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Characterization of nonuniform wavelets associated with AB-MRA on $L^2(\Lambda)$

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Abstract

Ahmad, Bhat and Sheikh characterized composite wavelets based on results of affine and quasi affine frames. We continued their study and provided the characterization of nonuniform composite wavelets based on results of affine and quasi affine frames. Moreover all the nonuniform composite wavelets associated with AB-MRA are characterized on $L^2(\Lambda)$.

Keywords: *Wavelets, Nouniform, Fourier transform, Multiresolution analysis, Dimension function.*

Mathematics Subject Classification (2000) *Primary 42C40; Secondary 65T60.*

1. Introduction

Wavelets are defined extensively and studied vigorously on the Euclidean spaces $\mathbf{R}$. Their characterization on the Hilbert space $L^2(\mathbf{R})$ was studied independently by Wang [17] and Gripenberg [10] in the shape of two basic equationsusing the techniques of Fourier transform of the wavelet (see also [7] and [14]). The obtained result was then generalized by Frazier, Garrigos, Wang, and Weiss [16] for dilation by 2 and by Calogero [5] for wavelets associated with a general dilation matrix. Bownik [3] used a new approach to characterize multiwavelets in $L^2(\mathbf{R})$. This characterization was attained with the help shift invariant systems and quasi-affine systems.

The concept of multiresolution analysis (MRA) is heart value of wavelets. It is a fact that wavelets are generated from MRA. But this not the case with all recipes. It were Gripenberg [10] and Wang [17] who proved that a wavelet arises from an MRA if and only if its dimension function is 1 a.e. Calogero and Garrigos [6] gave a characterization of wavelet families arising from biorthogonal MRAs of multiplicity d. This result was later on modified by Bownik and Garrigos in [4], where they provided this characterization in terms of the dimension function. More results in this direction are obtained in [2, 15] and the references therein. But in all these cases, the translation set is always a group. Recently, Gabardo and Nashed in [8, 9] defined a multiresolution analysis associated with a translation set $\{0, r/N\} + 2\mathbf{Z}$, where $N \geq 1$ is an integer, $1 \leq r \leq 2N-1, r$ is an odd integer and r, N are relatively prime, a discrete set which is not necessarily a group. They call this an *NUMRA*. As, the case $N = 1$ reduces to the standard definition of *MRA* with dyadic dilation.

Guo, Labate, Lim, Weiss, and Wilson [11, 12, 13] introduced the theory of composite dilation wavelets and detailed the extension of a multiresolution analysis (MRA) to this setting. Let $f_\ell \in L^2(\Lambda)$. Then the *nonuniform affine systems with composite dilations* are defined by

$$F_{AB} = \{D_a D_b T_\lambda f_\ell : \lambda \in \Lambda, b \in B, a \in A, \ell = 1, 2, \ldots, 2N-1\},$$

where the Translation operator T_λ is defined by $T_\lambda f(x) = f(x-\lambda)$, Dilation operator by $D_a f(x) = |\det a|^{-1/2} f(a^{-1}x)$. $A \subset GL_n(R)$ consist of elements having some expanding properties and $B \subset GL_n(R)$ consist elements having determinant of absolute value one. By choosing f_ℓ, A, B, appropriately, F_{AB} can be made orthonormal basis or more generally a Parseval frame for $L^2(\Lambda)$. Here we call $F = \{f_1, f_2, \ldots, f_{2N-1}\}$ an orthonormal AB-multiwavelet or a Parseval Frame AB-multiwavelet. For $L = 1$, i.e., when we have single generator, we have wavelet instead of multiwavelet.

Recently Ahmad et.al obtained the characterization of wavelets associated with the composite dialtion MRA on $L^2(\mathbf{R})$. We used their technique to obtain the characterization of nonuniform wavelets associated with AB-MRA on $L^2(\Lambda)$. This paper is organised in the following manner. In Section 2, we recall some basic results and use them to characterize composite wavelets. Here we also give another characterization of these wavelets. In Section 3, we characterize the wavelets associated with the AB-MRA on $L^2(\Lambda)$.

2. Characterization of Nonuniform Composite Wavelets

For an integer $N \geq 1$ and an odd integer r with $1 \leq r \leq 2N-1$ such that r and N are relatively prime, we define

$$\Lambda = \left\{0, \frac{r}{N}\right\} + 2\mathbf{Z} = \left\{\frac{rk}{N} + 2n : n \in \mathbf{Z}, k = 0, 1\right\}.$$

It is easy to verify that Λ is not necessarily a group nor a uniform discrete set, but is the union of $\mathbf{Z}$ and a translate of $\mathbf{Z}$. Moreover, the set Λ is the spectrum for the spectral set $\Gamma = \left[0, \frac{1}{2}\right) \cup \left[\frac{N}{2}, \frac{N+1}{2}\right)$ and the pair (Λ, Γ) is called a *spectral pair* [8,9].

Definition 2.1. Let $F = \left\{f^1, f^2, \ldots, f^{2N-1}\right\}$ be a finite family of functions in $L^2(\Lambda)$. The *affine system* generated by F is the collection

$$X(F) = \left\{f^{\ell}_{m,j,\lambda}(x) = q^{j/2} f^{\ell}\left(A^j B^m x - \lambda\right), j \in \mathbf{Z}, \lambda \in \Lambda, 1 \leq \ell \leq 2N-1, 1 \leq m \leq M, \right\}$$

where $M = \min\{r : r \geq 1, r \in \mathbf{Z}\}$, with the assumption $B^r = I$, A is an $n \times n$ expansive real matrix with eigenvalues λ satisfying $|\lambda| > 1$, B is a rotation matrix, $AB^m\lambda \in \Lambda(\forall\, \lambda \in \Lambda, 1 \leq m \leq M)$. It is clear that $X(F) = D_j T_\lambda f^\ell(x)$. The *quasi-affine system* generated by F is

$$\tilde{X}(F) = \left\{\tilde{f}^{\ell}_{m,j,\lambda} : j \in \mathbf{Z}, \lambda \in \Lambda, 1 \leq \ell \leq 2N-1, 1 \leq m \leq M\right\},$$

where

$$\tilde{f}^{\ell}_{j,\lambda}(x) = \begin{cases} D_j D_m T_\lambda f^\ell(x) = q^{j/2} f^\ell\left(A^j B^m x - \lambda\right), & j \geq 0, \\ \\ q^{j/2} T_\lambda D_j D_m f^\ell(x) = q^{j/2} f^\ell\left(A^j B^m (x - \lambda)\right), & j < 0. \end{cases} \tag{2.1}$$

We say that F is a set of basic wavelets of $L^2(\Lambda)$ if the affine system $X(F)$ forms an orthonormal basis for $L^2(\Lambda)$.

Definition 2.2. A subset X of $L^2(\Lambda)$ is called a Bessel family if there exists a constant $b > 0$ such that

$$\sum_{\eta\in X} |\langle f, \eta\rangle|^2 \leq b\left\|f\right\|^2 \quad \text{for all} \quad f \in L^2(\Lambda). \tag{2.2}$$

If, in addition, there exists a constant $a > 0, a \leq b$ such that

$$a\left\|f\right\|^2 \leq \sum_{\eta\in X} |\langle f, \eta\rangle|^2 \leq b\left\|f\right\|^2 \quad \text{for all} \quad f \in L^2(\Lambda), \tag{2.3}$$

then X is called a frame. The frame is *tight* if we can choose a and b such that $a = b$. The affine system $X(F)$ is an affine frame if (2.3) holds for $X = X(F)$. Similarly, the quasi-affine system $\tilde{X}(F)$ is a quasi-affine frame if (2.3) holds for $X = \tilde{X}(F)$.

