N}(ds, d\eta) - \sum_{0<t_k<T} S(T-t_k)I_k(x(t_k^-))\}(t). \end{aligned} \tag{3.3}$$

Put the control $u(.)$ into the stochastic control system (3.1) and obtain a non linear operator Π on $\mathcal{B}_T$ given by

$$\Pi(x)(t) = \begin{cases} \varphi(t), \quad \text{if } \ t \in (-\infty, 0], \\ S(t)(\varphi(0) - p(0, \varphi, 0)) + p(t, x_t, \int_0^t \theta_1(t, s, x_s)ds) \\ + \int_0^t AS(t-s)p(s, x_s, \int_0^s \theta_1(s, \eta, x_\eta)d\eta)ds \\ + \int_0^t S(t-s)Bu_x(s)ds + \int_0^t S(t-s)h(s, x_s, \int_0^s \theta_2(s, \eta, x_\eta)d\eta)ds \\ + \int_0^t S(t-s)\sigma(s)dB^H(s) + \int_0^t \int_{\mathcal{U}} S(t-s)g(s, x_s, \eta)\widetilde{N}(ds, d\eta) \\ + \sum_{0<t_k<t} S(t-t_k)I_k(x(t_k^-)), \ \text{if } \ t \in [0, T]. \end{cases}$$

Then it is clear that to prove the existence of mild solutions to equation (1.1) is equivalent to find a fixed point for the operator Π, and that $\Pi x(T) = x_1$, which means that the system is controllable, provided we can obtain a fixed point of the operator Π.

Let $y : (-\infty, T] \longrightarrow X$ be the function defined by

$$y(t) = \begin{cases} \varphi(t), & \text{if } \ t \in (-\infty, 0], \\ S(t)\varphi(0), & \text{if } \ t \in [0, T], \end{cases}$$

then, $y_0 = \varphi$. For each function $z \in \mathcal{B}_T$, set

$$x(t) = z(t) + y(t).$$

It is obvious that x satisfies the stochastic control system (3.1) if and only if z satisfies $z_0 = 0$ and

$$\begin{aligned} z(t) = \ & p(t, z_t + y_t, \int_0^t \theta_1(t, s, z_s + y_s)ds) - S(t)p(0, \varphi, 0) \\ & + \int_0^t AS(t-s)p(s, z_s + y_s, \int_0^s \theta_1(s, \eta, z_\eta + y_\eta)d\eta)ds \\ & + \int_0^t S(t-s)Bu_{z+y}(s)ds + \int_0^t S(t-s)h(s, z_s + y_s, \int_0^s \theta_2(s, \eta, z_\eta + y_\eta)d\eta)ds \\ & + \int_0^t S(t-s)\sigma(s)dB^H(s) + \int_0^t \int_{\mathcal{U}} S(t-s)g(s, z_s + y_s, \eta)\widetilde{N}(ds, d\eta) \\ & + \sum_{0<t_k<t} S(t-t_k)I_k(z(t_k^-) + y(t_k^-)), \quad \text{if } \ t \in [0, T], \end{aligned} \tag{3.4}$$

where $u_{z+y}(t)$ is obtained from (3.3) by replacing $x_t = z_t + y_t$. Set

$$\mathcal{B}_T^0 = \{z \in \mathcal{B}_T : z_0 = 0\};$$

for any $z \in B_T^0$, we have

$$\|z\|_{\mathcal{B}_T^0} = \mathbf{E}\|z_0\|_{\mathcal{B}} + \sup_{t\in[0,T]} (\mathbf{E}\|z(t)\|^2)^{\frac{1}{2}} = \sup_{t\in[0,T]} (\mathbf{E}\|z(t)\|^2)^{\frac{1}{2}}.$$

Then, $(\mathcal{B}_T^0, \|.\|_{\mathcal{B}_T^0})$ is a Banach space. Define the operator $\Phi : \mathcal{B}_T^0 \longrightarrow \mathcal{B}_T^0$ by

$$(\Phi z)(t) = \begin{cases} 0 \ \text{ if } \ t \in (-\infty, 0], \\ p(t, z_t + y_t, \int_0^t \theta_1(t, s, z_s + y_s)ds) - S(t)p(0, \varphi, 0) \\ + \int_0^t AS(t-s)p(s, z_s + y_s, \int_0^s \theta_1(s, \eta, z_\eta + y_\eta)d\eta)ds \\ + \int_0^t S(t-s)h(s, z_s + y_s, \int_0^s \theta_2(s, \eta, z_\eta + y_\eta)d\eta)ds \\ + \int_0^t S(t-s)Bu_{z+y}(s)ds + \int_0^t S(t-s)\sigma(s)dB^H(s) \\ + \int_0^t \int_{\mathcal{U}} S(t-s)g(s, z_s + y_s, \eta)\widetilde{N}(ds, d\eta) \\ + \sum_{0<t_k<t} S(t-t_k)I_k(z(t_k^-) + y(t_k^-)), \quad \text{if } \ t \in [0, T], \end{cases}$$

(3.5)

