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Dynamics of a second order three species nonlinear difference system with exponents

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Abstract

In this paper, we study the persistence, boundedness, convergence, invariance and global asymptotic behavior of the positive solutions of the second order difference system

$$x_{n+1} = \alpha_1 + ae^{-x_{n-1}} + by_n e^{-y_{n-1}},$$

$$y_{n+1} = \alpha_2 + ce^{-y_{n-1}} + dz_n e^{-z_{n-1}}, \tag{0.1}$$

$$z_{n+1} = \alpha_3 + he^{-z_{n-1}} + jx_n e^{-x_{n-1}}, \quad n = 0, 1, 2, \ldots.$$

Here x_n, y_n, z_n *can be considered as population densities of three species such that the population density of* x_n, y_n, z_n *depends on the growth of* y_n, z_n, x_n *respectively with growth rate* b, d, j *respectively. The positive real numbers* $\alpha_1, \alpha_2, \alpha_3$ *are immigration rate of* x_n, y_n, z_n *respectively, while* a, c, h *denotes the growth rate of* x_n, y_n, z_n *respectively, and the initial values* $x_{-1}, y_{-1}, z_{-1}, x_0, y_0, z_0$ *are nonnegative numbers.*

Keywords: *Local behavior, global behavior, invariance, persistence, boundedness.*

AMS Subject Classification 2000: *39A22.*

1. Introduction

The theory of discrete dynamical systems has many applications in applied sciences. Mathematical modeling of a physical, biological, or ecological problem mostly leads to a nonlinear difference system (See [4],[8],[17]-[19]). The biological models like prey-predator competition, SIR, SIRS, SEIR, SEIRS, SI, SIS, etc are mostly of the first order. Two species models are analyzed in [28] and [29]. Two species second order competition models with exponents can be found in [7]-[12].
Many authors worked on three species competition models. Dynamics of a food chain model

$$\begin{bmatrix} {}_{n+1} \\ y_{n+1} \\ z_{n+1} \end{bmatrix} = \begin{bmatrix} x_n(1 - x_n - y_n - z_n) \\ \beta y_n(x_n - z_n) \\ \gamma z_n y_n \end{bmatrix} \tag{1.1}$$

where α, β and γ are positive constants, with strong pressure on preys was discussed in [21].

Stability analysis of discrete version of the three species May-Leonard model

$$\begin{aligned} x_{n+1} &= x_n e^{r(1-x_n-ay_n-bz_n)}, \\ y_{n+1} &= y_n e^{r(1-y_n-az_n-bx_n)}, \\ z_{n+1} &= z_n e^{r(1-z_n-ax_n-by_n)}, \end{aligned} \tag{1.2}$$

where $r > 0$ is the growth rate and the constants $0 < a < 1 < b$ was studied in [24].

Complex discrete-time varying dynamical character of a two-prey one-predator system

$$\begin{aligned} x_{n+1} &= x_n e^{r_1(1-\alpha_1 x_n)-\frac{\beta_1 x_n p_n}{1+x_n^2}-\gamma_1 y_n} \\ y_{n+1} &= y_n e^{r_2(1-\alpha_2 y_n)-\frac{\beta_2 y_n p_n}{1+y_n^2}-\gamma_2 x_n} \\ p_{n+1} &= p_n e^{\frac{c_1 x_n^2}{1+x_n^2}+\frac{c_2 y_n^2}{1+y_n^2}-\delta p_n-\delta_1} \end{aligned} \tag{1.3}$$

with coupled with inter-specific competition among the prey due to overlap of diet, Holling Type-III functional response and intra-specific competition

among the predators was discussed in [20].
Persistence and global behavior of the three species model

$$
\begin{aligned}
J_{n+1} &= bB_n + (1-\gamma_1)s_{1_{J_n}} J_n \\
N_{n+1} &= \gamma_1 s_{1_{J_n}} J_n + (1-\gamma_2)s_{2_{N_n+B_n}} N_n \\
B_{n+1} &= \gamma_2 s_{2_{N_n+B_n}} N_n + S_{2_{N_n+B_n}} B_n
\end{aligned}
\tag{1.4}
$$

was studied in [23]. Here J_n, N_n and B_n are the juveniles, non-breeders and breeders respectively. The variable $b > 0$ is the birth rate, $0 < \gamma_1, \gamma_2 < 1$ are the fractions of juveniles that become non-breeders and non-breeders that become breeders respectively, while the sequences s_1, s_2 are the survivor sequences of juveniles and non-breeders respectively.

