Proyecciones Journal of Mathematics
Vol. 41, N^o 4, pp. 805-823, August 2022.
Universidad Católica del Norte
Antofagasta - Chile

10.22199/issn.0717-6279-4581

On the isotopic characterizations of generalized Bol loops

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Received : March 2021. Accepted : November 2021

Abstract

In this study, the notion of isotopy of generalized Bol loop is characterized. A loop isotope of a σ-generalized Bol loop is shown to be a σ'-generalized Bol loop if σ' fixes its (isotope) identity element where σ' is some conjugate of σ. A loop isotope of a σ-generalized Bol loop is shown to be a σ'-generalized Bol loop if and only if the image of the isotope's identity element under σ' is right nuclear (where σ' is some conjugate of σ). It is shown that a generalized Bol loop can be constructed using a group and a subgroup of it. A right conjugacy closed σ-generalized Bol loop is shown to be a σ-generalized right central loop.

2020 Mathematics Subject Classification: *Primary 20N02, 20N05.*

Keywords and Phrases: *Bol loops, generalized Bol loops.*

1. Introduction

Let L be a non-empty set. Define a binary operation $(\cdot)$ on L : If $x \cdot y \in L$ for all $x, y \in L$, $(L, \cdot)$ is called a groupoid. Let $(L, \cdot)$ be a groupoid and let a be a fixed element in L, then the left and right translations L_a and R_a of a are respectively defined by $xL_a = a \cdot x$ and $xR_a = x \cdot a$ for all $x \in L$. If the equations:

$$a \cdot x = b \quad and \quad y \cdot a = b$$

have unique solutions for x and y respectively, then $(L, \cdot)$ is called a quasigroup. It can now be seen that a groupoid $(L, \cdot)$ is a quasigroup if its left and right translation mappings are permutations. In a quasigroup $(L, \cdot)$, the self maps $J_\rho : x \mapsto x^\rho$ and $J_\lambda : x \mapsto x^\lambda$ are called the right and left inverse maps respectively such that $x \cdot x^\rho = e^\rho$ and $x^\lambda \cdot x = e^\lambda$ where x^ρ and x^λ are called the right and left inverse elements of $x \in L$ respectively. Here, $e^\rho \in L$ and $e^\lambda \in L$ satisfy the relations $x \cdot e^\rho = x$ and $e^\lambda \cdot x = x$ for all $x \in L$ and are respectively called the right and left identity elements. Now, if $e^\lambda = e^\rho = e$, then $e \in L$ is called the identity element and $(L, \cdot)$ is called a loop. In case $x^\lambda = x^\rho$, then, we simply write $x^\lambda = x^\rho = x^{-1}$ and refer to x^{-1} as the inverse of x. In what follows, we shall write xy instead of $x \cdot y$ and stipulate that $\cdot$ has lower priority than juxtaposition among factors to be multiplied. For example, $x \cdot yz = xy \cdot z$ or $x(yz) = (xy)z$ means $x \cdot (y \cdot z) = (x \cdot y) \cdot z$ for all $x, y, z \in L$.

A loop $(L, \cdot)$ is called a (right) Bol loop if it satisfies the identity

$$(xy \cdot z)y = x(yz \cdot y). \tag{1.1}$$

A loop $(L, \cdot)$ is called a left Bol loop if it satisfies the identity

$$y(z \cdot yx) = (y \cdot zy)x. \tag{1.2}$$

A loop $(L, \cdot)$ is called a Moufang loop if it satisfies the identity

$$(xy) \cdot (zx) = (x \cdot yz)x. \tag{1.3}$$

A loop $(L, \cdot)$ is called a right inverse property loop (RIPL) if it satisfies right inverse property (RIP)

$$(yx)x^\rho = y \tag{1.4}$$

A loop $(L, \cdot)$ is called a left inverse property loop (LIPL) if it satisfies left inverse property (LIP)

$$x^\lambda(xy) = y \tag{1.5}$$

A loop $(L, \cdot)$ is called an automorphic inverse property loop (AIPL) if it satisfies automorphic inverse property (AIP)

$$(xy)^{-1} = x^{-1}y^{-1}. \tag{1.6}$$

A loop $(L, \cdot)$ is called a right conjugacy closed loop (RCCL) if it satisfies the identity

$$yx \cdot (x\backslash z)x = (yz)x. \tag{1.7}$$

A loop $(L, \cdot)$ is called a (left or right) Bruck loop or a K-loop if it is both a Bol loop and either a AIPL or obeys the identity $xy^2 \cdot x = (yx)^2$.

Let $(L, \cdot)$ be a loop with a single valued self-map $\sigma : x \longrightarrow \sigma(x)$:

The triple $(L, \cdot, \sigma)$ is called a σ-generalized (right) Bol loop or simply a generalized (right) Bol loop or right B-loop $(L, \cdot)$ (where there is no confusion on the self map σ because we are just being silent on the self map σ) if it satisfies the identity

$$(xy \cdot z)\sigma(y) = x(yz \cdot \sigma(y)) \tag{1.8}$$

$(L, \cdot, \sigma)$ is called a σ-generalized left Bol loop or simply a generalized left Bol loop or left B-loop $(L, \cdot)$ (where there is no confusion on the self map σ because we are just being silent on the self map σ) if it satisfies the identity

$$\sigma(y)(z \cdot yx) = (\sigma(y) \cdot zy)x \tag{1.9}$$

$(L, \cdot, \sigma)$ is called a σ-generalized right central loop or simply a generalized right central loop or generalized RC-loop $(L, \cdot)$ (where there is no confusion on the self map σ because we are just being silent on the self map σ) if it satisfies the identity

$$(yx \cdot \sigma(x))z = y(x\sigma(x) \cdot z) \tag{1.10}$$

$(L, \cdot, \sigma) = (L, \cdot)$ is called an M-loop if it satisfies the identity

$$(xy) \cdot (z\sigma(x)) = (x \cdot yz)\sigma(x) \tag{1.11}$$

The right nucleus of $(L, \cdot)$ is defined by $N_\rho(L, \cdot) = \{x \in L \mid zy \cdot x = z \cdot yx \ \forall\ y, z \in L\}$.