Set

$$\mathcal{B}_k = \{z \in \mathcal{B}_T^0 : \|z\|^2_{\mathcal{B}_T^0} \le k\}, \qquad \text{for some } k \ge 0,$$

then $\mathcal{B}_k \subseteq \mathcal{B}_T^0$ is a bounded closed convex set, and for $z \in \mathcal{B}_k$, we have

$$\begin{aligned}\mathbf{E}\|z_t+y_t\|_{\mathcal{B}}^2 &\le 2(\mathbf{E}\|z_t\|_{\mathcal{B}}^2+\mathbf{E}\|y_t\|_{\mathcal{B}}^2)\\ &\le 2\Big[2K_T^2(\sup_{0\le s\le T}\mathbf{E}\|z(s)\|)^2+2N_T^2(\mathbf{E}\|z_0\|_{\mathcal{B}})^2\\ &+2K_T^2(\sup_{0\le s\le t}\mathbf{E}\|y(s)\|)^2+2N_T^2(\mathbf{E}\|\varphi\|_{\mathcal{B}})^2\Big]\\ &\le 4K_T^2(k+M\{\mathbf{E}\|\varphi(\mathbf{0})\|^{\mathbf{2}})+\mathbf{4N_T^2}\{\mathbf{E}\|\varphi\|_{\mathcal{B}}^{\mathbf{2}}\\ &:= r^*.\end{aligned}$$

From the assumptions mentioned above, we have for any $z\in\mathcal{B}_k$

$$\begin{aligned}\mathbf{E}\|u_{z+y}\|^2 \quad &\le 9M_w\{\|x_1\|^2+M\{\mathbf{E}\|\varphi(\mathbf{0})\|^{\mathbf{2}}+\mathbf{2Mc_1^2}[\mathbf{k_2}\|\mathbf{y}\|_{\mathcal{B}_\zeta}^{\mathbf{2}}+\overline{\mathbf{k}}_{\mathbf{2}}]\\ &+2(c_1^2+\tfrac{(M_{1-\beta}T^\beta)^2}{2\beta-1})[k_2(1+2k_1)r^*+2k_2\overline{k}_1+\overline{k}_2]\\ &+2MT^2[k_3(1+2k_4)r^*+2k_3\overline{k}_4+\overline{k}_3]+2MT^{2H-1}\int_0^T\|\sigma(s)\|_{\mathcal{L}_2^0}^2ds\\ &+2MT(k_5r^*+\overline{k}_5)+mM\textstyle\sum_{k=1}^m\widetilde{M}_k\}:=\mathcal{G},\end{aligned}\tag{3.6}$$

and for $z,v\in\mathcal{B}_k$

$$\begin{aligned}\mathbf{E}\|u_{z+y}-u_{v+y}\|^2 &\le 5M_w\{(c_1^2+\tfrac{(M_{1-\beta}T^\beta)^2}{2\beta-1})k_2(1+2k_1)+MT^2k_3(1+2k_4)\\ &+MTk_5+mM\textstyle\sum_{k=1}^m M_k\}\mathbf{E}\|z_t-v_t\|_{\mathcal{B}}^2.\end{aligned}\tag{3.7}$$

It is clear that the operator Π has a fixed point if and only if Φ has one. The proof will be given in following steps.

Step 1: We verify that $\Phi(x)(t)$ is a cadlag process on J. Let $t\in J$ and h be sufficiently small, then for all $z\in\mathcal{B}_T^0$, we have