Theorem 2.3 [16]. *Let $F = \left\{f^1, f^2, \ldots, f^{2N-1}\right\}$ be a finite subset of $L^2(\Lambda)$. Then*

(a) *$X(F)$ is a Bessel family if and only if $\tilde{X}(F)$ is a Bessel family. Furthermore, their exact upper bounds are equal.* (b) *$X(F)$ is an affine frame if and only if $\tilde{X}(F)$ is a quasi-affine frame. Furthermore, their lower and upper exact bounds are equal.*

Definition 2.4. Given $\{t_i : i \in \mathbf{N}\} \subset \mathbf{l^2(\Lambda)}$, where t_i are orthogonal, define the operator $H : l^2(\Lambda) \to l^2(\mathbf{N})$ by

$$H(v) = (\langle v, t_i\rangle)_{i\in\mathbf{N}} \quad \text{for } v = \Big(v(\lambda)\Big)_{\lambda\in\Lambda} \in l^2(\Lambda).$$

If H is bounded then $\tilde{G} = H^{\star}H : l^2(\Lambda) \to l^2(\Lambda)$ is called the dual Gramian of $\{t_i : i \in \mathbf{N}\}$. Observe that $\tilde{G}$ is a non negative definite operator on $l^2(\Lambda)$. Also, note that for $\lambda, \nu \in \Lambda$, we have

$$\left\langle \tilde{G}e_\lambda, e_\nu\right\rangle = \langle He_\lambda, He_\nu\rangle = \sum_{i\in\mathbf{N}} t_i(\lambda)\overline{t_i(p)},$$

where $\{e_i : i \in \mathbf{N}\}$ is the standard basis of $l^2(\Lambda)$.

Theorem 2.5 [16] *Let $\{g_i : i \in \mathbf{N}\} \subset \mathbf{l^2(\Lambda)}$ and for a.e. $\zeta \in \mathbf{T^n}$, let $\tilde{G}(\zeta)$ denote the dual Gramian of $\{t_i : i \in \mathbf{N}\} \subset \mathbf{l^2(\Lambda)}$. The system of translates*

$\{T_\lambda g_i : \lambda \in \Lambda, i \in \mathbf{N}\}$ *is a frame for* $L^2(\Lambda)$ *with constants* a *and* b *if and only if* $\tilde{G}(\zeta)$ *is bounded for a.e.* $\zeta \in \mathbf{T^n}$ *and*

$$a\left\|v\right\|^2 \le \left\langle \tilde{G}(\zeta)v, v\right\rangle \le b\left\|v\right\|^2 \quad \textit{for } v \in l^2(\Lambda), \textit{a.e.}, \zeta \in \mathbf{T^n}$$

that is, the spectrum of $\tilde{G}(\zeta)$ *is contained in* $[a,b]$ *for a.e.* $\zeta \in \mathbf{T^n}$.

Lemma 2.6. *Suppose that* $F = \left\{f^1, f^2, \ldots, f^{2N-1}\right\} \subset L^2(\Lambda)$. *The affine system* $X(F)$ *is orthonormal in* $L^2(\Lambda)$ *if and only if for* $j \ge 0$ *and* $1 \le \ell, \ell' \le 2N-1$,

$$\sum_{m=1}^{M}\sum_{\lambda\in\Lambda} \hat{f}^{\ell}(\zeta+\lambda)\overline{\hat{f}^{\ell'}\left(A^{*j}B^{*m}(\zeta+\lambda)\right)} = \delta_{\ell,\ell'}\delta_{j,0}\delta_{m,0}, \text{ for } a.e.\ \zeta \in \Lambda. \tag{2.4}$$

Proof. By a simple change of variables, one can observe that for $j, j' \in \mathbf{Z}, \lambda, \lambda' \in \Lambda, 1 \le \ell, \ell' \le 2N-1$ and $1 \le m, m' \le M$,

$$\left\langle f^{\ell}_{m,j,\lambda}, f^{\ell'}_{m',j',\lambda'}\right\rangle = \delta_{\ell,\ell'}\delta_{j,j'}\delta_{\lambda,\lambda'}\delta_{m,m'}$$

is equivalent to

$$\left\langle f^{\ell}_{m,j,\lambda}, f^{\ell'}_{0,0,0}\right\rangle = \delta_{\ell,\ell'}\delta_{j,0}\delta_{\lambda,0}\delta_{m,0}.$$

Taking any $j \ge 0, \lambda \in \Lambda, 1 \le \ell, \ell' \le 2N-1$ and $1 \le m \le M$, we have by Plancherel's formula

$$\begin{aligned}
\left\langle f^{\ell}_{m,j,\lambda}, f^{\ell'}_{0,0,0}\right\rangle &= \left\langle \hat{f}^{\ell}_{m,j,\lambda}, \hat{f}^{\ell'}_{0,0,0}\right\rangle \\
&= \int_{\Lambda} q^{-j/2}\hat{f}^{\ell}\left(A^{*-j}B^{*-m}\zeta\right)e^{-2\pi i A^{*-j}B^{*-m}k\zeta}\overline{\hat{f}^{\ell'}(\zeta)}d\zeta \\
&= q^{j/2}\int_{\Lambda}\hat{f}^{\ell}(\zeta)e^{-2\pi i\lambda\zeta}\overline{\hat{f}^{\ell'}\left(B^{*m}A^{*j}\zeta\right)}d\zeta \\
&= q^{j/2}\sum_{\sigma\in\Lambda}\int_{\mathbf{T^n}}\hat{f}^{\ell}(\zeta)\overline{\hat{f}^{\ell'}\left(B^{*m}A^{*j}\zeta\right)}e^{-2\pi i\lambda\zeta}d\zeta \\
&= q^{j/2}\int_{\mathbf{T^n}}\left\{\sum_{\sigma\in\Lambda}\hat{f}^{\ell}(\zeta+\sigma)\overline{\hat{f}^{\ell'}\left(B^{*m}A^{*j}(\zeta+\sigma)\right)}\right\}e^{-2\pi i\lambda\zeta}d\zeta.
\end{aligned}$$

If $\left\langle f^{\ell}_{m,j,\lambda}, f^{\ell'}_{0,0,0}\right\rangle = \delta_{\ell,\ell'}\delta_{j,0}\delta_{\lambda,0}\delta_{m,0}$ for all $j \geq 0, \lambda \in \Lambda, 1 \leq \ell, \ell' \leq 2N-1, 1 \leq m \leq M$, then the $L^1(\mathbf{T^n})$ functions

$$K(\zeta) = \sum_{\sigma\in\Lambda} \hat{f}^{\ell}(\zeta+\sigma)\overline{\hat{f}^{\ell'}\Big(B^{*m}A^{*j}(\zeta+\sigma)\Big)}$$

has the property that its Fourier coefficients are all zero except for the coefficient corresponding to $\lambda = 0$, which is 1 if $j = 0$ and $\ell = \ell'$. Hence, $K(\zeta) = \delta_{\ell,\ell'}\delta_{j,0}$ for a.e. $\zeta \in \mathbf{T^n}$. Conversely, if $K(\zeta) = \delta_{\ell,\ell'}\delta_{j,0}$, then the same calculation shows that $\left\langle f^{\ell}_{m,j,\lambda}, f^{\ell'}_{0,0,0}\right\rangle = \delta_{\ell,\ell'}\delta_{j,0}\delta_{\lambda,0}\delta_{m,0}$. This completes the proof of Lemma. $\square$