Consider $(G, \cdot)$ and $(H, \circ)$ being two distinct groupoids(quasigroups, loops). Let A, B and C be three bijective mappings that map G onto H.

The triple $\alpha = (A, B, C)$ is called an isotopism of $(G, \cdot)$ onto $(H, \circ)$ if and only if

$$xA \circ yB = (x \cdot y)C \ \forall \ x, y \in G.$$

If $(G, \cdot) = (H, \circ)$, then the triple $\alpha = (A, B, C)$ of bijections on $(G, \cdot)$ is called an autotopism of the groupoid(quasigroup, loop) $(G, \cdot)$. Such triples form a group $AUT(G, \cdot)$ called the autotopism group of $(G, \cdot)$. Furthermore, if $A = B = C$, then A is called an automorphism of the groupoid(quasigroup, loop) $(G, \cdot)$. Such bijections form a group $AUM(G, \cdot)$ called the automorphism group of $(G, \cdot)$.

For an overview of the theory of loops, readers may check [18, 36].

The birth of Bol loops can be traced back to Gerrit Bol [8] in 1937 when he established the relationship between Bol loops and Moufang loops, the latter which was discovered by Moufang Ruth [26]. Thereafter, a theory of Bol loops was evolved through the Ph.D. thesis of Robinson [31] in 1964 where he studied the algebraic properties of Bol loops, Moufang loops and Bruck loops, isotopy of Bol loop and some other notions on Bol loops. Some later results on Bol loops and Bruck loops can be found in Bruck [9], Solarin [41], Adéníran and Akinleye [2], Bruck [10], Burn [11], Gerrit Bol [8], Blaschke and Bol [7], Sharma [32, 33], Adéníran and Solarin [4]. In the 1980s, the study and construction of finite Bol loops caught the attention of many researchers among which are Burn [11, 12, 13], Solarin and Sharma [34, 37, 38, 39] and others like Chein and Goodaire [14, 15, 16], Foguel at. al. [17], Kinyon and Phillips [24, 25] in the present millennium. One of the most important results in the theory of Bol loops is the solution of the open problem on the existence of a simple Bol loop which was finally laid to rest by Nagy [27, 28, 29]. To any right Bol loop or left Bol loop, there corresponds a middle Bol loop and vice versa. Jaiyéọlá and David [20], Jaiyéọlá et al. [21, 22], Syrbu and Drapal [42], Syrbu and Grecu [43] and Syrbu [44] have studied the algebraic properties and structure of middle Bol loop.

In 1978, Sharma [34], Sharma and Sabinin [35] introduced and studied the algebraic properties of the notion of half-Bol loops(left B-lops). Thereafter, Adéníran [1], Adéníran and Akinleye [2], Adéníran and Solarin [5] studied the algebraic properties of generalized Bol loops. Also, Ajmal [6] introduced and studied the algebraic properties of generalized Bol loops and their relationship with M-loops. The latest study on the holomorph of generalized Bol loops can be found in Adéníran et. al. [3] and Jaiyéọlá and

Popoola [19]. Osoba and Jaiyéọlá [30] recently announced some algebraic connections between right and middle Bol loops and their cores.

Some of their results are highlighted below.

Theorem 1.1. *(Adéníran and Akinleye [2])*
If $(L, \cdot, \sigma)$ is a generalized Bol loop, then:

1. *$(L, \cdot)$ is a RIPL.*
2. *$x^\lambda = x^\rho$ for all $x \in L$.*
3. *$R_{y \cdot \sigma(y)} = R_y R_{\sigma(y)}$ for all $y \in L$.*
4. *$[xy \cdot \sigma(x)]^{-1} = (\sigma(x))^{-1} y^{-1} \cdot x^{-1}$ for all $x, y \in L$.*
5. *$(R_{y^{-1}}, L_y R_{\sigma(y)}, R_{\sigma(y)}) \in AUT(L, \cdot)$ for all $y \in L$.*

Theorem 1.2. *(Sharma [34])*
If $(L, \cdot, \sigma)$ is a half Bol loop, then:

1. *$(L, \cdot)$ is a LIPL.*
2. *$x^\lambda = x^\rho$ for all $x \in L$.*
3. *$L_{(x)} L_{(\sigma(x))} = L_{(\sigma(x)x)}$ for all $x \in L$.*
4. *$(\sigma(x) \cdot yx)^{-1} = x^{-1} \cdot y^{-1}(\sigma(x))^{-1}$ for all $x, y \in L$.*
5. *$(R_{(x)} L_{(\sigma(x))}, L_{(x)^{-1}}, L_{(\sigma(x))}), (R_{(\sigma(x))} L_{(x)^{-1}}, L_{\sigma(x)}, L_{(x)^{-1}}) \in AUT(L, \cdot)$ for all $x \in L$.*

Theorem 1.3. *(Ajmal [6])*
Let $(L, \cdot)$ be a loop. The following statements are equivalent:

1. *$(L, \cdot, \sigma)$ is an M-loop;*
2. *$(L, \cdot, \sigma)$ is both a left B-loop and a right B-loop;*
3. *$(L, \cdot, \sigma)$ is a right B-loop and satisfies the LIP;*
4. *$(L, \cdot, \sigma)$ is a left B-loop and satisfies the RIP;*

Theorem 1.4. *(Ajmal [6])*
Every isotope of a right B-loop with the LIP is a right B-loop.