$$\begin{aligned}
\mathbf{E}\|\Phi(z)(t+h) \quad & -\Phi(z)(t)\|^2 \leq 8\{\mathbf{E}\|[\mathbf{S(t+h)-S(t)]p(0,\varphi,0)}\|^{\mathbf{2}} \\
& +8\{\mathbf{E}\|\mathbf{p(t+h, z_{t+h}+y_{t+h}}, \textstyle\int_{\mathbf{0}}^{\mathbf{t+h}} \theta_{\mathbf{1}}\mathbf{(t+h,s,z_s+y_s)ds}) \\
& -p(t,z_t+y_t, \textstyle\int_0^t \theta_1(t,s,z_s+y_s)ds)\|^2 \\
& +8\{\mathbf{E}\| \textstyle\int_{\mathbf{0}}^{\mathbf{t+h}} \mathbf{AS(t+h-s)p(s,z_s+y_s}, \textstyle\int_{\mathbf{0}}^{\mathbf{s}} \theta_{\mathbf{1}}\mathbf{(s},\eta,\mathbf{z}_\eta+\mathbf{y}_\eta)\mathbf{d}\eta)\mathbf{ds} \\
& - \textstyle\int_0^t AS(t-s)p(s,z_s+y_s, \textstyle\int_0^s \theta_1(s,\eta,z_\eta+y_\eta)d\eta)ds\|^2 \\
& +8\{\mathbf{E}\| \textstyle\int_{\mathbf{0}}^{\mathbf{t+h}} \mathbf{S(t+h-s)h(s,z_s+y_s}, \textstyle\int_{\mathbf{0}}^{\mathbf{s}} \theta_{\mathbf{2}}\mathbf{(s},\eta,\mathbf{z}_\eta+\mathbf{y}_\eta)\mathbf{d}\eta)\mathbf{ds} \\
& - \textstyle\int_0^t S(t-s)h(s,z_s+y_s, \textstyle\int_0^s \theta_2(s,\eta,z_\eta+y_\eta)d\eta)ds\|^2 \\
& +8\{\mathbf{E}\| \textstyle\int_{\mathbf{0}}^{\mathbf{t+h}} \mathbf{S(t+h-s)Bu_{z+y}(s)ds} - \textstyle\int_{\mathbf{0}}^{\mathbf{t}} \mathbf{S(t-s)Bu_{z+y}(s)ds}\|^{\mathbf{2}} \\
& +8\{\mathbf{E}\| \textstyle\int_{\mathbf{0}}^{\mathbf{t+h}} \mathbf{S(t+h-s)}\sigma\mathbf{(s)dB^H(s)} - \textstyle\int_{\mathbf{0}}^{\mathbf{t}} \mathbf{S(t-s)}\sigma\mathbf{(s)dB^H(s)}\|^{\mathbf{2}} \\
& +8\{\mathbf{E}\| \textstyle\int_{\mathbf{0}}^{\mathbf{t+h}} \textstyle\int_{\mathcal{U}} \mathbf{S(t+h-s)g(s,z_s+y_s},\eta)\widetilde{\mathbf{N}}(\mathbf{ds},\mathbf{d}\eta) \\
& - \textstyle\int_0^t \textstyle\int_{\mathcal{U}} S(t-s)g(s,z_s+y_s,\eta)\widetilde{N}(ds,d\eta)\|^2 \\
& +8\{\mathbf{E}\|\textstyle\sum_{\mathbf{0<t_k<t+h}}\mathbf{S(t+h-t_k)I_k(z(t_k^-)+y(t_k^-))} \\
& -\textstyle\sum_{0<t_k<t} S(t-t_k)I_k(z(t_k^-)+y(t_k^-))\|^2 \\
& = 8\textstyle\sum_{i=1}^{8}\mathbf{E}\|N_i(h)\|^2.
\end{aligned}$$

By the strong continuity of $S(t)$, we have

$$\lim_{h\to 0}(S(t+h)-S(t))p(0,\phi,0) = 0.$$

And from the condition $\mathcal{H}.1$, we get

$$\mathbf{E}\|(S(t+h)-S(t))p(0,\phi,0)\| \leq 2M\{\mathbf{E}\|\mathbf{p(0},\phi,\mathbf{0})\|^{\mathbf{2}}.$$

Then the Lebesgue dominated theorem implies that

$$\lim_{h\to 0}\mathbf{E}\|N_1(h)\|^2 = 0.$$

From condition (iii) in $\mathcal{H}.2$ and since the operator $(-A)^{-\beta}$ is bounded, we conclude that

$$\lim_{h\to 0}\mathbf{E}\|N_2(h)\|^2 = 0.$$

For the third terme $N_3(h)$, we have

$$\begin{aligned}\|N_3(h)\| &\le \|\textstyle\int_0^t A[S(t+h-s)-S(t-s)]p(s,z_s+y_s,\int_0^s\theta_1(s,\tau,z_\tau+y_\tau)d\tau)ds\|\\ &+\|\textstyle\int_t^{t+h} AS(t+h-s)p(s,z_s+y_s,\int_0^s\theta_1(s,\tau,z_\tau+y_\tau)d\tau)ds\|\\ &= \|N_{31}(h)\|+\|N_{32}(h)\|.\end{aligned}$$

By using Holder's inequality, we get

$$\mathbf{E}\|N_{31}(h)\|^2 \le t\int_0^t\{\mathbf{E}\|\mathbf{A}[\mathbf{S}(\mathbf{t}+\mathbf{h}-\mathbf{s})-\mathbf{S}(\mathbf{t}-\mathbf{s})]\mathbf{p}(\mathbf{s},\mathbf{z_s}+\mathbf{y_s},\int_\mathbf{0}^\mathbf{s}\theta_\mathbf{1}(\mathbf{s},\tau,\mathbf{z}_\tau+\mathbf{y}_\tau)\mathbf{d}\tau)\|^\mathbf{2}\mathbf{ds}$$

By using condition $\mathcal{H}.1$, (i) and (ii) in $\mathcal{H}.2$, one has that

$$\begin{aligned}&\mathbf{E}\|A[S(t+h-s)-S(t-s)]p(s,z_s+y_s,\textstyle\int_0^s\theta_1(s,\tau,z_\tau+y_\tau)d\tau)\|^2\\ &\le \|S(h)-I\|^2\frac{M_{1-\beta}^2}{(t-s)^{2(1-\beta)}}\Big[k_2(1+2k_1)r^*+2k_2\overline{k}_1+\overline{k}_2\Big].\end{aligned}$$