Suppose $F = \left\{f^1, f^2, \ldots, f^{2N-1}\right\}$ be a finite family of functions in $L^2(\Lambda)$. For $j \geq 0$ and $1 \leq m \leq M$, let $\mathcal{D}_j$ be a set of representatives of distinct cosets of $\Lambda \setminus A^jB^m\Lambda$. For $j < 0$, we define $\mathcal{D}_j = \{0\}$. Since the quasi affine system $\tilde{X}(F)$ is invariant under integer, we have

$$\begin{aligned} \tilde{X}(F) &= \{\Big\{T_\lambda g : \lambda \in \Lambda, g \in \mathcal{A}\Big\}, \\ \mathcal{A} &:= \left\{\tilde{f}^{\ell}_{m,j,d} : j \in \mathbf{Z}, d \in \mathcal{D}_j, 1 \leq \ell \leq 2N-1, 1 \leq m \leq M\right\}. \end{aligned} \tag{2.5}$$

The dual Gramian $\tilde{G}(\zeta)$ of the quasi affine system $\tilde{X}(F)$ at $\zeta \in \mathbf{T^n}$ is defined as the dual Gramian of $\left\{\Big(\hat{g}(\zeta+\lambda)\Big)_{\lambda\in\Lambda} : g \in \mathcal{A}\right\} \subset l^2(\Lambda)$, where $\mathcal{A}$ is defined by (2.5). We now compute $\tilde{G}(\zeta)$ in terms of Fourier transforms of functions in F and show that it does not depend upon the choice of representatives $\mathcal{D}_j$.

For $\sigma \in \Lambda \setminus AB\Lambda$, define the function

$$t_\sigma(\zeta) = \sum_{\ell=1}^{2N-1}\sum_{m=1}^{M}\sum_{j=0}^{\infty} \hat{f}^{\ell}\Big(A^{*j}B^{*m}\zeta\Big)\overline{\hat{f}^{\ell}\Big(A^{*j}B^{*m}(\zeta+\sigma)\Big)},\ \zeta \in \Lambda. \tag{2.6}$$

Lemma 2.7. *Let $F = \left\{f^1, f^2, \ldots, f^{2N-1}\right\} \subset L^2(\Lambda)$ and $\tilde{G}(\zeta)$ be the dual Gramian of $\tilde{X}(F)$ at $\zeta \in \mathbf{T^n}$. Then*

$$\left\langle \tilde{G}(\zeta)e_\lambda, e_\lambda\right\rangle = \sum_{\ell=1}^{2N-1}\sum_{m=1}^{M}\sum_{j\in\mathbf{Z}} \left|\hat{f}^{\ell}\Big(A^{*j}B^{*m}(\zeta+\lambda)\Big)\right|^2, \quad \text{for } \zeta \in \Lambda, \tag{2.7}$$

$$\left\langle \tilde{G}(\zeta)e_\lambda, e_\nu \right\rangle = t_{B^{*-m}A^{*-m}(\nu-\lambda)}\Big(B^{*-m}A^{*-m}\zeta + B^{*-m}A^{*-m}\lambda\Big), \quad \text{for } \lambda \neq \nu \in \Lambda, \tag{2.8}$$

where $m = \max\Big\{j \in \mathbf{Z} : B^{*-m}A^{*-j}(\nu-\lambda) \in \Lambda\Big\}$ *and the functions* $t_\sigma, \sigma \in \Lambda \setminus AB\Lambda$, *are given by (2.6).*

Proof. For $\lambda, \nu \in \Lambda$, we have

$$\begin{aligned}
\left\langle \tilde{G}(\zeta)e_\lambda, e_\nu \right\rangle &= \sum_{g \in \mathcal{A}} \hat{g}(\zeta+\lambda)\overline{\hat{g}(\zeta+\nu)} \\
&= \sum_{\ell=1}^{2N-1}\sum_{m=1}^{M}\sum_{j<0} \hat{f}^\ell\Big(A^{*-j}B^{*-m}(\zeta+\lambda)\Big)\overline{\hat{f}^\ell\Big(A^{*-j}B^{*-m}(\zeta+\nu)\Big)} \\
&\quad + \sum_{\ell=1}^{2N-1}\sum_{m=1}^{M}\sum_{j\geq 0} \hat{f}^\ell\Big(A^{*-j}B^{*-m}(\zeta+\lambda)\Big)\overline{\hat{f}^\ell\Big(A^{*-j}B^{*-m}(\zeta+\nu)\Big)} \\
&\qquad \times \sum_{d \in \mathcal{D}_j} q^{-j} e^{-2\pi i d B^{*-m}A^{*-j}(\nu-\lambda)} \\
&= \sum_{\ell=1}^{2N-1}\sum_{m=1}^{M}\sum_{j=-\infty}^{r} \hat{f}^\ell\Big(A^{*-j}B^{*-m}(\zeta+\lambda)\Big)\overline{\hat{f}^\ell\Big(A^{*-j}B^{*-m}(\zeta+\nu)\Big)},
\end{aligned}$$

where $r = \max\Big\{j \in \mathbf{Z} : B^{*-m}A^{*-j}(\nu-\lambda) \in \Lambda\Big\}$ and $r = \infty$ when $\lambda = \nu$. The sum over $\mathcal{D}_j$ is equal to 1 if $(\lambda - \nu) \in A^{*j}B^{*m}\Lambda$ and 0 otherwise. Therefore, if $\lambda = \nu$, then (2.7) holds. If $\lambda \neq \nu$, then

$$\begin{aligned}
&\Big\langle \tilde{G}(\zeta)e_\lambda, e_\nu \Big\rangle \\
&\quad = \sum_{\ell=1}^{2N-1} \sum_{m=1}^{M} \sum_{j\geq 0} \hat{f}^\ell\Big(A^{*-j-r}B^{*-m}(\zeta+\lambda)\Big)\overline{\hat{f}^\ell\Big(A^{*-j-r}B^{*-m}(\zeta+\nu)\Big)} \\
&\quad = \sum_{\ell=1}^{2N-1} \sum_{m=1}^{M} \sum_{j\geq 0} \hat{f}^\ell\Big(A^{*-j}B^{*-m}(A^{*-r}\zeta + A^{*-r}\lambda)\Big) \\
&\qquad\qquad \times \overline{\hat{f}^\ell\Big(A^{*-j}B^{*-m}\Big(A^{*-r}\zeta + A^{*-m}\lambda + A^{*-r}(\nu-\lambda)\Big)\Big)}
\end{aligned}$$

which proves (2.8) and hence completes the proof. □

Theorem 2.8. *Suppose that* $F = \Big\{f^1, f^2, \dots, f^{2N-1}\Big\} \subset L^2(\Lambda)$. *The affine system* $X(F)$ *is tight frame with constant* 1 *for* $L^2(\Lambda)$ *i.e.,*

$$\sum_{\ell=1}^{2N-1} \sum_{m=1}^{M} \sum_{j\in\mathbf{Z}} \sum_{\lambda\in\Lambda} \Big|\langle f, f^\ell_{m,j,\lambda}\rangle\Big|^2 = \Big\|f\Big\|_2^2 \quad \text{for all } f \in L^2(\Lambda)$$

if and only if the functions $f^1, f^2, \dots, f^{2N-1}$ *satisfy the following two conditions:*

$$\sum_{\ell=1}^{2N-1} \sum_{m=1}^{M} \sum_{j\in\mathbf{Z}} \Big|\hat{f}^\ell\Big(A^{*j}B^{*m}\zeta\Big)\Big|^2 = 1, \quad \text{for a.e. } \zeta \in \Lambda, \tag{2.9}$$

and

$$t_m(\zeta) = 0, \quad \text{for a.e. } \zeta \in \Lambda, m \in \Lambda \setminus AB\Lambda. \tag{2.10}$$

In particular, F *is a set of basic wavelets of* $L^2(\Lambda)$ *if and only if* $\Big\|f^\ell\Big\|_2 = 1$ *for* $\ell = 1, 2, \cdots, 2N-1$ *and (2.9) and (2.10) hold.*