Example 1.1. *(Sharma [34])*

Let R be a ring of characteristic 3 which possesses at least one set of elements a, b, c so that $ca^2b+caba+cba^2 \neq cb^2a+cbab+cab^2$. Let $Q = R\times R$ and define

$$(u, f) \cdot (v, g) = (u + v, f + g + vu^3)$$

for all $(u, f), (v, g) \in Q$. $(Q, \circ)$ is an half-Bol loop with $\sigma(x) = x \circ x$ for all $x \in Q$ which is not a left Bol loop.

For instance, if R is the ring of all 2×2 matrices taken over the field of three elements. Then $(Q, \cdot)$ is a loop which is not a left Bol loop but which is a half-Bol loop with $\sigma : x \mapsto x^2$.

Bruck loops have applications in special relativity (see Ungar [6], 2002). Left Bruck loops are equivalent to Ungar's 2002 gyrocommutative gyrogroups, even though the two structures are defined differently. K-loops are non-associative generalizations of abelian groups. Over the years, a few papers have been written on K-loops, but Bol loops have tremendous attention. There was a twist after Ungar showed that the set of admissible velocities with the addition of velocities in special relativity forms a K-loop. Ungars discovery sparked a rapid development of the theory of K-loops (Kiechle [6]).

Besides the application to special relativity, another important source of motivation for the study of K-loops is the problem of existence of a proper neardomain. This question is closely related with the structure of sharply 2-transitive groups. Frobenius groups with many involutions seem to be a reasonable generalization of sharply 2-transitive groups. Hence, since Bol loops have been generalized and the study of generalized Bol loops has already began, then it is important to continue the study of generalized Bol loops and initiate the study of generalized Bruck loops by first of all characterizing them with the possibility of finding a relationship between the self mapping σ; the generalizing factor in a σ-generalized Bol loop and the gyration (gyrator) in a gyrogroup. It would be recalled that in gyrogroup, the gyrators are actually left inner automorphisms. In fact, one of our results here show that the loop isotope H of a given generalized Bol loop G is a σ-generalized Bol loop if σ fixes the identity element in H; hence a $\sigma \in AUM(H)$ is a pleasant choice.

In this study, the notion of isotopy of generalized Bol loop is characterized. A loop isotope of a σ-generalized Bol loop is shown to be a σ'-generalized Bol loop if σ' fixes its (isotope) identity element where σ' is some conjugate of σ. A loop isotope of a σ-generalized Bol loop is shown to be a σ'-generalized Bol loop if and only if the image of the isotope's identity element under σ' is right nuclear (where σ' is some conjugate of σ). It is shown that a generalized Bol loop can be constructed using a group and a subgroup of it. A right conjugacy closed σ-generalized Bol loop is shown to be a σ-generalized right central loop.

2. Main Results

Theorem 2.1. *Let $(G,\cdot,1,\sigma)$ be a σ-generalized Bol loop. Then*

1. $\sigma(1) \in N_\rho(G,\cdot)$.
2. *If $(G,\cdot,1,\sigma)$ is isomorphic to a loop $(H,*)$ under θ, then $\left(H,*,\theta\sigma\theta^{-1}\right)$ is a $\theta\sigma\theta^{-1}$-generalized Bol loop.*
3. *If $(G,\cdot,\sigma) \xrightarrow[Isotopism]{(\alpha,\beta,I)} (G,\circ,1_\circ)$, then $\left(G,\circ,\beta\sigma\beta^{-1}\right)$ is a $\sigma'=\beta\sigma\beta^{-1}$-generalized Bol loop whenever $\sigma'(1_\circ)=1_\circ$.*

Proof.

1. Substitute $y=1$ into (1.8) to get $(x\cdot z)\sigma(1)=x(z\cdot\sigma(1))$, which implies $\sigma(1)\in N_\rho(G,\cdot)$.
2. If $(G,\cdot,\sigma)\overset{\theta}{\cong}(H,*)$, then $(G,\cdot,\sigma)$ is a σ-generalized Bol loop implies that
   $[(x\cdot y)\cdot z]\cdot\sigma(y)=x\cdot[(y\cdot z)\cdot\sigma(y)]$
   $\Rightarrow \theta\{[(x\cdot y)\cdot z]\cdot\sigma(y)\}=\theta\{x\cdot[(y\cdot z)\cdot\sigma(y)]\}$

   $$\Rightarrow [(\theta(x)*\theta(y))*\theta(z)]*\theta\sigma(y)=\theta(x)*[(\theta(y)*\theta(z))*\theta\sigma(y)] \tag{2.1}$$

   Let $\theta(x)=\bar{x},\theta(y)=\bar{y}$, then $y=\theta^{-1}(\bar{y}),\theta(z)=\bar{z}$ and substitute into equation (2.1). So, $(G,\cdot,\sigma)$ is a σ-generalized Bol loop implies that

   $$[(\bar{x}*\bar{y})*\bar{z}]*\theta\sigma\theta^{-1}(\bar{y})=\bar{x}*[(\bar{y}*\bar{z})*\theta\sigma\theta^{-1}(\bar{y})] \tag{2.2}$$

   which implies that $(H,*)$ is a $\theta\sigma\theta^{-1}$-generalized Bol loop.