Then by using the strong continuity of $S(t)$ and by applying the Lebesgue dominated theorem, we obtain

$$\lim_{h\to 0}\mathbf{E}\|N_{31}(h)\|^2 = 0$$

Through the use of $\mathcal{H}.1$, (i) and (ii) in $\mathcal{H}.2$ along with the Holder's inequality, we get

$$\mathbf{E}\|N_{32}(h)\|^2 \quad \le hM_{1-\beta}^2\Big[k_2(1+2k_1)r^*+2k_2\overline{k}_1+\overline{k}_2\Big]\textstyle\int_0^T(t+h-s)^{2\beta-2}ds,$$

therefore

$$\lim_{h\to 0}\mathbf{E}\|N_3(h)\|^2 = 0.$$

A similar calculation to the one made previously gives

$$\lim_{h\to 0}\mathbf{E}\|N_i(h)\|^2 = 0, \qquad i=4,5.$$

Moreover, we have

$$\begin{aligned} N_6(h) & \leq \left\| \int_0^t (S(t+h-s)-S(t-s))\sigma(s)dB_H(s) \right\| \\ & + \left\| \int_t^{t+h} S(t+h-s)\sigma(s)dB_H(s) \right\| \\ & \leq N_{61}(h)+N_{62}(h). \end{aligned}$$

From Lemma 2.2, we get that

$$\mathbf{E}\,|N_{61}(h)|^2 \leq 2Ht^{2H-1}\int_0^t \|(S(t+h-s)-S(t-s))\sigma(s)\|^2_{\mathcal{L}_2}ds.$$

By using condition $\mathcal{H}.1$ and since $S(t)$ is strongly continuous we conclude, by the dominated convergence theorem that,

$$\lim_{h\to 0}\mathbf{E}\,|N_{61}(h)|^2 = 0.$$

Again by virtue of Lemma 2.2 , we get that

$$\mathbf{E}\,|N_{62}(h)|^2 \leq 2MHh^{2H-1}\int_t^{t+h} \|\sigma(s)\|^2_{\mathcal{L}_2}ds \to 0 \text{ as } h\to 0.$$

By assumption $\mathcal{H}.1$ and $\mathcal{H}.4$, we get

$$\begin{aligned} \mathbf{E}\|N_7(h)\|^2 & \leq \mathbf{E}\left\|\int_0^t\int_{\mathcal{U}}[S(t+h-s)-S(t-s)]g(s,z_s+y_s,\eta)\widetilde{N}(ds,d\eta)\right\|^2 \\ & + \mathbf{E}\left\|\int_t^{t+h}\int_{\mathcal{U}}S(t+h-s)g(s,z_s+y_s,\eta)\widetilde{N}(ds,d\eta)\right\|^2 \\ & \leq 4MT\|S(h)-I\|^2[k_5r^*+\bar{k}_5]+4Mh[k_5r^*+\bar{k}_5]. \end{aligned} \tag{3.8}$$

Using the inequality (3.8) together with the strong continuity of $S(t)$, we obtain that

$$\lim_{h\to 0}\mathbf{E}\|N_7(h)\|^2 = 0.$$

Now, for the last term, we have

$$\begin{aligned} \mathbf{E}\|N_8(h)\|^2 \leq\ & 2\mathbf{E}\left\|\textstyle\sum_{0<t_i<t}(S(h)-I)S(t-t_i)I_i(z(t_i^-)+y(t_i^-))\right\|^2 \\ & +2\mathbf{E}\left\|\textstyle\sum_{t\leq t_i<t+h}S(t+h-t_i)I_i(z(t_i^-)+y(t_i^-))\right\|^2. \end{aligned}$$

From condition $\mathcal{H}.1$ and $\mathcal{H}.6$, we have

$$\left\|(S(h)-I)S(t-t_i)I_i(z(t_i^-)+y(t_i^-))\right\|^2 \leq \|(S(h)-I)\|^2M\widetilde{M}_k,$$

and

$$\left\|S(t+h-t_i)I_i(z(t_i^-)+y(t_i^-))\right\|^2 \leq M\widetilde{M}_k.$$

So, we obtain that

$$\mathbf{E}\left\|N_8(h)\right\|^2 \to 0 \text{ as } h \to 0$$

The above arguments show that $\lim_{h\to 0}\mathbf{E}\|\Phi(x)(t+h)-\Phi(x)(t)\|^2 = 0$.