Proof. It follows from Theorem 2.3 that $X(F)$ is a tight frame with constant 1 if and only if $\tilde{X}(F)$ is a tight frame with constant 1. By Theorem 2.5, this is equivalent to the spectrum of $\tilde{G}(\zeta)$ consisting of a single point 1, i.e., $\tilde{G}(\zeta)$ is identity on $l^2(\Lambda)$ for a.e. $\zeta \in \mathbf{T^n}$. By Lemma 2.7, this is equivalent to the fact that Eqs. (2.9) and (2.10) hold. The second assertion follows since a tight frame $X(F)$ with constant 1 is an orthonormal basis if and only if $\left\|f^\ell\right\|_2 = 1$ for $\ell = 1, 2, \cdots, 2N-1$ (see Theorem 1.8, section 7.1 in [12]). This completes the proof. $\square$

Theorem 2.9. *Suppose that* $F = \left\{f^1, f^2, \ldots, f^{2N-1}\right\} \subset L^2(\Lambda)$. *Assume that* $X(F)$ *is a Bessel family with constant* 1. *Then the following are equivalent:* (a) $X(F)$ *is a tight frame with constant* 1. (b) F *satisfies equality (2.9).* (c) F *satisfies*

$$\sum_{\ell=1}^{2N-1} \int_\Lambda \left|\hat{f}^\ell(\zeta)\right|^2 \frac{d\zeta}{\rho(\zeta)} = 1, \tag{2.11}$$

for some quasi-norm ρ *associated with* B^*A^*.

Proof. It is obvious from Theorem 2.8 that (a) $\Rightarrow$ (b). To show (b) implies (c), assume that (2.10) holds. Then, since $\left\{A^{*j}B^{*m}S : 1 \leq m \leq M, j \in \mathbf{Z}\right\}$ is a partition of Λ (modulo sets of measure zero), for any $S \subset \Lambda$, we have

$$\begin{aligned}
\sum_{\ell=1}^{2N-1} \int_\Lambda \left|\hat{f}^\ell(\zeta)\right|^2 \frac{d\zeta}{\rho(\zeta)} &= \sum_{\ell=1}^{2N-1} \sum_{m=1}^{M} \sum_{j\in\mathbf{Z}} \int_{A^{*j}B^{*m}S} \left|\hat{f}^\ell(\zeta)\right|^2 \frac{d\zeta}{\rho(\zeta)} \\
&= \sum_{\ell=1}^{2N-1} \int_S \sum_{m=1}^{M} \sum_{j\in\mathbf{Z}} \left|\hat{f}^\ell\left(A^{*j}B^{*m}\zeta\right)\right|^2 \frac{d\zeta}{\rho(\zeta)} \\
&= 1.
\end{aligned}$$

To prove (c) $\Rightarrow$ (a), we assume that (2.11) holds. Since $X(F)$ is a Bessel family with constant 1, so is $\tilde{X}(F)$, by condition (a) of Theorem 2.3. Let $\tilde{G}(\zeta)$ be the dual Gramian of $\tilde{X}(F)$ at $\zeta \in \mathbf{T^n}$. By Theorem 2.5, we have $\|\tilde{G}(\zeta)\| \leq 1$ for a.e. $\zeta \in \mathbf{T^n}$. In particular, $\|\tilde{G}(\zeta)e_\lambda\| \leq 1$. Hence,

$$1 \geq \left\|\tilde{G}(\zeta)\right\|^2 = \sum_{\nu\in\Lambda} \left|\left\langle \tilde{G}(\zeta)e_\lambda, e_\nu\right\rangle\right|^2 = \left|\left\langle \tilde{G}(\zeta)e_\lambda, e_\nu\right\rangle\right|^2 + \sum_{\nu\in\Lambda,\nu\neq\lambda} \left|\left\langle \tilde{G}(\zeta)e_\lambda, e_\nu\right\rangle\right|^2. \tag{2.12}$$

By Lemma 2.7, we have

$$\sum_{\ell=1}^{2N-1} \sum_{m=1}^{M} \sum_{j\in\mathbf{Z}} \left|\hat{f}^\ell\left(A^{*j}B^{*m}(\zeta+\lambda)\right)\right|^2 \leq 1, \quad \text{for } \lambda \in \Lambda,\ \zeta \in \mathbf{T^n}.$$

Hence,

$$1 = \int_S \sum_{\ell=1}^{2N-1} \sum_{m=1}^{M} \sum_{j\in\mathbf{Z}} \left|\hat{f}^\ell\left(A^{*j}B^{*m}(\zeta)\right)\right|^2 \frac{d\zeta}{\rho(\zeta)} \le \int_D \frac{d\zeta}{\rho(\zeta)} = 1,$$

From this it follows that $\sum_{\ell=1}^{2N-1}\sum_{m=1}^{M}\sum_{j\in\mathbf{Z}} \left|\hat{f}^\ell\left(A^{*j}B^{*m}\zeta\right)\right|^2 = 1$ for a.e. $\zeta \in D$ and hence for a.e. $\zeta \in \Lambda$. This means that equation (2.9) holds. By Lemma 2.7 and equality (2.9), $\left|\left\langle \tilde{G}(\zeta)e_\lambda, e_\nu\right\rangle\right|^2 = 1$ for all $\lambda \in \Lambda$. Thus, by (2.12), it follows that $\left\langle \tilde{G}(\zeta)e_\lambda, e_\nu\right\rangle = 0$ for $\lambda \neq \nu$ so that $\tilde{G}(\zeta)$ is the identity operator on $l^2(\Lambda)$. Hence, by Theorem 2.5, $\tilde{X}(F)$ is a tight frame with constant 1. Therefore, $X(F)$ is also a tight frame with constant 1, by Theorem 2.3. This completes the proof. □

In the consequence of above theorem, we provide a new characterization of wavelets.

Theorem 2.9. *Suppose* $F = \left\{f^1, f^2, \ldots, f^{2N-1}\right\} \subset L^2(\Lambda)$. *Then the following are equivalent:*

(a) *F is a set of basic wavelets of $L^2(\Lambda)$.*
(b) *satisfies (2.4) and (2.9).*
(c) *satisfies (2.4) and (2.11).*

Proof. It follows from Theorem 2.8 and Lemma 2.7 that (a)⇒(b)⇒(c).We now prove that (c) implies (a). Assume that F satisfies (2.4) and (2.11). The equation (2.4) implies that $X(F)$ is an orthonormal system, hence it is a Bessel family with constant 1. By Theorem 2.8 and (2.11), $X(F)$ is a tight frame with constant 1. Since each f^ℓ has L^2 norm 1, it follows that $X(F)$ is an orthonormal basis for $L^2(\Lambda)$. That is, F is a set of basic wavelets of $L^2(\Lambda)$.

3. Characterization of Composite MRA Wavelets

As usual, we construct wavelets from multiresolution analysis(MRA).

Definition 3.1. A closed subspaces sequence $\{V_j\}_{j\in\mathbf{Z}}$ of $L^2(\Lambda)$ is called a nonuniform AB-multiresolution analysis or nonuniform composite multiresolution analysis with A and B same as in Section 2, if the following conditions are satisfied:

(1) $V_j \subset V_{j+1}$, $\forall j \in \mathbf{Z}$;
(2) $\overline{\bigcup_{j\in\mathbf{Z}} V_j} = L^2(\Lambda)$;

(3) $\bigcap_{j\in\mathbf{Z}} V_j = \{0\}$;
(4) $f(x) \in V_j$ if and only if $f(2NAx) \in V_{j+1}$;
(5) there exists a function $g(x) \in V_0$, such that $\{g_{0,\ell,\lambda}\}_{\lambda\in\Lambda}$ is an orthonormal basis of $V_{0,\ell}$, in addition, $V_0 = \oplus_{\ell=1}^{2N-1} V_{0,\ell}$, where $\{V_{0,\ell}\}_{1\le\ell\le 2N-1}$ are mutually orthogonal. Here function $g(x)$ is called the *scaling function (or generator)*.