Non-permanence for three-species Lotka-Volterra cooperative difference systems

$$
\begin{aligned}
x_{n+1} &= x_n e^{r_1 - a_1 x_n + a_2 y_n + a_3 z_n}, \\
y_{n+1} &= y_n e^{r_2 + b_1 x_n - b_2 y_n + b_3 z_n}, \\
z_{n+1} &= z_n e^{r_3 + c_1 x_n + c_2 y_n - c_3 z_n},
\end{aligned}
\tag{1.5}
$$

was discussed in [25].

In [22], authors considered the discrete system

$$
\begin{aligned}
x_{n+1} &= a_1 z_n, \\
y_{n+1} &= s_1 x_n + a_2 z_n, \\
z_{n+1} &= s_2(y_n + z_n)y_n + s_3(y_n + z_n)z_n,
\end{aligned}
\tag{1.6}
$$

as a three stage model. Here x_n, y_n, z_n are the number of seeds, juveniles, adults respectively, s_1, s_2, s_3 are the survival rate of the above said, and a_1,a_2 are rate of seeds produced by adults, rate of juveniles produced respectively.

As seen in (1.1), (1.2), (1.3), (1.4), (1.5) and (1.6) much work has happened related to three species first order systems. In [26] and [27] we have analyzed a two species second order system wherein we assume interaction between species of exponential form, a choice which has prominent importance in population biology. Not much work has happened related to the three species second order systems.

In this paper, we propose a second order three species competitive model (0.1) which is an extension of [26]. Here x_n, y_n, z_n can be considered as population densities of three species such that population density of x_n, y_n, z_n depends on the growth of y_n, z_n, x_n respectively with growth rate b, d, j respectively. The positive real numbers $\alpha_1, \alpha_2, \alpha_3$ are immigration rate of x_n, y_n, z_n respectively, a, c, h denotes the growth rate of x_n, y_n, z_n respectively, while the initial conditions $x_{-1}, y_{-1}, z_{-1}, x_0, y_0, z_0$ are arbitrary nonnegative numbers. We investigate the persistence, boundedness, convergence, invariance, and global asymptotic character of the positive solutions of (0.1).

2. Main Results

The following theorem proposes a condition for the persistence and boundedness of the positive solution (x_n, y_n, z_n) of (0.1).

Theorem 2.1. *Every positive solution (x_n, y_n, z_n) of (0.1) is bounded and persists whenever*

$$B = bdje^{-\alpha_1-\alpha_2-\alpha_3} < 1. \tag{2.1}$$

Proof. $x_n \geq \alpha_1, y_n \geq \alpha_2, z_n \geq \alpha_3$, $n = 1, 2, \ldots.$

Hence (x_n, y_n, z_n) of system (0.1) persists.
For $n = 4, 5, \ldots$, (0.1) becomes

$$\begin{aligned} x_{n+1} &\leq \alpha_1 + ae^{-\alpha_1} + be^{-\alpha_2}[\alpha_2 + dz_{n-1}e^{-z_{n-2}} + ce^{-y_{n-2}}] \\ &\leq A + Bx_{n-2}, \end{aligned} \tag{2.2}$$

where $A = \alpha_1 + ae^{-\alpha_1} + b\alpha_2 e^{-\alpha_2} + bce^{-\alpha_2-\alpha_2} + bd\alpha_3 e^{-\alpha_2-\alpha_3} + bdhe^{-\alpha_2-\alpha_3-\alpha_3}$ and $B = bdje^{-\alpha_1-\alpha_2-\alpha_3}$.