3. Since $x \cdot y = \alpha(x) \circ \beta(y)$ and $(G, \cdot, \sigma)$ is a σ-generalized Bol loop, then

$$(xy \cdot z)\sigma(y) = x(yz \cdot \sigma(y))$$

$$\iff [(\alpha(x) \circ \beta(y)) \cdot z]\sigma(y) = x \cdot [(\alpha(y) \circ \beta(z)) \circ \sigma(y)] \tag{2.3}$$

$$\iff \alpha[\alpha\{(\alpha(x)) \circ \beta(y)\} \circ \beta(z)] \circ \beta\sigma(y) = \alpha(x) \circ \beta[\alpha\{(\alpha(y)) \circ \beta(z)\} \circ \beta\sigma(y)] \tag{2.4}$$

Let $\alpha(x) = \bar{x}, \beta(y) = \bar{y}$ then $y = \beta^{-1}(\bar{y}), \beta(z) = \bar{z}$ and use these in equation (2.1) to get

$$\alpha[\alpha(\bar{x} \circ \bar{y}) \circ \bar{z}] \circ \beta\sigma\beta^{-1}(\bar{y}) = \bar{x} \circ \beta[\alpha\{\alpha\beta^{-1}(\bar{y}) \circ \bar{z}\} \circ \beta\sigma\beta^{-1}(\bar{y})] \tag{2.5}$$

Put $\bar{x} = 1_\circ$ in equation (2.5), to get

$$\alpha[(\alpha(\bar{y}) \circ \bar{z})] \circ \beta\sigma\beta^{-1}(\bar{y}) = \beta[\alpha\{\alpha\beta^{-1}(\bar{y}) \circ \bar{z}\} \circ \beta\sigma\beta^{-1}(\bar{y})] \tag{2.6}$$

By substituting (2.6) into the right side of equation (2.5), we have

$$\alpha[\alpha(\bar{x} \circ \bar{y}) \circ \bar{z}] \circ \beta\sigma\beta^{-1}(\bar{y}) = \bar{x} \circ [\alpha(\alpha(\bar{y}) \circ \bar{z}) \circ \beta\sigma\beta^{-1}(\bar{y})] \tag{2.7}$$

Putting $\bar{y} = 1_\circ$ and $\sigma_1 = \beta\sigma\beta^{-1}$ in equation (2.7), we have

$$\alpha(\alpha(\bar{x}) \circ \bar{z}) \circ \sigma_1(1_\circ) = \bar{x} \circ [\alpha(\alpha(1_\circ) \circ \bar{z}) \circ \sigma_1(1_\circ)] \tag{2.8}$$

which gives

$$\alpha(\alpha(\bar{x}) \circ \bar{z}) = \bar{x} \circ [\alpha(\alpha(1_\circ) \circ \bar{z})] \tag{2.9}$$

Let

$$\alpha(\alpha(1_\circ) \circ \bar{z}) = \delta(\bar{z}) \tag{2.10}$$

where δ is some bijections on G. By substituting equation (2.10) in equation (2.9), we get

$$\alpha[\alpha(\bar{x}) \circ \bar{z}] = \bar{x} \circ \delta(\bar{z}) \tag{2.11}$$

By substituting equation (2.11) in (2.7), we get

$$[(\bar{x} \circ \bar{y}) \circ \delta(\bar{z})] \circ \sigma_1(\bar{y}) = \bar{x} \circ [(\bar{y} \circ \delta(\bar{z})) \circ \sigma_1(\bar{y})] \tag{2.12}$$

Replacing $\bar{x}$ by x,$\bar{y}$ by y,$\delta(\bar{z})$ by z in equation (2.12), we have

$$[(x \circ y) \circ z] \circ \sigma_1(y) = x \circ [(y \circ z) \circ \sigma_1(y) \tag{2.13}$$

which means that $(G, \circ, \sigma_1)$ is a σ_1-generalized Bol loop.

□

Corollary 2.1. *Let $(G, \cdot, \sigma)$ be a σ-generalized Bol loop. A loop isotope $(H, *, \sigma_2)$ of $(G, \cdot, \sigma)$ is a σ_2-generalized Bol loop if $\sigma_2(e_H) = e_H$ where σ_2 is some conjugate of σ.*

Proof. Let $(H, *, \sigma_2)$ be a loop isotope of a σ-generalized Bol loop $(G, \cdot, \sigma)$ and let the triple (A, B, C) be an isotopism from $(G, \cdot, \sigma)$ to $(H, *, \sigma_2)$. There exists a principal isotope $(G, \circ)$ of $(G, \cdot)$ under the isotopism (α, β, I) such that $(G, \circ) \overset{\gamma}{\cong} (H, *)$. By 2. of Theorem 2.1, $(G, \circ, \sigma_1)$ is a σ_1-generalized Bol loop if and only if

$$[(\bar{x} * \bar{y}) * \bar{z}] * \gamma\sigma_1\gamma^{-1}(\bar{y}) = \bar{x} * [(\bar{y} * \bar{z}) * \gamma\sigma_1\gamma^{-1}(\bar{y})] \tag{2.14}$$

if and only if $(H, *, \sigma_2)$ is a σ_2-generalized Bol loop with a single valued self map $\sigma_2 = \gamma\sigma_1\gamma^{-1}$ on H.