Step 2: Next, we claim that $\Phi(\mathcal{B}_k) \subset \mathcal{B}_k$. If it is not true, then for each positive number k, there is a function $z^k(.) \in \mathcal{B}_k$, such that $\Phi(z^k) \notin \mathcal{B}_k$, that is $\mathbf{E}\|\Phi(z^k)(t)\|^2 > k$ for some $t \in J$. on the other hand, we have

$$\begin{aligned}
k &< \mathbf{E}\|\Phi(z^k)(t)\|^2 \\
&\leq 8\{2Mc_1^2(k_2\|\varphi\|_{\mathcal{B}}^2+\overline{k}_2)+2(c_1^2+\tfrac{(M_{1-\beta}T^{\beta})^2}{2\beta-1}[k_2(1+2k_1)r^*+2k_2\overline{k}_1+\overline{k}_2] \\
&+MM_bT^2\mathcal{G}+2MT^2(k_3(1+2k_4)r^*+2k_3\overline{k}_4+\overline{k}_3)+2MT^{2H-1}\textstyle\int_0^T\|\sigma(s)\|_{\mathcal{L}_2^0}^2ds \\
&+2MT(k_5r^*+\overline{k}_5)+M\textstyle\sum_{k=1}^m\widetilde{M}_k\} \\
&\leq 8(1+9MM_bM_wT^2)\{2Mc_1^2(k_2\|\varphi\|_{\mathcal{B}}^2+\overline{k}_2)+2(c_1^2+\tfrac{(M_{1-\beta}T^{\beta})^2}{2\beta-1}[k_2(1+2k_1)r^* \\
&+2k_2\overline{k}_1+\overline{k}_2]+2MT^2[k_3(1+2k_4)r^*+2k_3\overline{k}_4+\overline{k}_3]+2MT^{2H-1}\textstyle\int_0^T\|\sigma(s)\|_{\mathcal{L}_2^0}^2ds \\
&+2MT(k_5r^*+\overline{k}_5)+M\textstyle\sum_{k=1}^m\widetilde{M}_k\}+8\times 9MM_bM_wT^2(\|x_1\|^2+M\{\mathbf{E}\|\varphi(\mathbf{0})\|^{\mathbf{2}}) \\
&\leq N+8(1+9MM_bM_wT^2)\{2(c_1^2+\tfrac{(M_{1-\beta}T^{\beta})^2}{2\beta-1})k_2(1+2k_1)r^* \\
&+2MT[k_3(1+2k_4)T+k_5]r^*\},
\end{aligned}$$

where

$$\begin{aligned}
N &= 8(1+9MM_bM_wT^2)\{2Mc_1^2(k_2\|\varphi\|_{\mathcal{B}}^2+\overline{k}_2)+2(c_1^2+\tfrac{(M_{1-\beta}T^{\beta})^2}{2\beta-1}(2k_2\overline{k}_1+\overline{k}_2)) \\
&+2MT^2(2k_3\overline{k}_4+\overline{k}_3)+2MT^{2H-1}\textstyle\int_0^T\|\sigma(s)\|_{\mathcal{L}_2^0}^2ds+2MT\overline{k}_5+mM\textstyle\sum_{k=1}^m\widetilde{M}_k\} \\
&+8\times 9MM_bM_wT^2(\|x_1\|^2+M\{\mathbf{E}\|\varphi(\mathbf{0})\|^{\mathbf{2}})\}
\end{aligned}$$

is independent of k. Dividing both sides by k and taking the limit as $k \longrightarrow \infty$, we get

$$8K_T^2(1+9MM_bM_wT^2)\{8(c_1^2+\frac{(M_{1-\beta}T^\beta)^2}{2\beta-1})k_2(1+2k_1)+8MT[k_3(1+2k_4)T+k_5]\} \geq 1.$$

This contradicts (3.2). Hence for some positive k,

$$(\Phi)(\mathcal{B}_k) \subseteq \mathcal{B}_k.$$

Step 3: Φ is a contraction. Let $t \in J$ and $z^1, z^2 \in \mathcal{B}_T^0$, we have

$$\begin{aligned} \mathbf{E}\|\Phi z^1(t) - \Phi z^2(t)\|^2 \quad & \leq 6\{\mathbf{E}\|\mathbf{p}(\mathbf{t}, \mathbf{z_t^1} + \mathbf{y_t}, \textstyle\int_\mathbf{0}^\mathbf{t} \theta_\mathbf{1}(\mathbf{t}, \mathbf{s}, \mathbf{z_s^1} + \mathbf{y_s})\mathbf{ds}) \\ & -p(t, z_t^2 + y_t, \textstyle\int_0^t \theta_1(t, s, z_s^2 + y_s)ds)\|^2 \\ & +6\{\mathbf{E}\| \textstyle\int_\mathbf{0}^\mathbf{t} \mathbf{AS}(\mathbf{t}-\mathbf{s})[\mathbf{p}(\mathbf{s}, \mathbf{z_s^1} + \mathbf{y_s}, \textstyle\int_\mathbf{0}^\mathbf{s} \theta_\mathbf{1}(\mathbf{s}, \eta, \mathbf{z}_\eta^\mathbf{1} + \mathbf{y}_\eta)\mathbf{d}\eta) \\ & -p(s, z_s^2 + y_s, \textstyle\int_0^s \theta_1(s, \eta, z_\eta^2 + y_\eta)d\eta)]ds\|^2 \\ & +6\{\mathbf{E}\| \textstyle\int_\mathbf{0}^\mathbf{t} \mathbf{S}(\mathbf{t}-\mathbf{s})\mathbf{B}[\mathbf{u}_{\mathbf{z^1}+\mathbf{y}}(\mathbf{s}) - \mathbf{u}_{\mathbf{z^2}+\mathbf{y}}(\mathbf{s})]\mathbf{ds}\|^\mathbf{2} \\ & +6\{\mathbf{E}\| \textstyle\int_\mathbf{0}^\mathbf{t} \mathbf{S}(\mathbf{t}-\mathbf{s})[\mathbf{h}(\mathbf{s}, \mathbf{z_s^1} + \mathbf{y_s}, \textstyle\int_\mathbf{0}^\mathbf{s} \theta_\mathbf{2}(\mathbf{s}, \eta, \mathbf{z}_\eta^\mathbf{1} + \mathbf{y}_\eta)\mathbf{d}\eta) \\ & -h(s, z_s^2 + y_s, \textstyle\int_0^s \theta_2(s, \eta, z_\eta^2 + y_\eta)d\eta)]ds\|^2 \\ & +6\{\mathbf{E}\| \textstyle\int_\mathbf{0}^\mathbf{t} \int_\mathcal{U} \mathbf{S}(\mathbf{t}-\mathbf{s})[\mathbf{g}(\mathbf{s}, \mathbf{z_s^1} + \mathbf{y_s}, \eta) \\ & -g(s, z_s^2 + y_s, \eta)]\widetilde{N}(ds, d\eta)\|^2 \\ & +6\{\mathbf{E}\| \textstyle\sum_{\mathbf{0}<\mathbf{t_k}<\mathbf{T}} \mathbf{S}(\mathbf{T}-\mathbf{t_k}) \\ & [I_k(z^1(t_k^-) + y(t_k^-)) - I_k(z^2(t_k^-) + y(t_k^-))]\|^2 \end{aligned}$$