Similarly,

$$y_{n+1} \leq C + By_{n-2}, \tag{2.3}$$

where $C = \alpha_2 + ce^{-\alpha_2} + d\alpha_3 e^{-\alpha_3} + dhe^{-\alpha_3-\alpha_3} + dj\alpha_1 e^{-\alpha_1-\alpha_3} + djae^{-\alpha_1-\alpha_1-\alpha_3}$.

Also,

$$z_{n+1} \leq H + Bz_{n-2}, \tag{2.4}$$

where $H = \alpha_3 + he^{-\alpha_3} + j\alpha_1 e^{-\alpha_1} + aje^{-\alpha_1-\alpha_1} + jb\alpha_2 e^{-\alpha_1-\alpha_2} + bjce^{-\alpha_1-\alpha_2-\alpha_2}$.

Now, consider the difference equations

$$u_{n+1} = A + Bu_{n-2},$$

$$v_{n+1} = C + Bv_{n-2},$$

$$w_{n+1} = H + Bw_{n-2}, \quad n = 4, 5, \dots \tag{2.5}$$

Therefore an arbitrary solution (u_n, v_n, w_n) of (2.5) can be written as

$$u_n = r_1 B^{n/3} + r_2 B^{n/3} \cos\left(\frac{n\pi}{2}\right) + r_3 B^{n/3} \sin\left(\frac{n\pi}{2}\right) + \frac{A}{1-B}, \quad n = 5, 6, \dots, \tag{2.6}$$

$$v_n = s_1 B^{n/3} + s_2 B^{n/3} \cos\left(\frac{n\pi}{2}\right) + s_3 B^{n/3} \sin\left(\frac{n\pi}{2}\right) + \frac{C}{1-B}, \quad n = 5, 6 \dots, \tag{2.7}$$

$$w_n = p_1 B^{n/3} + p_2 B^{n/3} \cos\left(\frac{n\pi}{2}\right) + p_3 B^{n/3} \sin\left(\frac{n\pi}{2}\right) + \frac{H}{1-B}, \quad n = 5, 6 \dots, \tag{2.8}$$

where r_1, r_2, r_3, s_1, s_2, s_3, p_1, p_2, p_3 rely on u_4, v_4, w_4 respectively.

Hence, (u_n, v_n, w_n) is bounded.

Let us examine the solution (u_n, v_n, w_n) such that $u_{-1} = x_{-1}, v_{-1} = y_{-1}, w_{-1} = z_{-1}, u_0 = x_0, v_0 = y_0, w_0 = z_0$.

Hence by induction, $x_n \leq u_n, y_n \leq v_n$ and $z_n \leq w_n, \quad n = 5, 6, \dots$.

Therefore, we get (x_n, y_n, z_n) is bounded. □

We give an example to verify Theorem 2.1.

Example 2.2. *Let* $a = 0.5, b = 0.5, c = 0.25, d = 0.005, h = 0.01, j = 0.03, \alpha_1 = 0.08, \alpha_2 = 0.09, \alpha_3 = 0.04,$ $x_{-1} = 3.0, x_0 = 1.0, y_{-1} = 9.0, y_0 = 2.0, z_{-1} = 1.0,$ *and* $z_0 = 2.0$.

Here $B = 6.079381844776403 \times 10^{-5} < 1$, *i.e., (2.1) is satisfied. In this case after 63 iterations* x_n *stabilizes to* 0.4914889223167139.

Similarly y_n *converges to 0.2793453372189754 and* z_n *converges to* 0.058451770505985656.

Hence we can see that the solution is bounded and persists.

The following theorems confirm the existence of invariant boxes of (0.1).