Let $1_\circ$ and 1_* be the identity elements of $(G, \circ)$ and $(H, *)$ respectively. Then, given that $\sigma_2(1_*) = 1_*$, we argue that $\sigma_1(1_\circ) = \gamma^{-1}\sigma_2\gamma(1_\circ) = \gamma^{-1}\sigma_2(1_*) = \gamma^{-1}(1_*) = 1_\circ$. By 3. of Theorem 2.1, $(G, \circ, \sigma_1)$ is a σ_1-generalized Bol loop. Therefore $(H, *, \sigma_2)$ is a σ_2-generalized Bol loop. □

Corollary 2.2. *Let $(G, \cdot, \sigma)$ be a σ-generalized Bol loop such that $\sigma \in AUM(G, \cdot)$. If $(G, \cdot, 1) \xrightarrow[Isotopism]{(\alpha, \beta, I)} (G, \circ, 1_\circ)$ such that $\beta(1) = 1_\circ$, then $\left(G, \circ, \beta\sigma\beta^{-1}\right)$ is a $\beta\sigma\beta^{-1}$-generalized Bol loop.*

Proof. This follows from 3. of Theorem 2.1. □

Corollary 2.3. *Let* $(G,\cdot,\sigma)$ *be a* σ*-generalized Bol loop such that* $\sigma \in Inn(G,\cdot)$. *If* $(G,\cdot,1) \xrightarrow[Isotopism]{(\alpha,\beta,I)} (G,\circ,1_\circ)$ *such that* $\beta(1)=1_\circ$, *then* $\Big(G,\circ,\beta\sigma\beta^{-1}\Big)$ *is a* $\beta\sigma\beta^{-1}$*-generalized Bol loop.*

Proof. $Inn(H)$ is the inner mapping group of H. The rest follows from 3. of Theorem 2.1. □

Corollary 2.4. *Let* $(G,\cdot,\sigma)$ *be a* σ*-generalized Bol loop such that* $\sigma = R_{(x,y)}$ or $\sigma = L_{(x,y)}$ *or* $\sigma = T_{(x)}$ *for any fixed* $x,y \in H$. *If* $(G,\cdot,1) \xrightarrow[Isotopism]{(\alpha,\beta,I)} (G,\circ,1_\circ)$ *such that* $\beta(1)=1_\circ$, *then* $\Big(G,\circ,\beta\sigma\beta^{-1}\Big)$ *is a* $\beta\sigma\beta^{-1}$*-generalized Bol loop.*

Proof. $R_{(x,y)}, L_{(x,y)}, T_{(x)}$ are the right, left and middle inner mappings of $(G,\cdot)$. The rest follows from 3. Theorem 2.1. □

Theorem 2.2. *Let* G *be a* σ*-generalized Bol loop. Any loop isotope* H *of* G *with identity element* e *is a* ω*-generalized Bol loop if and only if* $\omega(e) \in N_\rho(H)$ *where* ω *is some conjugate of* σ.

Proof. Let $(H,*)$ be an arbitrary loop isotope of a generalized Bol loop $(G,\cdot,\sigma)$ and let the triple (A,B,C) be an isotopism from $(G,\cdot)$ to $(H,*)$. There exists a principal isotope $(G,\circ)$ of $(G,\cdot)$ under the isotopism (α,β,I) such that $(G,\circ) \overset{\gamma}{\cong} (H,*)$. Let $1_\circ$ and 1_* be the identity elements of $(G,\circ)$ and $(H,*)$ respectively.

Since $x\cdot y = \alpha(x)\circ\beta(y)$ and $(G,\cdot,\sigma)$ is a σ-generalized Bol loop, then we shall now follow the procedure of the proof of Theorem (2.1) from (2.7) to (2.13). At (2.8), we assume that $\sigma_1(1_\circ) \in N_\rho(G,\circ)$ which gives (2.13) where $\sigma_1 = \beta\sigma\beta^{-1}$. Thus, $(G,\circ,\sigma_1)$ is a σ_1-generalized Bol loop. Also note that $\sigma_2 = \gamma\sigma_1\gamma^{-1}$ and so $\sigma_2 = \gamma\beta\sigma(\gamma\beta)^{-1}$. Therefore $(H,*,\sigma_2)$ is a σ_2-generalized Bol loop. In fact, $(G,\circ,\sigma_1)$ is a σ_1-generalized Bol loop if and only if $(H,*,\sigma_2)$ is a σ_2-generalized Bol loop.

Now, assuming that $(G,\circ,\sigma_1)$ is a σ_1-generalized Bol loop, then

$$[(x\circ y)\circ z]\circ\sigma_1(y) = x\circ[(y\circ z)\circ\sigma_1(y)] \tag{2.15}$$

Substitute $y = 1_\circ$ in (2.15) to get $(x \circ z) \circ \sigma_1(1_\circ) = x \circ (z \circ \sigma_1(1_\circ))$ which implies that $\sigma_1(1_\circ) \in N_\rho(G, \circ)$.

Thus, we have shown that $(G, \circ, \sigma_1)$ is a σ_1-generalized Bol loop if and only if $(H, *, \sigma_2)$ is a σ_2-generalized Bol loop if and only if $\sigma_1(1_\circ) \in N_\rho(G, \circ)$.

Recall that $(G, \circ) \overset{\gamma}{\cong} (H, *)$. So, $\sigma_1(1_\circ) \in N_\rho(G, \circ) \Leftrightarrow \gamma\sigma_1(1_\circ) \in N_\rho(H, *)$
$\Leftrightarrow \sigma_2\gamma(1_\circ) \in N_\rho(H, *) \Leftrightarrow \sigma_2(1_*) \in N_\rho(H, *)$.