(3.9)

On the other hand from $(\mathcal{H}.1)-(\mathcal{H}.8)$ combined with (3.7), we obtain

$$\begin{aligned}
\mathbf{E}\|\Phi z^1(t) \quad & -\Phi z^2(t)\|^2 \leq 6(1+5MM_bM_wT^2)[(c_1^2+\tfrac{(M_{1-\beta}T^{\beta})^2}{2\beta-1})k_2(1+2k_1) \\
& +MT^2k_3(1+k_4)+MTk_5+mM\textstyle\sum_{k=1}^m M_k]\mathbf{E}\|z_t^1-z_t^2\|_{\mathcal{B}}^2 \\
& \leq 6(1+5MM_bM_wT^2)[(c_1^2+\tfrac{(M_{1-\beta}T^{\beta})^2}{2\beta-1})k_2(1+2k_1) \\
& +MT^2k_3(1+k_4)+MTk_5+mM\textstyle\sum_{k=1}^m M_k] \\
& \times\{2K_T^2\sup_{0\leq s\leq T}\mathbf{E}\|z^1(s)-z^2(s)\|^2+2N_T^2\{\mathbf{E}\|\mathbf{z_0^1}-\mathbf{z_0^2}\|_{\mathcal{B}}^{\mathbf{2}}\} \\
& \leq \omega\sup_{0\leq s\leq T}\mathbf{E}\|z^1(s)-z^2(s)\|^2) \quad (\text{ since } \ z_0^1=z_0^2=0)
\end{aligned}$$

Taking supremum over t,

$$\|\Phi z^1-\Phi z^2\|_{\mathcal{B}_T^0}\leq\omega\|z^1-z^2\|_{\mathcal{B}_T^0},$$

where

$$\begin{aligned}
\omega = \quad & 12K_T^2(1+5MM_bM_wT^2)[(c_1^2+\tfrac{(M_{1-\beta}T^{\beta})^2}{2\beta-1})k_2(1+2k_1) \\
& +MT^2k_3(1+k_4)+MTk_5+mM\textstyle\sum_{k=1}^m M_k].
\end{aligned}$$

By condition $(\mathcal{H}.8)$, we have $\omega<1$, hence Φ is a contraction mapping on $\mathcal{B}_T^0$ and therefore has a unique fixed point, which is a mild solution of equation (1.1) on $(-\infty,T]$. Clearly, $(\Phi x)(T)=x_1$ which implies that the system (1.1) is controllable on $(-\infty,T]$. This completes the proof. □

Remark 3.4. *When the impulses disappear, that is $M_k=\tilde{M}_k=0$, $k=1,...,m$ then the system* (1.1) *reduces to the following neutral stochastic integro-differential equation:*

$$(3.10)\quad \begin{cases}
d[x(t) & -p(t,x_t,\int_0^t\theta_1(t,s,x_s)ds)]=[Ax(t)+h(t,x_t,\int_0^t\theta_2(t,s,x_s)ds) \\
& +Bu(t)]dt+\sigma(t)dB^H(t)+\int_{\mathcal{U}}g(t,x_t,\eta)\widetilde{N}(dt,d\eta),\ t\in J, \\
& x_0=\varphi\in\mathcal{B}
\end{cases}$$

where the operators $A,p,h,g,\theta_1,\theta_2$ and σ are defined as same as before.