Theorem 2.3. *Let (2.1) hold. Let (x_n, y_n, z_n) denote a positive solution of (0.1). Then* $[\alpha_1, \frac{A}{1-B}] \times [\alpha_2, \frac{C}{1-B}] \times [\alpha_3, \frac{H}{1-B}]$ *is an invariant set for (0.1) where A, C, H are defined as in the proof of Theorem 2.1.*

Proof. Let $I_1 = [\alpha_1, \frac{A}{1-B}], I_2 = [\alpha_2, \frac{C}{1-B}]$, and $I_3 = [\alpha_3, \frac{H}{1-B}]$.
Let $x_{-1}, x_0 \in I_1$, $y_{-1}, y_0 \in I_2$ and $z_{-1}, z_0 \in I_3$.
Then $x_1 \leq \alpha_1 + ae^{-\alpha_1} + be^{-\alpha_2} y_0$

$$\leq \alpha_1 + ae^{-\alpha_1} +$$

$$be^{-\alpha_2}[\frac{\alpha_2 + ce^{-\alpha_2} + d\alpha_3 e^{-\alpha_3} + dhe^{-\alpha_3-\alpha_3} + dj\alpha_1 e^{-\alpha_1-\alpha_3} + djae^{-\alpha_1-\alpha_1-\alpha_3}}{1 - bdje^{-\alpha_1-\alpha_2-\alpha_3}}].$$

Hence we get

$$x_1 \leq \frac{\alpha_1 + ae^{-\alpha_1} + b\alpha_2 e^{-\alpha_2} + bce^{-\alpha_2-\alpha_2} + bd\alpha_3 e^{-\alpha_2-\alpha_3} + bdhe^{-\alpha_2-\alpha_3-\alpha_3}}{1 - bdje^{-\alpha_1-\alpha_2-\alpha_3}}].$$

i.e., $x_1 \in I_1$. Similarly we get $y_1 \in I_2$ and $z_1 \in I_3$.
Therefore the proof follows by applying the method of induction. □

Theorem 2.4. *Let (2.1) hold. Let A, C, H are defined as in the proof of Theorem 2.1. Consider the intervals $I_4 = [\alpha_1, \frac{A+\epsilon}{1-B}]$, $I_5 = [\alpha_2, \frac{C+\epsilon}{1-B}]$ and $I_6 = [\alpha_3, \frac{H+\epsilon}{1-B}]$ where ϵ is an arbitrary positive number. If (x_n, y_n, z_n) is any arbitrary solution of (0.1), then there exists an $N \in \mathbf{N}$ such that $x_n \in I_4, y_n \in I_5$ and $z_n \in I_6, n \geq N$.*

Proof. Let (x_n, y_n, z_n) denote an arbitrary solution of (0.1).
Then by Theorem 2.1, $\limsup_{n\to\infty} x_n = M < \infty, \limsup_{n\to\infty} y_n = L < \infty$ and $\limsup_{n\to\infty} z_n = T < \infty$.
Hence, from Theorem 2.1, we get $x_{n+1} \leq A + bdjx_{n-2}e^{-\alpha_1-\alpha_2-\alpha_3}$, $y_{n+1} \leq C + bdjy_{n-2}e^{-\alpha_1-\alpha_2-\alpha_3}$ and $z_{n+1} \leq H + bdjz_{n-2}e^{-\alpha_1-\alpha_2-\alpha_3}$.
Hence, $M \leq \frac{A}{1 - bdje^{-\alpha_1-\alpha_2-\alpha_3}}$, $L \leq \frac{C}{1 - bdje^{-\alpha_1-\alpha_2-\alpha_3}}$ and
$T \leq \frac{H}{1 - bdje^{-\alpha_1-\alpha_2-\alpha_3}}$.

Hence, there exists an $N \in \mathbf{N}$ such that the theorem is satisfied. $\square$

Now we state a lemma which is an alteration of Theorem 1.16 in [16] and an extension of Lemma 5 in [26]. The proof is similar to the proof in [26] and hence omitted.