Let $F = \left\{f^1, f^2, \ldots, f^{2N-1}\right\}$ be a set of basic wavelets of $L^2(\Lambda)$. We define the spaces $W_j, j \in \mathbf{Z}$, by $W_j = \overline{\operatorname{span}}\{f^\ell_{m,j,\lambda} : 1 \le \ell \le 2N-1, 1 \le m \le M, \lambda \in \Lambda\}$. We also define $V_j = \oplus_{m<j} W_m, j \in \mathbf{Z}$. Then it follows that $\{V_j : j \in \mathbf{Z}\}$ satisfies the properties (a)-(d) in the definition of a MRA. Hence, $\{V_j : j \in \mathbf{Z}\}$ will form a MRA of $L^2(\Lambda)$ if we can find a function $g \in L^2(\Lambda)$ such that the system $\{g(x-\lambda) : \lambda \in \Lambda\}$ is an orthonormal basis for V_0. In this case, we say that F is associated with a MRA, or simply that F is a MRA-wavelet.

Now suppose that $\left\{f^1, f^2, \ldots, f^{2N-1}\right\}$ is a set of basic wavelets for $L^2(\Lambda)$ associated with a MRA $\{V_j : j \in \mathbf{Z}\}$. Let $g \in L^2(\Lambda)$ be the corresponding scaling function. Then in view of [1], we have

$$g(A^{-1}x) = \sum_{m=1}^{M} \sum_{\lambda\in\Lambda} d_{1,m,\lambda}\, g\Big(B^m x - \lambda\Big), \tag{3.1}$$

for any $\{d_{1,m,\lambda}\}_{1\le m\le M,\lambda\in\Lambda} \in l^2(\mathbf{N_0})$. Taking Fourier transform of equation (3.1), we get

$$\hat{g}\Big(A^*\zeta\Big) = \sum_{m=1}^{M} h_0^{(m)}(\zeta)\hat{g}\Big(B^{*-m}\zeta\Big), \tag{3.2}$$

where

$$h_0^{(m)}(\zeta) = \sum_{\lambda\in\Lambda} d_{1,m,\lambda} e^{-2\pi i\lambda\zeta}$$

is an integral periodic function in $L^\infty(\mathbf{T^n})$. Also, since $\{f^1, f^2, \ldots, f^{2N-1}\}$ are the wavelets associated with a MRA corresponding to the scaling function g, there exist integral-periodic functions $h_{1,\ell}^{(m)}, 1 \le m \le M, 1 \le \ell \le 2N-1$, such that the matrix

$$\mathcal{M}^{(m)}(\zeta) = \Big[h_{1,\ell_1}^{(m)}\Big(\zeta + \ell_2\Big)\Big]_{\ell_1,\ell_2=0}^{2N-1}$$

is unitary for a.e. $\zeta \in [0, 2\pi]$ and

$$\hat{f}^{\ell}\Big(A^{*}\zeta\Big) = \sum_{m=1}^{M} h_{1,\ell}^{(m)}(\zeta)\hat{g}\Big(B^{*-m}\zeta\Big), \tag{3.3}$$

where

$$h_{1,\ell}^{(m)}(\zeta) = \sum_{\lambda\in\Lambda} c_{\ell,m,\lambda} e^{-2\pi i\lambda\zeta}.$$

Hence, by (3.2), we have

$$\begin{aligned}\Big|\hat{g}\Big(A^{*}\zeta\Big)\Big|^{2} + \sum_{\ell=1}^{2N-1}\Big|\hat{f}\Big(A^{*}\zeta\Big)\Big|^{2} &= \Big|\sum_{m=1}^{M} h_{0}^{(m)}(\zeta)\hat{g}\Big(B^{*-m}\zeta\Big)\Big|^{2} \\ &\quad + \textstyle\sum_{\ell=1}^{2N-1}\Big|\sum_{m=1}^{M} h_{1,\ell}^{(m)}(\zeta)\hat{g}\Big(B^{*-m}\zeta\Big)\Big|^{2} \\ &= \sum_{m=1}^{M}\Big|g\Big(B^{*-m}\zeta\Big)\Big|^{2}\left(\sum_{\ell=0}^{2N-1}\Big|h_{1,\ell}^{(m)}(\zeta)\Big|^{2}\right).\end{aligned}$$

Since $\mathcal{M}^{(m)}(\zeta)$ is unitary for each $m, 1 \le m \le M$, we have

$$\Big|\hat{g}\Big(A^{*}\zeta\Big)\Big|^{2} + \sum_{\ell=1}^{2N-1}\Big|\hat{f}\Big(A^{*}\zeta\Big)\Big|^{2} = \sum_{m=1}^{M}\Big|g\Big(B^{*-m}\zeta\Big)\Big|^{2}.$$

Thus equality holds for for a.e, $\zeta \in \Lambda$. Hence, we have

$$|\hat{g}(\zeta)|^{2} = \sum_{m=1}^{M}\left(\Big|\hat{g}\Big(A^{*}B^{*m}\zeta\Big)\Big|^{2} + \sum_{\ell=1}^{2N-1}\Big|f^{\ell}\Big(A^{*}B^{*m}\zeta\Big)\Big|^{2}\right)$$

Iterating for any integer $N \ge 1$, we get,

$$|\hat{g}(\zeta)|^{2} = \sum_{m=1}^{M}\left(\Big|\hat{g}\Big(A^{*N}B^{*m}\zeta\Big)\Big|^{2} + \sum_{\ell=1}^{2N-1}\sum_{j=1}^{N} f^{\ell}\Big(A^{*j}B^{*m}\zeta\Big)\right).$$

Since $|\hat{g}(\zeta)|^{2} \le 1$, the sequence $\{\sum_{\ell=1}^{2N-1}\sum_{m=1}^{M}\sum_{j=1}^{N} f^{\ell}\Big(A^{*j}B^{*m}\zeta\Big) :$ $N \ge 1\}$ of real numbers is increasing and is bounded by 1, hence it converges. Therefore $\lim_{N\to\infty}\sum_{m=1}^{M}\Big|\hat{g}\Big(A^{*N}B^{*m}\zeta\Big)\Big|^{2}$ also exists. Now

$$\int_{\Lambda}\sum_{m=1}^{M}\Big|\hat{g}\Big(A^{*N}B^{*m}\zeta\Big)\Big|^{2}\zeta = q^{-N}\int_{\Lambda}|\hat{g}(\zeta)|^{2}\,d\zeta \to 0 \text{ as } N \to \infty.$$

Hence, by Fatou's Lemma, we have

$$\int_\Lambda \lim_{N\to\infty} \sum_{m=1}^{M} \left|\hat{g}\left(A^{*N}B^{*m}\zeta\right)\right|^2 d\zeta \le \lim_{N\to\infty} \int_\Lambda \sum_{m=1}^{M} \left|\hat{g}\left(A^{*N}B^{*m}\zeta\right)\right|^2 d\zeta = 0.$$

This shows that $\lim_{N\to\infty} \sum_{m=1}^{M} \left|\hat{g}\left(A^{*N}B^{*m}\zeta\right)\right|^2 = 0$. Hence, we get

$$|\hat{g}(\zeta)|^2 = \sum_{\ell=1}^{2N-1} \sum_{m=1}^{M} \sum_{j=1}^{\infty} \left|\hat{f}^\ell\left(A^{*j}B^{*m}\zeta\right)\right|^2$$