Therefore, $(H, *, \sigma_2)$ is a σ_2-generalized Bol loop if and only if $\sigma_2(1_*) \in N_\rho(H, *)$. $\square$

Corollary 2.5. *Let G be a σ-generalized Bol loop. Any loop isotope H of G is a σ'-generalized Bol loop if and only if $g\backslash\sigma(g) \in N_\rho(G)$ for some $g \in G$ where σ' is some conjugate of σ.*

Proof. From Theorem 2.2, $(G, \circ, \sigma_1)$ is a σ_1-generalized Bol loop if and only if $(H, *, \sigma_2)$ is a σ_2-generalized Bol loop if and only if $\sigma_1(1_\circ) \in N_\rho(G, \circ)$. It is known that $(\alpha, \beta, I) = (R_g, L_f, I)$ for some $f, g \in G$. Thus, $\sigma_1 = \beta\sigma\beta^{-1} = L_f\sigma L_f^{-1}$. It is also known that $N_\rho(G, \cdot) \overset{(L_gL_f)}{\cong} N_\rho(G, \circ)$, which implies that $N_\rho(G, \circ) \overset{(L_gL_f)^{-1}}{\cong} N_\rho(G, \cdot)$.

So, $\sigma_1(1_\circ) \in N_\rho(G, \circ) \Leftrightarrow L_f\sigma L_f^{-1}(1_\circ) = L_f\sigma L_f^{-1}(fg) \in N_\rho(G, \circ) \Leftrightarrow f\sigma(g) \in N_\rho(G, \circ) \Leftrightarrow [f\sigma(g)]L_f^{-1}L_g^{-1} \in N_\rho(G, \cdot) \Leftrightarrow g\backslash\sigma(g) \in N_\rho(G)$. $\square$

Theorem 2.3. *Let H be a subgroup of a group G and let $< g_1, g_2 >= g_1g_2g_1^{-1}g_2^{-1}$, $[g_1, g_2] = g_1^{-1}g_2^{-1}g_1g_2$ and $g_2^{g_1} = g_1^{-1}g_2g_1$ denote the left commutator, right commutator of $g_1, g_2 \in G$ and conjugate of g_2 by g_1 respectively. Define $'\circ'$ on $H \times G$ such that for all $x, y \in A$, $x = (h_1, g_1)$ and $y = (h_2, g_2)$,*

$$x \circ y = (h_1, g_1) \circ (h_2, g_2) = (h_1h_2, h_2g_1h_2^{-1}g_2) \tag{2.16}$$

Let $\sigma : A \to A \uparrow \sigma(h, g) = (\delta_1 h, \delta_2 g)$ where $\delta_1, \delta_2 : G \longrightarrow G$ are single valued mappings. The following are true.

1. *$(A, \circ)$ is a group if and only if $< h, h' > h'hg = h'hg\left\langle h^{-1}, h'^{-1}\right\rangle$ for all $h, h' \in H$ and $g \in G$.*

2. *If H is an abelian subgroup of G, then $(A, \circ)$ is a group.*

3. *$(A,\circ,\sigma)$ is a σ-generalized Bol loop if and only if* $\left[g,\left(\delta_1(h')h'h\right)^{-1}\right]=\left[g,\left(hh'\delta_1(h')\right)^{-1}\right]$ *for all $h,h'\in H$ and $g\in G$.*

4. *If for all $h_1,h_2\in H$,*

$$< h_1,h_2 >=\left\langle(\delta_1 h_2)^{-1},h_2h_1\right\rangle \tag{2.17}$$

then, $(A,\circ,\sigma)$ is a σ-generalized Bol loop.

Proof.

1. It is easy to check that $(A,\circ)$ is a loop with identity (e,e), where e is the identity element of G. Let $x=(h_1,g_1), y=(h_2,g_2)$ and $z=(h_3,g_3)$ be elements of A. Then,

$$\begin{aligned} x\circ(y\circ z) &= (h_1,g_1)\circ[(h_2,g_2)\circ(h_3,g_3)]\\ &= (h_1,g_1)\circ[h_2g_3,h_3g_2h_3^{-1}g_3]\\ &= (h_1h_2h_3,h_2h_3g_1(h_2h_3)^{-1}h_3g_2h_3^{-1}g_3)\end{aligned}$$

$$= (h_1h_2h_3,h_2h_3g_1h_3^{-1}h_2^{-1}h_3g_2h_3^{-1}g_3) \tag{2.18}$$

Also,

$$\begin{aligned} (x\circ y)\circ z &= [(h_1,g_1)\circ(h_2,g_2)]\circ(h_3,g_3)\\ &= (h_1h_2,h_2g_1h_2^{-1}g_2)\circ(h_3,g_3)\end{aligned}$$

$$= (h_1h_2h_3,h_3h_2g_1h_2^{-1}g_2h_3^{-1}g_3). \tag{2.19}$$

For all $h,h'\in H$, note that $[h',h]=\left\langle h'^{-1},h^{-1}\right\rangle$. So by (2.18) and (2.19), $(A,\circ)$ is a group if and only if

$$\begin{aligned} &x\circ(y\circ z)=(x\circ y)\circ z \Leftrightarrow h_2h_3g_1h_3^{-1}h_2^{-1}h_3=h_3h_2g_1h_2^{-1}\Leftrightarrow h_3h_2\\ &=h_2h_3g_1h_3^{-1}h_2^{-1}h_3h_2g_1^{-1}\Leftrightarrow< h_3,h_2> h_2h_3\\ &=h_2h_3g_1[h_3,h_2]g_1^{-1}\Leftrightarrow< h_3,h_2> h_2h_3g_1=h_2h_3g_1\left\langle h_3^{-1},h_2^{-1}\right\rangle\end{aligned}$$

2. This follows from 1.

3. Given that $\sigma(h,g)=(\delta_1 h,\delta_2 g)$ where $\delta_1,\delta_2:G\longrightarrow G$ are single valued mappings:

$x \circ y = (h_1 \circ g_1) \circ (h_2, g_2) = (h_1h_2, h_2g_1h_2^{-1}g_2),$
$[(x \circ y) \circ z] = (h_1h_2, h_2g_1h_2^{-1}g_2) \circ (h_3, g_3)$
$= (h_1h_2h_3, h_3h_2g_1h_2^{-1}g_2h_3^{-1}g_3)[(x \circ y) \circ z] \circ \sigma(y)$
$= [(x \circ y) \circ z] \circ \sigma(h_2, g_2)$
$= [(x \circ y) \circ z] \circ (\delta_1h_2, \delta_2g_2)$
$= [h_1h_2h_3, h_3h_2g_1h_2^{-1}g_2h_3^{-1}g_3] \circ (\delta_1h_2, \delta_2g_2)$

$$= [h_1h_2h_3(\delta_1h_2), (\delta_1h_2)h_3h_2g_1h_2^{-1}g_2h_3^{-1}g_3(\delta_1h_2)^{-1}(\delta_2g_2)] \tag{2.20}$$

Similarly,

$y \circ z = (h_2, g_2) \circ (h_3, g_3) = (h_2h_3, h_3g_2h_3^{-1}g_3).$
$(y \circ z) \circ \sigma(y) = (y \circ z) \circ \sigma(h_2, y_2) = (y \circ z) \circ (\delta h_2, \delta g_2)$
$= (h_2h_3, h_3g_2h_3^{-1}g_3 \circ (\delta_1h_2, \delta_2g_2)$
$= \Big(h_2h_3(\delta_1h_2), (\delta h_2)h_3g_2h_3^{-1}g_3(\delta_1h_2)^{-1}(\delta_2g_2)\Big).$
$x \circ [(y \circ z) \circ \sigma(y)] = (h_1, g_1) \circ [(y \circ z) \circ \sigma(y)]$
$= (h_1, g_1) \circ \Big(h_2h_3(\delta_1h_2), (\delta_1h_2)h_3g_2h_3^{-1}g_3(\delta_1h_2)^{-1}(\delta_2g_2)\Big)$
$= \Big(h_1h_2h_3(\delta_1h_2), h_2h_3(\delta_1h_2)g_1(h_2h_3(\delta_1h_2))^{-1}(\delta_1h_2)h_3g_2h_3^{-1}$
$g_3(\delta_1h_2)^{-1}(\delta_2g_2)\Big)$

$$= \Big(h_1h_2h_3(\delta_1h_2), h_2h_3(\delta_1h_2)g_1(\delta_1h_2)^{-1}h_3^{-1}h_2^{-1}(\delta_1h_2)h_3g_2h_3^{-1} \\ g_3(\delta_1h_2)^{-1}(\delta_2g_2)\Big) \tag{2.21}$$

$(A, \circ, \sigma)$ is a σ-generalized Bol loop if and only if (2.20) and (2.21) are equal. This is true if and only if

$(\delta_1h_2)h_3h_2g_1h_2^{-1}g_2h_3^{-1}g_3(\delta_1h_2)^{-1}(\delta_2g_2) =$
$h_2h_3(\delta_1h_2)g_1(\delta_1h_2)^{-1}h_3^{-1}h_2^{-1}(\delta_1h_2)h_3g_2h_3^{-1}g_3(\delta_1h_2)^{-1}(\delta_2g_2) \Leftrightarrow$
$(\delta_1h_2)h_3h_2g_1h_2^{-1} = h_2h_3(\delta_1h_2)g_1(\delta_1h_2)^{-1}h_3^{-1}h_2^{-1}(\delta_1h_2)h_3 =$
$h_2h_3(\delta_1h_2)g_1[h_2h_3(\delta_1h_2)]^{-1}(\delta_1h_2)h_3 \Leftrightarrow$
$(\delta_1h_2)h_3h_2g_1h_2^{-1}h_3^{-1}(\delta_1h_2)^{-1} = h_2h_3(\delta_1h_2)g_1[h_2h_3(\delta_1h_2)]^{-1} \Leftrightarrow$

$$(\delta_1h_2)h_3h_2g_1[(\delta_1h_2)h_3h_2]^{-1} = h_2h_3(\delta_1h_2)g_1[h_2h_3(\delta_1h_2)]^{-1} \tag{2.22}$$

$g_1^{((\delta_1h_2)h_3h_2)^{-1}} = g_1^{(h_2h_3(\delta_1h_2))^{-1}} \Leftrightarrow g_1\Big[g_1, ((\delta_1h_2)h_3h_2)^{-1}\Big]$
$= g_1\Big[g_1, (h_2h_3(\delta_1h_2))^{-1}\Big] \Leftrightarrow \Big[g_1, ((\delta_1h_2)h_3h_2)^{-1}\Big] = \Big[g_1, (h_2h_3(\delta_1h_2))^{-1}\Big]$

4. Equation (2.22) is true if

$$(\delta_1 h_2)h_3h_2 = h_2h_3(\delta_1 h_2) \Leftrightarrow h_3h_2 = (\delta_1 h_2)^{-1}(h_2h_3)(\delta_1 h_2) \Leftrightarrow$$
$$h_3h_2h_3^{-1}h_2^{-1} = (\delta_1 h_2)^{-1}(h_2h_3)(\delta_1 h_2)h_3^{-1}h_2^{-1}$$
$$= (\delta_1 h_2)^{-1}(h_2h_3)(\delta_1 h_2)(h_2h_3)^{-1} \Leftrightarrow$$

$$< h_3, h_2 >= \left\langle (\delta_1 h_2)^{-1}, h_2h_3 \right\rangle \tag{2.23}$$

Therefore, $(A, \circ, \sigma)$ is a σ-generalized Bol loop.