Set

$$\bar{\omega} = 10K_T^2(1+4MM_bM_wT^2)[(c_1^2+\frac{(M_{1-\beta}T^\beta)^2}{2\beta-1})k_2(1+2k_1)$$
$$+MT^2k_3(1+k_4)+MTk_5].$$

By replacing the constant ω in hypothesis $\mathcal{H}.8$ by $\bar{\omega}$, and using the same technique in Theorem 3.3, we can easily deduce the following corollary.

Corollary 3.5. *Suppose that $(\mathcal{H}.1)-(\mathcal{H}.4)$ and $(\mathcal{H}.6)-(\mathcal{H}.8)$ hold. Then, the system (3.10) is controllable on $(-\infty,T]$ provide that*
$7\mathrm{K}_T^2(1+8MM_bM_wT^2)\{8(c_1^2+\frac{(M_{1-\beta}T^\beta)^2}{2\beta-1})k_2(1+2k_1)+8MT[k_3(1+2k_4)T+k_5]\}<1.$

4. Example

In this section, we gives an application for our theoretical result. Let $X = Y = U = L^2([0,\pi])$, and define the operator $A : D(A) \subset X \longrightarrow X$ by $A = \frac{\partial^2}{\partial^2\xi}$ with

$$D(A) = \{y \in X : y' \text{ is absolutely continuous}, y'' \in X, \quad y(0) = y(\pi) = 0\},$$

furthermore,

$$Ay = \sum_{n=1}^{\infty} n^2 < y, e_n >_X e_n, y \in D(A),$$

where $e_n := \sqrt{\frac{2}{\pi}}\sin nx,\ n = 1, 2, ...$ is an orthogonal set of eigenvector of $-A$. The bounded linear operator $(-A)^{\frac{3}{4}}$ is given by

$$(-A)^{\frac{3}{4}}y = \sum_{n=1}^{\infty} n^{\frac{3}{2}} < y, e_n >_X e_n,$$

with domain $D((-A)^{\frac{3}{4}}) = \{y \in X, \sum_{n=1}^{\infty} n^{\frac{3}{2}} < y, e_n >_X e_n \in X\}$, and $\|(-A)^{\frac{3}{4}}\| = 1$.

Then, A generates an analytic semigroup $\{S(t)\}_{t\geq 0}$ in X given by (see [9])

$$S(t)y = \sum_{n=1}^{\infty} e^{-n^2t} < y, e_n > e_n,$$

for $y \in X$ and $t \geq 0$. Since the semigroup $\{S(t)\}_{t\geq 0}$ is analytic, there exists a constant $M > 0$ such that $\|S(t)\|^2 \leq M$ for every $t \geq 0$. In other words, the condition $(\mathcal{H}.1)$ holds.

Consider the stochastic control system:

$$
(4.1) \quad \begin{cases}
\frac{\partial}{\partial t}[y(t,\xi) - R_1(t, y(t-b,\xi), \int_0^t r_1(t,s,y(s-b,\xi))ds)] \\
= \frac{\partial^2}{\partial^2 \xi} y(t,\xi) + R_2(t, y(t-b,\xi), \int_0^t r_2(t,s,y(s-b,\xi))ds) \\
+ c(\xi)u(t) + \sigma(t)\frac{dB^H(t)}{dt} + \int_{\mathcal{U}} \Gamma(t, y(t-b,\xi), \eta)\widetilde{N}(dt, d\eta), \\
0 \leq t \leq T,\ t \neq t_j,\ 0 \leq \xi \leq \pi \\
\Delta y(t_j,\xi) = y(t_j^+,\xi) - y(t_j^-,\xi) = I_j(y(t_j^-)), \quad j = 1, 2, ..., m; \\
y(t,0) = y(t,\pi) = 0, \qquad 0 \leq t \leq T, \\
y(s,\xi) = \varphi(s,\xi),\ -\infty < s \leq 0, \quad 0 \leq \xi \leq \pi,
\end{cases}
$$

where $0 < t_1 < t_2 < ... < t_m < T$ are prefixed numbers, and $\varphi \in \mathcal{B}$. For $(t,\phi) \in [0,T] \times \mathcal{B}$, where $\phi(\tau)(\xi) = \phi(\tau,\xi)$, $(\tau,\xi) \in (-\infty,0] \times [0,\pi]$, we put $y(t)(\xi) = y(t,\xi)$. Let $B : U \longrightarrow X$ be a bounded linear operator defined by $Bu(t)(\xi) = c(\xi)u(t)$, $0 \leq \xi \leq \pi$, $u \in L^2([0,T],U)$.

Moreover, we assume that The linear operator $W : L^2([0,T],U) \longrightarrow X$ given by

$$
Wu(\xi) = \int_0^T S(T-s)c(\xi)u(t)ds,\ \ 0 \leq \xi \leq \pi,
$$

is a bounded linear operator but not necessarily one-to-one. Let $Ker\, W = \{x \in L^2([0,T],U),\ Wx = 0\}$ be the null space of W and $[Ker\, W]^\perp$ be its orthogonal complement in $L^2([0,T],U)$. Let $\widetilde{W} : [Ker\, W]^\perp \longrightarrow Range(W)$ be the restriction of W to $[Ker\, W]^\perp$, $\widetilde{W}$ is necessarily one-to-one operator. The inverse mapping theorem says that $\widetilde{W}^{-1}$ is bounded since $[Ker\, W]^\perp$ and $Range(W)$ are Banach spaces. So that W^{-1} is bounded and takes values in $L^2([0,T],U) \setminus Ker\, W$, and hence hypothesis $(\mathcal{H}.7)$ is satisfied.