Since $\{g(x-\lambda) : \lambda \in \Lambda\}$ is an orthonormal system, we get for a.e. $\zeta \in \Lambda$,

$$1 = \sum_{\lambda\in\Lambda} |\hat{g}(\zeta+\lambda)|^2 = \sum_{\ell=1}^{2N-1} \sum_{m=1}^{M} \sum_{j=1}^{\infty} \sum_{\lambda\in\Lambda} \left|\hat{f}^\ell\left(A^{*j}B^{*m}(\zeta+\lambda)\right)\right|^2$$

Definition 3.2. Suppose $F = \{f^1, f^2, \ldots, f^{2N-1}\}$ is a set of basic wavelets for $L^2(\Lambda)$. The *dimension function* of F is defined as

$$D_F(\zeta) = \sum_{\ell=1}^{2N-1} \sum_{m=1}^{M} \sum_{j=1}^{\infty} \sum_{\lambda\in\Lambda} \left|\hat{f}^\ell\left(A^{*j}B^{*m}(\zeta+\lambda)\right)\right|^2. \tag{3.4}$$

Note that if $f^1, f^2, \ldots, f^{2N-1} \in L^2(\Lambda)$, then

$$\int_{[0,2\pi]} \sum_{m=1}^{M} \sum_{j=1}^{\infty} \sum_{\lambda\in\Lambda} \left|\hat{f}^\ell\left(A^{*j}B^{*m}(\zeta+\lambda)\right)\right|^2 d\zeta = \sum_{j=1}^{\infty} \int_{\mathbf{R}} \left|\hat{f}^\ell(\zeta)\right|^2 d\zeta < \infty. \tag{3.5}$$

Then D_F is well defined for a.e. $\zeta \in \Lambda$. In particular, $\sum_{\lambda\in\Lambda} \left|\hat{f}^\ell\left(A^{*j}B^{*m}(\zeta+\lambda)\right)\right|^2 < \infty$ for a.e. $\zeta \in \Lambda$. Thus for all $j \ge 1, 1 \le \ell \le L, 1 \le m \le M$, and a.e. $\zeta \in \Lambda$, we can define the vector $\zeta^\ell_{j,m}(\zeta) \in l^2(\Lambda)$, where

$$\zeta^\ell_{j,m}(\zeta) = \left\{\hat{f}^\ell\left(A^{*j}B^{*m}(\zeta+\lambda)\right) : \lambda \in \Lambda\right\}.$$

Hence, D_F can also be written as

$$D_F(\zeta) = \sum_{\ell=1}^{2N-1} \sum_{m=1}^{M} \sum_{j=1}^{\infty} \left\|\zeta^\ell_{j,m}(\zeta)\right\|^2_{l^2(\Lambda)}. \tag{3.6}$$

We have thus proved that if $F=\{f^1,f^2,\ldots,f^{2N-1}\}$ is a set of basic wavelets associated with a MRA of $L^2(\Lambda)$, then it is necessary that $D_F(\zeta)=1$ a.e. Our aim is to show that this condition is also sufficient. We will show that if $F=\{f^1,f^2,\ldots,f^{2N-1}\}$ is a set of basic wavelets of $L^2(\Lambda)$ and $D_F(\zeta)=1$ a.e., then F is an AB-MRA wavelet. To prove this we need the following lemma.

Lemma 3.3 *For all $j\geq 1, 1\leq \ell\leq L$, and a.e. $\zeta\in\Lambda$, we have*

$$\zeta^{\ell}_{j,m}(\zeta)=\sum_{h=1}^{2N-1}\sum_{m=1}^{M}\sum_{i=1}^{\infty}\Big\langle \zeta^{\ell}_{j,m}(\zeta),\zeta^{h}_{i,m}(\zeta)\Big\rangle\,\zeta^{h}_{i,m}(\zeta). \tag{3.7}$$

Proof. The series appearing in the lemma converges absolutely by (3.5) for a.e. $\zeta\in\Lambda$. We first show that

$$\hat{f}^{\ell}\Big(A^{*j}B^{*m}\zeta\Big)=\sum_{h=1}^{2N-1}\sum_{m=1}^{M}\sum_{j=1}^{\infty}\sum_{\lambda\in\Lambda}\hat{f}^{\ell}\Big(A^{*j}B^{*m}(\zeta+\lambda)\Big)\overline{\hat{f}^{h}(A^{*i}B^{*m}(\zeta+\lambda))}\hat{f}^{h}\Big(A^{*j}B^{*m}\zeta\Big). \tag{3.8}$$

Let us denote the series on the right of (3.8) by $G^{\ell}_{j,m}(\zeta)$. Then by using Lemma 2.6 and equation (2.6), we have

$$\begin{aligned}
G^{\ell}_{j,m}(\zeta) &= \sum_{\lambda\in\Lambda}\sum_{m=1}^{M}\hat{f}^{\ell}\Big(A^{*j}B^{*m}(\zeta+\lambda)\Big)\sum_{h=1}^{2N-1}\sum_{i=1}^{\infty}\overline{\hat{f}^{h}(A^{*i}B^{*m}(\zeta+\lambda))}\hat{f}^{h}\\
&\quad \Big(A^{*j}B^{*m}\zeta\Big)\\
&= \sum_{\lambda\in\Lambda}\sum_{m=1}^{M}\hat{f}^{\ell}\Big(A^{*j}B^{*m}(\zeta+\lambda)\Big)\left\{t_{\lambda}(\zeta)-\sum_{h=1}^{2N-1}\sum_{i=1}^{\infty}\overline{\hat{f}^{h}((\zeta+\lambda))}\hat{f}^{h}(\zeta)\right\}\\
&= \sum_{\lambda\in AB\Lambda}\sum_{m=1}^{M}\hat{f}^{\ell}\Big(A^{*j}B^{*m}(\zeta+\lambda)\Big)t_{\lambda}(\zeta)\\
&= \sum_{h=1}^{2N-1}\sum_{\lambda\in\Lambda}\sum_{m=1}^{M}\sum_{i=0}^{\infty}\hat{f}^{\ell}(A^{*j}B^{*m}(\zeta+B^{*}A^{*}\lambda))\\
&\quad \overline{\hat{f}^{h}(A^{*i}B^{*m}(\zeta+B^{*}A^{*}\lambda))}\hat{f}^{h}\Big(A^{*j}B^{*m}\zeta\Big)\\
&= \sum_{h=1}^{2N-1}\sum_{\lambda\in\Lambda}\sum_{m=1}^{M}\sum_{i=1}^{\infty}\hat{f}^{\ell}(A^{*j+1}B^{*m+1}(A^{*-1}B^{*-1}\zeta+\lambda))\\
&\qquad \times\overline{\hat{f}^{h}(A^{*i}B^{*m}(A^{*-1}B^{*-1}(\zeta+\lambda)))}\hat{f}^{h}(A^{*j}B^{*m}A^{*-1}B^{*-1}\zeta)\\
&= G^{\ell}_{j+1,m+1}(A^{*-1}B^{*-1}\zeta).
\end{aligned}$$