Lemma 2.5. *Let $[a,b],[c,d]$ and $[p,q]$ denote intervals of real numbers. Let $f:[a,b]\times[c,d]\times[c,d]\to[a,b]$, $g:[c,d]\times[p,q]\times[p,q]\to[c,d]$ and $l:[a,b]\times[a,b]\times[p,q]\to[p,q]$ be continuous functions. Consider the difference system*

$$
\begin{aligned}
x_{n+1} &= f(x_{n-1},y_n,y_{n-1}),\\
y_{n+1} &= g(y_{n-1},z_n,z_{n-1}),\\
z_{n+1} &= l(x_n,x_{n-1},z_{n-1}), \quad n=0,1,2,\dots
\end{aligned}
\tag{2.9}
$$

such that the initial values $x_{-1},x_0\in[a,b], y_{-1},y_0\in[c,d]$ and $z_{-1},z_0\in[p,q]$. (or $x_{n_0},x_{n_0+1}\in[a,b]$, $y_{n_0},y_{n_0+1}\in[c,d], z_{n_0},z_{n_0+1}\in[p,q], n_0\in\mathbf{N}$). Suppose the following are true.

(i) If $f(x,y,z)$ is nonincreasing in x, $f(x,y,z)$ is nondecreasing in y and $f(x,y,z)$ is nonincreasing in z.

(ii) If $g(x,y,z)$ is nonincreasing in x, $g(x,y,z)$ is nondecreasing in y and $g(x,y,z)$ is nonincreasing in z.

(iii) If $l(x,y,z)$ is nondecreasing in x, $l(x,y,z)$ is nonincreasing in y and $l(x,y,z)$ is nonincreasing in z.

(iv) If $(m_1,M_1,m_2,M_2,m_3,M_3)\in[a,b]^2\times[c,d]^2\times[p,q]^2$ satisfies the systems $m_1=f(M_1,m_2,M_2), M_1=f(m_1,M_2,m_2); m_2=g(M_2,m_3,M_3)$, $M_2=g(m_2,M_3,m_3)$ and $m_3=l(m_1,M_1,M_3), M_3=l(M_1,m_1,m_3)$ then $m_1=M_1, m_2=M_2$ and $m_3=M_3$,

then there exists a unique equilibrium solution $(\bar{x},\bar{y},\bar{z})$ of (2.9) with $\bar{x}\in[a,b]$,$\bar{y}\in[c,d]$and $\bar{z}\in[p,q]$. Also every solution of (2.9) converges to $(\bar{x},\bar{y},\bar{z})$.

The following theorem proposes conditions for the convergence of the equilibrium solution of (0.1).

Theorem 2.6. *Let (2.1) hold. Suppose*

$$ce^{-\alpha_2} < 1, ae^{-\alpha_1} < 1, he^{-\alpha_3} < 1 \tag{2.10}$$

and

$$\frac{B[1+\frac{A}{1-B}][1+\frac{C}{1-B}][1+\frac{H}{1-B}]}{[1-ae^{-\alpha_1}][1-ce^{-\alpha_2}][1-he^{-\alpha_3}]} < 1, \tag{2.11}$$

where A, C, H are defined as in the proof of Theorem 2.1. Then (0.1) has a unique positive equilibrium $E(\bar{x}, \bar{y}, \bar{z})$. Also every solution of (0.1) converges to $E(\bar{x}, \bar{y}, \bar{z})$.