□

Theorem 2.4. *Let H be a subgroup of a group G and $A = H \times G$. Define $'\circ'$ on $H \times G$ such that for all $x, y \in A$, $x = (h_1, g_1)$ and $y = (h_2, g_2)$,*

$$x \circ y = (h_1, g_1) \circ (h_2, g_2) = (h_1h_2, h_2g_1h_2^{-1}g_2) \tag{2.24}$$

Let $\sigma : A \to A \uparrow \sigma(h, g) = (\delta_1 h, \delta_2 g)$ where $\delta_1, \delta_2 : G \longrightarrow G$ are single valued mappings. Then, $(A, \circ, \sigma)$ is a σ-generalized Bol loop if δ_1 fixes the elements of H pointwisely.

Proof. The proof is similar to the proof of Theorem 2.3 up till equation (2.22). □

Theorem 2.5. *Let H be a subgroup of a group G,*
$AUM_H(G) = \{\alpha \in AUM(G) | x\alpha = x \ \forall \ x \in H\}$, $G_H = \{x \in G | x\alpha = x \ \forall \ \alpha \in AUM_H(G)\}$,
$A = G_H \times G$ and $B = H \times G$. Define $'\circ'$ on A such that for all $x, y \in A$, $x = (h_1, g_1)$ and $y = (h_2, g_2)$,

$$x \circ y = (h_1, g_1) \circ (h_2, g_2) = (h_1h_2, h_2g_1h_2^{-1}g_2) \tag{2.25}$$

Let $\sigma : A \to A \uparrow \sigma(h, g) = (\delta_1 h, \delta_2 g)$ where $\delta_1, \delta_2 : G \longrightarrow G$ are single valued mappings and $\delta_1 \in AUM_H(G)$. Then,

1. *$AUM_H(G) \leq AUM(G)$ and $AUM_H(G)$ is a H-automorphism group of G.*
2. *$H \leq G_H \leq G$ and G_H is a fixed subgroup of $AUM_H(G)$.*
3. *$(A, \circ, \sigma)$ and $(B, \circ, \sigma)$ are σ-generalized Bol loops.*
4. *$(B, \circ, \sigma)$ is a σ-generalized Bol subloop of $(A, \circ, \sigma)$.*

Proof.

1. and 2. are easy. The proof of 3. follows from Theorem 2.4. $\square$

Theorem 2.6. *An RCC σ-generalized Bol loop is a σ-generalized RC-loop.*

Proof. Let $(Q,\cdot,\sigma)$ be an RCC σ-generalized Bol loop, then $A = (R_x, L_x^{-1}R_x, R_x), B = (R_{x^{-1}}, L_xR_{\sigma(x)}, R_{\sigma(x)}) \in AUT(Q,\cdot)$ for all $x \in Q$. Thus, we have

$$\begin{aligned} C &= AB = (R_x, L_x^{-1}R_x, R_x)(R_x^{-1}L_xR_{\sigma(x)}, R_{\sigma(x)}) \\ &= (R_xR_{x^{-1}}, L_x^{-1}R_xL_xR_{\sigma(x)}, R_xR_{\sigma(x)}) \\ &= (I, L_x^{-1}R_xL_xR_{\sigma(x)}, R_xR_{\sigma(x)}) \in AUT(Q,\cdot)\ forall\ x \in Q. \end{aligned}$$

Since Q is a RIPL, then

$$C_\mu = (R_xR_{\sigma(x)}, J_\rho L_x^{-1}R_xL_xR_{\sigma(x)}J_\rho, I) \in AUT(Q,\cdot). \tag{2.26}$$

which implies that for all $y, z \in Q$,

$$yR_xR_{\sigma(x)} \cdot zJ_\rho L_x^{-1}R_xL_xR_{\sigma(x)}J_\rho = yz \tag{2.27}$$

Put $y = e$ in (2.27),

$eR_xR_{\sigma(x)} \cdot zJ_\rho L_x^{-1}R_xL_xR_{\sigma(x)}J_\rho = z \Longrightarrow x \cdot \sigma(x) \cdot zJ_\rho L_x^{-1}R_xL_xR_{\rho(x)}J_\rho$
$= z \Longrightarrow zJ_\rho L_x^{-1}R_xL_xR_{\sigma(x)}J_\rho L_{(x\cdot\sigma(x))}$
$= z \Longrightarrow zJ_\rho L_x^{-1}R_xL_xR_{\sigma(x)}J_\rho = zL^{-1}_{(x\cdot\sigma(x))} \Longrightarrow$

$$J_\rho L_x^{-1}R_xL_xR_{\sigma(x)}J_\rho = L^{-1}_{(x\cdot\sigma(x))} \tag{2.28}$$

By putting (2.28) into (2.26), we have $C_\mu = (R_xR_{\sigma(x)}, L^{-1}_{x\cdot\sigma(x)}, I) \in AUT(Q,\cdot)$ for all $x \in Q$. Thus, $yR_xR_{\sigma(x)} \cdot zL^{-1}_{(x\cdot\sigma(x))} = yz$ for all $y, z \in Q$.

Let $\overline{z} = zL^{-1}_{x\cdot\sigma(x)}$ then $z = \overline{z}L_{(x\cdot\sigma(x))}$. Then

$yR_xR_{\sigma(x)} \cdot \overline{z} = y \cdot \overline{z}L_{(x\cdot\sigma(x))} \Longrightarrow (yx \cdot \sigma(x)) \cdot \overline{z} = y((x\cdot\sigma(x))\overline{z})$
which is equation (1.10). Therefore, $(Q,\cdot,\sigma)$ is a generalized σ-RC-loop. $\square$

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