This is equivalent to $G^{\ell}_{j,m}(\zeta)=G^{\ell}_{j-1,m-1}(A^{*}B^{*}\zeta)$. Iterating this equation, we obtain, $G^{\ell}_{j,m}(\zeta)=G^{\ell}_{1,m}(A^{*j-1}B^{*m-1}\zeta)$. We now calculate $G^{\ell}_{1,m}(\zeta)$. We have

$$\begin{aligned}
G^{\ell}_{1,m}(\zeta) &= \sum_{\lambda\in\Lambda} \hat{f}^{\ell}(A^{*}B^{*}(\zeta+\lambda)) \sum_{h=1}^{2N-1}\sum_{m=1}^{M}\sum_{i=1}^{\infty} \overline{\hat{f}^{h}(A^{*i}B^{*m}(\zeta+\lambda))}\hat{f}^{h} \\
&\quad (A^{*i}B^{*m}\zeta) \\
&= \sum_{\lambda\in\Lambda} \hat{f}^{\ell}(A^{*}B^{*}\zeta + A^{*}B^{*}\lambda)) \sum_{h=1}^{2N-1}\sum_{m=1}^{M}\sum_{i=1}^{\infty} \\
&\quad \overline{\hat{f}^{h}(A^{*i}B^{*m}(A^{*}B^{*}\zeta + A^{*}B^{*}\lambda))} \times \hat{f}^{h}(A^{*i}B^{*m}A^{*}B^{*}\zeta) \\
&= \sum_{\lambda\in AB\Lambda} \hat{f}^{\ell}(A^{*}B^{*}\zeta+\lambda) \sum_{h=1}^{2N-1}\sum_{m=1}^{M}\sum_{i=1}^{\infty} \overline{\hat{f}^{h}(A^{*i}B^{*m}(A^{*}B^{*}\zeta+\lambda))} \\
&\quad \times \hat{f}^{h}(A^{*i}B^{*m}A^{*}B^{*}\zeta) \\
&= \sum_{h=1}^{2N-1}\sum_{m=1}^{M}\sum_{i=1}^{\infty} \hat{f}^{h}(A^{*i}B^{*m}A^{*}B^{*}\zeta)\delta_{i,0}\delta_{m,0}\delta_{\ell,h} \\
&= \hat{f}^{\ell}(A^{*}B^{*}\zeta).
\end{aligned}$$

Thus $G^{\ell}_{j}(\zeta) = \hat{f}^{\ell}(A^{*-j}B^{*-m}\zeta)$ a.e. $\zeta \in \Lambda$. Since $\left\langle \zeta^{\ell}_{j}(\zeta), \zeta^{h}_{i}(\zeta) \right\rangle$ is integral periodic, (3.7) follows. This completes the proof. □

Lemma 3.4. *Let $\{\nu_j : j \geq 1\}$ be a family of vectors in a Hilbert space H such that* (i) $\sum_{n=1}^{\infty} \left\|\nu_n\right\|^2 = C < \infty$, (ii) $\nu_n = \sum_{n=1}^{\infty} \langle \nu_n, \nu_m\rangle \nu_m$ *for all $n \geq 1$. Let $\mathbf{F} = \overline{span}\{\nu_j : j \geq 1\}$. Then*

$$\dim \mathbf{F} = \sum_{j=1}^{\infty} \left\|\nu_j\right\|^2 = C.$$

Theorem 3.5. *A wavelet $F = \left\{f^1, f^2, \ldots, f^{2N-1}\right\} \subset L^2(\Lambda)$ is an AB-MRA wavelet if only if $D_F(\zeta) = 1$ for almost every $\zeta \in \Lambda$.*

Proof. We have already observed that $D_F(\zeta) = 1$ for almost every $\zeta \in \Lambda$ when F is an AB-MRA wavelet. We now prove the converse. Assume that

$D_F(\zeta) = 1$ for almost every $\zeta \in \Lambda$. Let E be the subset of $\mathbf{T^n}$ on which $D_F(\zeta)$ is finite and (3.7) is satisfied. Then $\zeta^{\ell}_{j,m}$ are well-defined on E. For $\zeta \in E$, we define the space

$$\mathcal{F}(\zeta) = \overline{\text{span}}\Big\{\zeta^{\ell}_{j,m}(\zeta) : 1 \le \ell \le 2N-1, 1 \le m \le M, j \ge 1\Big\}.$$

Then, by Lemmas 3.3 and 3.4, we have

$$\dim \mathcal{F}(\zeta) = \sum_{\ell=1}^{2N-1}\sum_{m=1}^{M}\sum_{j=1}^{\infty}\Big\|\zeta^{\ell}_{j,m}(\zeta)\Big\|_2^2 = D_F(\zeta) = 1. \tag{3.9}$$

That is, for each $\zeta \in E, \mathcal{F}(\zeta)$ is generated by a single unit vector $U(\zeta)$. We now choose a suitable vector. For $j \ge 1$, let us define

$$X_j = \Big\{\zeta \in E : \zeta^{\ell}_{j,m}(\zeta) \ne 0 \text{ and } \zeta^{\ell}_{m,m}(\zeta) = 0, \forall\, m < j$$

$$\text{and } 1 \le \ell \le 2N-1, 1 \le m \le M\Big\}$$

and

$$X_0 = \Big\{\zeta \in \mathbf{T^n} : \zeta^{\ell}_{\mathbf{j,m}}(\zeta) \ne \mathbf{0}, \forall\, \mathbf{j} \ge \mathbf{1},\ \textbf{and } \mathbf{1} \le \ell \le \mathbf{2N-1}, \mathbf{1} \le \mathbf{m} \le \mathbf{M}\Big\}.$$

Then $\{X_j : j = 0, 1, 2, \ldots\}$ forms a partition of E. Note that $X_0 = \{\zeta \in \mathbf{T^n} : \mathbf{D_F}(\zeta) = \mathbf{0}\}$. So for a.e. $\zeta \in E \setminus X_0$, there exists $j \ge 1$ such that $\zeta \in X_j$. Hence, there exists at least one $\ell, 1 \le \ell \le 2N-1$, and one $m, 1 \le m \le M$ such that $\zeta^{\ell}_{j,m}(\zeta) \ne 0$. Choose the smallest such ℓ and m define

$$U(\zeta) = \frac{\zeta^{\ell}_{j,m}(\zeta)}{\Big\|\zeta^{\ell}_{j,m}(\zeta)\Big\|_{l^2}}.$$

Thus, $U(\zeta)$ is well defined and $\Big\|U(\zeta)\Big\|_{l^2} = 1$ for a.e. $\zeta \in \mathbf{T^n}$. We write $U(\zeta) = \{u_\lambda(\zeta) : \zeta \in \Lambda\}$. Now, define $\hat{g}(\zeta) = u_\lambda(\zeta - \lambda)$, where k is the unique integer in Λ such that $\zeta \in \mathbf{T^n} + \lambda$. This defines $\hat{g}$ on Λ.We first show that $g \in L^2(\Lambda)$ and $\{g(x-\lambda) : \lambda \in \Lambda\}$ is an orthonormal system in

$L^2(\Lambda)$. We have

$$\begin{aligned} \left\|\hat{g}\right\|_2^2 &= \int_\Lambda |\hat{g}(\zeta)|^2 \, d\zeta \\ &= \int_{\mathbf{T^n}} \sum_{\lambda\in\Lambda} |\hat{g}(\zeta+\lambda)|^2 \, d\zeta \\ &= \sum_{\lambda\in\Lambda} \int_{\mathbf{T^n}} |u_\lambda(\zeta)|^2 \, d\zeta \\ &= \int_{\mathbf{T^n}} \left\|U(\zeta)\right\|_{l^2}^2 d\zeta \\ &= 1. \end{aligned}$$

Thus $g \in L^2(\Lambda)$. Also,

$$\sum_{\lambda\in\Lambda} |\hat{g}(\zeta+\lambda)|^2 = \sum_{\lambda\in\Lambda} |u_\lambda(\zeta)|^2 = \left\|U(\zeta)\right\|_{l^2}^2 = 1. \tag{3.10}$$

This is equivalent to the fact that $\{g(x-\lambda) : \lambda \in \Lambda\}$ is an orthonormal system. We now define $V_0^{\#} = \overline{\operatorname{span}}\{g(x-\lambda) : \lambda \in \Lambda\}$. Let $W_j = \overline{\operatorname{span}}\{f^\ell_{m,j,\lambda} : 1 \le \ell \le 2N-1, 1 \le m \le M, \lambda \in \Lambda\}$ and $V_0 = \oplus_{j<0} W_j$. If we can show that $V_0^{\#} = V_0$, then it will follow that $\{V_j : j \in \mathbf{Z}\}$ is the required MRA .