Proof. Let $f : \mathbf{R}^+ \times \mathbf{R}^+ \times \mathbf{R}^+ \to \mathbf{R}^+, g : \mathbf{R}^+ \times \mathbf{R}^+ \times \mathbf{R}^+ \to \mathbf{R}^+, l : \mathbf{R}^+ \times \mathbf{R}^+ \times \mathbf{R}^+ \to \mathbf{R}^+$ be continuous functions such that $f(x, y, z) = \alpha_1 + ae^{-x} + bye^{-z}$, $g(x, y, z) = \alpha_2 + ce^{-x} + dye^{-z}$,$l(x, y, z) = \alpha_3 + he^{-z} + jxe^{-y}$. Let $M_1, m_1, M_2, m_2, M_3, m_3$ be positive real numbers such that $m_i \le M_i, i = 1, 2, 3$ satisfying

$$m_1 = \alpha_1 + ae^{-M_1} + bm_2e^{-M_2}, M_1 = \alpha_1 + ae^{-m_1} + bM_2e^{-m_2},$$

$$m_2 = \alpha_2 + ce^{-M_2} + dm_3e^{-M_3}, M_2 = \alpha_2 + ce^{-m_2} + dM_3e^{-m_3}$$

and

$$m_3 = \alpha_3 + he^{-M_3} + jm_1e^{-M_3}, M_3 = \alpha_3 + he^{-m_3} + jM_1e^{-m_1}. \tag{2.12}$$

Therefore, $M_1 - m_1 = a[e^{-m_1} - e^{-M_1}] + b[M_2e^{-m_2} - m_2e^{-M_2}]$.

$$M_1 - m_1 = a[e^{-m_1} - e^{-M_1}] + be^{-m_2-M_2}[M_2e^{M_2} - m_2e^{m_2}]. \tag{2.13}$$

Also, there exists a ζ, $m_2 \le \zeta \le M_2$ satisfying

$$M_2e^{M_2} - m_2e^{m_2} = (1+\zeta)e^{\zeta}(M_2 - m_2). \tag{2.14}$$

From (2.13) and (2.14) we get,

$$M_1 - m_1 = a[e^{-m_1} - e^{-M_1}] + be^{-m_2-M_2+\zeta}(1+\zeta)[M_2 - m_2]. \tag{2.15}$$

Now, $a[e^{-m_1} - e^{-M_1}] = ae^{-m_1-M_1}[e^{M_1} - e^{m_1}]$.
Also, there exists a λ, $m_1 \le \lambda \le M_1$ satisfying

$$a[e^{-m_1} - e^{-M_1}] = ae^{-m_1-M_1+\lambda}[M_1 - m_1]. \tag{2.16}$$

Since $M_1, m_1 \geq \alpha_1$,

$$a[e^{-m_1} - e^{-M_1}] \leq ae^{-\alpha_1}[M_1 - m_1]. \tag{2.17}$$

Thus from (2.15) and (2.17) we get,

$$M_1 - m_1 \leq ae^{-\alpha_1}[M_1 - m_1] + be^{-m_2 - M_2 + \zeta}(1+\zeta)[M_2 - m_2]. \tag{2.18}$$

Since $M_2, m_2 \geq \alpha_2$, (2.18) becomes

$$M_1 - m_1 \leq ae^{-\alpha_1}[M_1 - m_1] + be^{-\alpha_2}(1+\zeta)[M_2 - m_2]. \tag{2.19}$$

i.e.,

$$[1 - ae^{-\alpha_1}][M_1 - m_1] \leq be^{-\alpha_2}(1+\zeta)[M_2 - m_2]. \tag{2.20}$$

Also, (2.12) can be written as

$$M_2 = \alpha_2 + ce^{-m_2} + d[\alpha_3 + he^{-m_3} + jM_1e^{-m_1}]e^{-m_3}. \tag{2.21}$$

i.e.,

$$M_2 \leq \frac{C}{1 - bdje^{-\alpha_1-\alpha_2-\alpha_3}}. \tag{2.22}$$

Since $\zeta \leq M_2$ we get,

$$\zeta \leq \frac{C}{1 - bdje^{-\alpha_1-\alpha_2-\alpha_3}}. \tag{2.23}$$

Therefore, (2.20) becomes

$$[1 - ae^{-\alpha_1}][M_1 - m_1] \leq be^{-\alpha_2}[1 + \frac{C}{1 - bdje^{-\alpha_1-\alpha_2-\alpha_3}}][M_2 - m_2]. \tag{2.24}$$