Let $q(t)_{t>0}$ be a Poisson point process with a σ-finite measure $\nu(d\eta)$. Let denote by $N(dt,d\eta)$ the Poisson counting measure, wich is induced by $q(\cdot)$, then $\widetilde{N}(dt,d\eta) = N(dt,d\eta) - dt\nu(d\eta)$ is the compensating martingale measure. Let $Q : Y := L^2([0,\pi],) \longrightarrow Y$, we choose a sequence $\{\lambda_n\}_{n\in} \subset^+$, set $Qe_n = \lambda_n e_n$, and assume that $tr(Q) = \sum_{n=1}^\infty \sqrt{\lambda_n} < \infty$. Define the fBm in Y by $B^H(t) = \sum_{n=1}^\infty \sqrt{\lambda_n} b_n^H(t) e_n$, where $H \in (\frac{1}{2}, 1)$ and $\{b_n^H\}_{n\in}$ is a sequence of one-dimensional fBm mutually independent. Let us assume the function $\sigma : [0,+\infty) \rightarrow$

$\mathcal{L}_2^0(L^2([0,\pi]), L^2([0,\pi]))$ satisfies $\int_0^T \|\sigma(s)\|^2_{\mathcal{L}_2^0} ds < \infty, \quad \forall T > 0$. Then the condition $(\mathcal{H}.6)$ is satisfied.

Define the functions $p : [0,T] \times \mathcal{B} \times X \to X$, $h : [0,T] \times \mathcal{B} \times X \to X$ and $g : [0,T] \times \mathcal{B} \times \mathcal{U} \to X$ as follow

$$\begin{aligned}
p(t,\phi,\textstyle\int_0^t \theta_1(t,s,\phi)ds)(\xi) &= R_1(t,\phi(\theta,\xi),\textstyle\int_0^t r_1(t,s,\phi(\tau,\xi))ds),\\
h(t,\phi,\textstyle\int_0^t \theta_2(t,s,\phi)ds)(\xi) &= R_2(t,\phi(\tau,\xi),\textstyle\int_0^t r_2(t,s,\phi(\tau,\xi))ds),\\
g(t,\phi,\eta)(\xi) &= \Gamma(t,\phi(s,\xi),\eta),
\end{aligned}$$

Thus the above system (4.1) can be written in the abstract form (1.1). Further, if we assume that there exist positive constants $k_1, k_2, k_3, k_4, k_5, M_k$ and $\widetilde{M}_k$ such that

$$\begin{aligned}
\mathbf{E}\|\textstyle\int_0^t [r_1(t,s,z_1) - r_1(t,s,z_2)]ds\|^2 &\leq k_1 \mathbf{E}\|z_1 - z_2\|^2_{\mathcal{B}},\\
\mathbf{E}\|(-A)^\beta R_1(t,z_1,y_1) - (-A)^\beta R_1(t,z_2,y_2)\|^2 &\leq k_2[\mathbf{E}\|z_1 - z_2\|^2_{\mathcal{B}} + \mathbf{E}\|y_1 - y_2\|^2],\\
\mathbf{E}\|R_2(t,z_1,y_1) - R_2(t,z_2,y_2)\|^2 &\leq k_3[\mathbf{E}\|z_1 - z_2\|^2_{\mathcal{B}} + \mathbf{E}\|y_1 - y_2\|^2]\\
\mathbf{E}\|\textstyle\int_0^t [r_2(t,s,z_1) - r_2(t,s,z_2)]ds\|^2 &\leq k_4 \mathbf{E}\|z_1 - z_2\|^2_{\mathcal{B}}\\
\textstyle\int_{\mathcal{U}} \mathbf{E}\|\Gamma(t,z_1,\eta) - \Gamma(t,z_2,\eta)\|^2 \nu(d\eta) &\leq k_5\{\mathbf{E}\|\mathbf{z_1} - \mathbf{z_2}\|^{\mathbf{2}}_{\mathcal{B}}\\
\|I_j(z_1) - I_j(z_2)\|^2 &\leq M_k\{\mathbf{E}\|\mathbf{z_1} - \mathbf{z_2}\|^{\mathbf{2}}_{\mathcal{B}}\\
\|I_j(z_1)\|^2 &\leq \widetilde{M}_k,
\end{aligned}$$

for $t \in [0,T]$, $z_1, z_2 \in \mathcal{B}$ and $y_1, y_2 \in X$, then all assumptions on Theorem 3.3 are satisfied and hence, the system (4.1) is controllable on $(-\infty, T]$.

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