We first show that $V_0^{\#} \subset V_0$. It is sufficient to verify that $f^\ell_{m,j,\lambda} \in V_0^{\#}, \lambda \in \Lambda, j < 0, 1 \le \ell \le 2N-1, 1 \le m \le M$. For each $j \ge 1$, there exists a measurable function $\nu^\ell_{j,m}$ on $\mathbf{T^n}$ such that $\zeta^\ell_{j,m}(\zeta) = \nu^\ell_{j,m}(\zeta)U(\zeta)$ for a.e. $\zeta \in \mathbf{T^n}$. That is,

$$\hat{f}^\ell\Big(A^{*j}B^{*m}(\zeta+\lambda)\Big) = \nu^\ell_{j,m}(\zeta)\hat{g}(\zeta+\lambda) \quad \text{for all } \zeta \in \mathbf{T^n}, \lambda \in \boldsymbol{\Lambda}.$$

Therefore, by (3.10), for a.e. $\zeta \in \mathbf{T^n}$, we have

$$\sum_{\lambda\in\Lambda} \left|\hat{f}^\ell\Big(A^{*j}B^{*m}(\zeta+\lambda)\Big)\right|^2 = \sum_{\lambda\in\Lambda} \left|\nu^\ell_{j,m}(\zeta)\right|^2 |\hat{g}(\zeta+\lambda)|^2 = \left|\nu^\ell_{j,m}(\zeta)\right|^2. \tag{3.11}$$

This shows that $\nu^\ell_{j,m} \in L^2(\mathbf{T^n})$ so that we can write its Fourier series expansion. Thus, for $j \ge 1$, there exists $\{a^\ell_{m,j,\lambda} : \lambda \in \Lambda\} \in l^2(\Lambda)$ such that

$\nu^{\ell}_{j,m}(\zeta) = \sum_{\lambda\in\Lambda} a^{\ell}_{m,j,\lambda} e^{-2\pi i\lambda\zeta}$, with convergence in $L^2(\mathbf{T^n})$. Extending $\nu^{\ell}_{j,m}$ integer periodically, we have

$$\hat{f}^{\ell}\Big(A^{*j}B^{*m}\zeta\Big) = \nu^{\ell}_{j,m}(\zeta)\hat{g}(\zeta), \quad \text{for a. e. } \zeta\in\Lambda, j\geq 1. \tag{3.12}$$

Taking inverse Fourier transform, we get

$$f^{\ell}_{-j,-m,0}(x) = q^{j/2}\sum_{\lambda\in\Lambda} a^{\ell}_{m,j,\lambda} g(\zeta-\lambda), \quad j\geq 1.$$

Hence, $f^{\ell}_{-j,-m,0}\in V_0^{\#}$ for $j\geq 1$. Moreover, since $V_0^{\#}$ is invariant under translations by $k,\lambda\in\Lambda$, we have $f^{\ell}_{m,j,\lambda}\in V_0^{\#}, j<0, \lambda\in\Lambda, 1\leq\ell\leq 2N-1, 1\leq m\leq M$.

To show the reverse inclusion, it suffices to show that $V_0^{\#}\perp W_j$, for $j\geq 0$. For $j\geq 0, \lambda\in\Lambda, 1\leq\ell\leq 2N-1, 1\leq m\leq M$, we have

$$\begin{aligned}
\Big\langle g, f^{\ell}_{m,j,\lambda}\Big\rangle &= \Big\langle \hat{g}, \hat{f}^{\ell}_{m,j,\lambda}\Big\rangle \\
&= \int_{\Lambda}\hat{g}(\zeta)q^{-j/2}\overline{\hat{f}^{\ell}\Big(A^{*j}B^{*m}\zeta\Big)}e^{-2\pi i A^{*j}B^{*m}\lambda\zeta}d\zeta \\
&= q^{j/2}\int_{\Lambda}\hat{g}(B^{*-m}A^{*-j}\zeta)\overline{\hat{f}^{\ell}(\zeta)}e^{-2\pi i\lambda\zeta}d\zeta \\
&= q^{j/2}\int_{\mathbf{T^n}}\sum_{n\in\Lambda}\hat{g}(B^{*-m}A^{*-j}(\zeta+n))\overline{\hat{f}^{\ell}(\zeta+n)}e^{-2\pi i\lambda\zeta}d\zeta.
\end{aligned} \tag{3.13}$$

Using Equation (3.11), we get

$$\sum_{\ell=1}^{2N-1}\sum_{m=1}^{M}\sum_{j=1}^{\infty}\left|\nu^{\ell}_{j,m}(\zeta)\right|^2 = \sum_{\ell=1}^{2N-1}\sum_{m=1}^{M}\sum_{j=1}^{\infty}\sum_{\lambda\in\Lambda}\left|\hat{f}^{\ell}\Big(A^{*j}B^{*m}(\zeta+\lambda)\Big)\right|^2 = 1$$

for a. e. $\zeta\in\Lambda$.

Hence, for such ζ and for all $j\geq 0$, there exists $j_0\geq 1$ such that $\nu^{\ell}_{j,m}\Big(A^{*j}B^{*m}\zeta\Big)\neq 0$. Thus, (3.12) implies that
$\hat{f}^{\ell}\Big(A^{*j+j_0}B^{*m}\zeta\Big) = \nu^{\ell}_{j_0,m}\Big(B^{*-m}A^{*-j}\zeta\Big)\hat{g}\Big(B^{*-m}A^{*-j}\zeta\Big)$. Therefore, for $\lambda\in\Lambda$, we get

$$\hat{f}^{\ell}\Big(A^{*j+j_0}B^{*m}(\zeta+\lambda)\Big) = \nu^{\ell}_{j_0,m}\Big(B^{*-m}A^{*-j}(\zeta+\lambda)\Big)\hat{g}\Big(B^{*-m}A^{*-j}(\zeta+\lambda)\Big)$$

Using integral periodicity of $\nu^{\ell}_{j_0}$, we get

$$\hat{g}\Big(B^{*-m}A^{*-j}(\zeta+\lambda)\Big) = \frac{1}{\nu^{\ell}_{j_0,m}\Big(B^{*-m}A^{*-j}\zeta\Big)}\hat{f}^{\ell}\Big(A^{*j+j_0}B^{*m}(\zeta+\lambda)\Big).$$

Therefore, using Lemma 2.6, for any h with $1 \le h \le 2N-1$ and for $1 \le m \le M$, we have

$$\begin{aligned}\sum_{\lambda\in\Lambda}\hat{g}\Big(B^{*-m}A^{*-j}(\zeta+\lambda)\Big)\overline{\hat{f}(\zeta+\lambda)} &= \frac{1}{\nu^{\ell}_{j_0,m}\Big(B^{*-m}A^{*-j}\zeta\Big)} \\ &\quad \textstyle\sum_{\lambda\in\Lambda}\hat{f}^{\ell}\Big(A^{*j+j_0}B^{*m}(\zeta+\lambda)\Big)\overline{\hat{f}(\zeta+\lambda)} \\ &= 0,\end{aligned}$$

since $j+j_0 \ge 1$. Substituting this in (3.12), we get $\Big\langle g, f^{\ell}_{m,j,\lambda}\Big\rangle = 0$ for $j \ge 0, \lambda \in \Lambda, 1 \le \ell \le 2N-1, 1 \le m \le M$. From this we conclude that $V_0^{\#} \subset V_0$. This completes the proof of theorem. □

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