Similarly we get,

$$[1 - ce^{-\alpha_2}][M_2 - m_2] \leq de^{-\alpha_3}[1 + \frac{H}{1 - bdje^{-\alpha_1-\alpha_2-\alpha_3}}][M_3 - m_3] \tag{2.25}$$

and

$$[1 - he^{-\alpha_3}][M_3 - m_3] \leq je^{-\alpha_1}[1 + \frac{A}{1 - bdje^{-\alpha_1-\alpha_2-\alpha_3}}][M_1 - m_1]. \tag{2.26}$$

From (2.24), (2.25) and (2.26) we get,

$$[M_1 - m_1] \leq \frac{bdje^{-\alpha_1-\alpha_2-\alpha_3}}{[1 - ae^{-\alpha_1}][1 - ce^{-\alpha_2}][1 - he^{-\alpha_3}]} \times [1 + \frac{A}{1 - bdje^{-\alpha_1-\alpha_2-\alpha_3}}]$$

$$\times [1+\frac{C}{1-bdje^{-\alpha_1-\alpha_2-\alpha_3}}]\times[1+\frac{H}{1-bdje^{-\alpha_1-\alpha_2-\alpha_3}}][M_1-m_1]. \tag{2.27}$$

Therefore from (2.11) and (2.27), we get $M_1 = m_1$.
Similarly we get $M_2 = m_2$ and $M_3 = m_3$.
Therefore by applying Lemma 2.5, the result is obtained. □

Example 2.7. *Let a, b, c, d, h, j and $\alpha_1, \alpha_2, \alpha_3$ are as in Example 2.2. Then (2.10) implies* $ce^{-\alpha_2} = 0.22848279631780705 < 1$, $ae^{-\alpha_1} = 0.4615581731933179 < 1$, $he^{-\alpha_3} = 0.009607894391523233 < 1$. *Also (2.11) implies* $0.0003513665471576195 < 1$. *Therefore (2.10) and (2.11) hold.*

For the initial conditions $x_{-1} = 11.0$, $x_0 = 5.0$, $y_{-1} = 12.0$, $y_0 = 0.2$, $z_{-1} = 15.0$ *and* $z_0 = 3.8$, *we can see that after few iterations* x_n, y_n *and* z_n *stabilizes to* 0.4914889223167139, 0.2793453372189754 *and* (0.058451770505985656) *respectively. Here* $E(\bar{x}, \bar{y}, \bar{z}) = (0.4914889223167139, 0.2793453372189754, 0.058451770505985656)$. *Also we can see that the solution of (0.1) converges to* $E(\bar{x}, \bar{y}, \bar{z})$.

In the next theorem, we derive conditions for the global asymptotic stability of the equilibrium solution of (0.1).

Theorem 2.8. *Assume (2.1), (2.10) and (2.11) hold. If*

$$ae^{-\alpha_1}[1+ce^{-\alpha_2}]+ce^{-\alpha_2}[1+he^{-\alpha_3}]+he^{-\alpha_3}[1+ae^{-\alpha_1}]+ache^{-\alpha_1}e^{-\alpha_2}e^{-\alpha_3}$$
$$+\frac{B}{(1-B)^3}[(1-B)^3+A(1-B)^2+C(1-B)^2+H(1-B)^2$$
$$+AC(1-B)+CH(1-B)+AH(1-B)+ACH]<1, \tag{2.28}$$

where A, C, H are defined as in the proof of Theorem 2.1, then the unique equilibrium $E(\bar{x}, \bar{y})$ is globally asymptotically stable.

Proof. We have to only show that $E(\bar{x}, \bar{y})$ is locally asymptotically stable. The characteristic equation of the Jacobian $JF(\bar{x}, \bar{y})$ about the equilibrium point $E(\bar{x}, \bar{y})$ is given by

$$\lambda^6+\lambda^4(ae^{-\bar{x}}+ce^{-\bar{y}}+he^{-\bar{x}})+\lambda^3(bdje^{-\bar{x}-\bar{y}-\bar{z}})+\lambda^2(bdj\bar{x}e^{-\bar{x}-\bar{y}-\bar{z}}$$
$$+bdj\bar{y}e^{-\bar{x}-\bar{y}-\bar{z}}+bdj\bar{z}e^{-\bar{x}-\bar{y}-\bar{z}}+ace^{-\bar{x}}e^{-\bar{y}}+ahe^{-\bar{x}}e^{-\bar{z}}+che^{-\bar{y}}e^{-\bar{z}})+\lambda e^{-\bar{x}-\bar{y}-\bar{z}}$$
$$(-bdj\bar{x}\bar{y}-bdj\bar{y}\bar{z}-bdj\bar{x}\bar{z})+e^{-\bar{x}-\bar{y}-\bar{z}}(ach+bdj\bar{x}\bar{y}\bar{z})=0.$$

By Remark 1.3.1 of [15], we write
$|ae^{-\bar{x}}| + |ce^{-\bar{y}}| + |he^{-\bar{x}}| + |bdje^{-\bar{x}-\bar{y}-\bar{z}}| + |bdj\bar{x}e^{-\bar{x}-\bar{y}-\bar{z}}| + |bdj\bar{y}e^{-\bar{x}-\bar{y}-\bar{z}}| + |bdj\bar{z}e^{-\bar{x}-\bar{y}-\bar{z}}| + |ace^{-\bar{x}}e^{-\bar{y}}| + |ahe^{-\bar{x}}e^{-\bar{z}}| + |che^{-\bar{y}}e^{-\bar{z}}| + |bdj\bar{x}\bar{y}e^{-\bar{x}-\bar{y}-\bar{z}}| + |bdj\bar{y}\bar{z}e^{-\bar{x}-\bar{y}-\bar{z}}| + |bdj\bar{x}\bar{z}e^{-\bar{x}-\bar{y}-\bar{z}}| + |ache^{-\bar{x}-\bar{y}-\bar{z}}| + |bdj\bar{x}\bar{y}\bar{z}e^{-\bar{x}-\bar{y}-\bar{z}}| < 1$
is satisfied whenever

$$ae^{-\alpha_1}[1 + ce^{-\alpha_2}] + ce^{-\alpha_2}[1 + he^{-\alpha_3}] + he^{-\alpha_3}[1 + ae^{-\alpha_1}]+$$

$$ache^{-\alpha_1}e^{-\alpha_2}e^{-\alpha_3} + B[1 + \bar{x} + \bar{y} + \bar{z} + \bar{x}\bar{y} + \bar{y}\bar{z} + \bar{x}\bar{z} + \bar{x}\bar{y}\bar{z}] < 1. \tag{2.29}$$

Since $E(\bar{x}, \bar{y})$ is the equilibrium point of (0.1), by Theorem 2.1, we get

$$\bar{x} \leq \frac{A}{1 - B}. \tag{2.30}$$

Similarly

$$\bar{y} \leq \frac{C}{1 - B} \tag{2.31}$$

and

$$\bar{z} \leq \frac{H}{1 - B}. \tag{2.32}$$

Substituting (2.30), (2.31) (2.32) in (2.29) and by Remark 1.3.1 of [15] we get the result. Therefore by using Theorem 2.6 we obtain the conditions for global asymptotic stability. □

Example 2.9. *Let a, b, c, d, h, j and $\alpha_1, \alpha_2, \alpha_3$ are as in Example 2.2. Now (2.28) implies* $0.8128945983253889 < 1$. *From Example 2.2 and Example 2.7 we can see that all the conditions of Theorem 2.8 are satisfied. Also we can see that $E(\bar{x}, \bar{y}, \bar{z})$*
$= (0.4914889223167139, 0.2793453372189754, 0.058451770505985656)$ *is a global attractor.*

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