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On the cohomological equation of a linear contraction *

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Abstract

In this paper, we study the discrete cohomological equation of a contracting linear automorphism A of the Euclidean space $\mathbf{R}^d$. More precisely, if δ is the cobord operator defined on the Fréchet space $E = C^l(\mathbf{R}^d)$ $(0 \leq l \leq \infty)$ by: $\delta(h) = h - h \circ A$, we show that:

• *If $E = C^0(\mathbf{R}^d)$, the range $\delta(E)$ of δ has infinite codimension and its closure is the hyperplane E_0 consisting of the elements of E vanishing at 0. Consequently, $H^1(A, E)$ is infinite dimensional non Hausdorff topological vector space and then the automorphism A is not cohomologically C^0-stable.*

• *If $E = C^l(\mathbf{R}^d)$, with $1 \leq l \leq \infty$, the space $\delta(E)$ coincides with the closed hyperplane E_0. Consequently, the cohomology space $H^1(A, E)$ is of dimension 1 and the automorphism A is cohomologically C^l-stable.*

Keywords: *Fréchet space, Cohomological equation*

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1. Introduction

Let M be a connected differentiable manifold. The space $E = C^l(M)$ of C^l functions on M $(0 \leq l \leq \infty)$ is a Fréchet space for the C^l topology (the topology of the uniform convergence of all the derivatives up to the order l, on compact subsets). A C^l action of a discrete group Γ (supposed of finite presentation) on M induces a natural action on E given by:

$$\forall \gamma \in \Gamma,\ \forall f \in E,\ \gamma.f = f \circ \gamma^{-1}.$$

This makes E a Γ-module. Then one can consider the cohomolgy $H^*(\Gamma, E)$ of the discrete group Γ with values in E.

One can show that $H^*(\Gamma, E) = 0$ for $* \geq 1$ in the case Γ is finite [6] or Γ acting freely and properly on M [5]. In the case Γ is generated by a single element γ we can easily show that the space $H^1(\Gamma, E)$ that we will denote $H^1(\gamma, E)$ is the Cokernel of the cobord operator:

$$\delta : E \to E,\ f \mapsto \delta(f) = f - f \circ \gamma$$

The calculation of the Cokernel $E/\delta(E)$ of δ amounts to solving the following equation:

$$f - f \circ \gamma = g \text{ where } \begin{cases} f \in E \ \text{ is unknown} \\ \ \text{and} \\ g \in E \ \text{ is given} \end{cases}$$

called the cohomological equation associated to the discrete dynamical system (M, γ). We say that the automorphism $\gamma : M \to M$ is cohomologically C^l-stable when $\delta(E)$ is a closed subspace of E, that is when the topological vector space $H^1(\gamma, E) = E/\delta(E)$ is Hausdorff [2].

Different works give an idea of what may represent this cohomological equation in some areas of mathematics. For instance that of D. V. Anosov [1], A. Avila and A. Kocsard [2], A. Dehghan-Nezhad and A. El Kacimi [3], Katok [7] and S. Marmi, P. Moussa, J.-C. Yoccoz [8].

In what follows, we study this cohomological equation in the case of a linear automorphism:

$$\gamma : \mathbf{R}^d \to \mathbf{R}^d,\ x \mapsto \gamma(x) = Ax$$

which satisfies the following contraction property:

$$\|A\| := \sup_{\|x\|=1} \|Ax\| < 1.$$

(Of course, the vector space $\mathbf{R}^d$ ($d \in \mathbf{N}^*$) being provided by its canonical Euclidean structure and its associated norm $||\cdot||$.)

We consider the Fréchet space $E = C^l(\mathbf{R}^d)$ ($0 \leq l \leq \infty$) and its subspace:

$$E_0 = \{h \in E, \mid h(0) = 0\}$$

We are interested in the image $\delta(E)$ of the cobord operator:

$$\delta : E \to E, \; h \mapsto \delta(h) = h - h \circ A$$

and the discrete cohomological equation associated to the dynamical system $(\mathbf{R}^d, \gamma)$:

$$\textbf{(e)} \qquad f - f \circ \gamma = g \;\text{ where }\; \begin{cases} f \in E := C^l(\mathbf{R}^d) \;\text{ is unknown} \\ \text{and} \\ g \in E \;\text{ is given} \end{cases}$$

The purpose of this work is to establish the following results:

1.1. Theorem

Let E be the Fréchet space $E = C^0(\mathbf{R}^d)$. Then:

(i) *The cohomological equation (e) admits a solution for the data g if and only if the series $\sum_{k\geq 0} g \circ A^k$ converges in E.*

In addition, the linear operator $S : \delta(E) \to E, \; g \mapsto S(g) = \sum_{k\geq 0} g \circ A^k$ satisfies $\delta\,[S(g)] = g$ for all $g \in \delta(E)$.

(ii) *The range $\delta\,(E)$ of δ has infinite codimension and its closure is the hyperplane E_0 consisting of the elements of E vanishing at 0. Consequently, $H^1\,(A, E)$ is infinite dimensional non Hausdorff topological vector space and then the automorphism A is not cohomologically C^0-stable.*

1.2. Theorem

Let E be the Fréchet space $E = C^l(\mathbf{R}^d)$ where $1 \leq l \leq \infty$. Then:

(i) *The cohomological equation (e) admits a solution for the data g if and only if $g(0) = 0$. In other words, the range of the cobord operator $\delta : E \to E, \; h \mapsto \delta(h) = h - h \circ \gamma$ is exactly the subspace E_0.*

(ii) *The operator δ induces, by restriction, an automorphism of the Fréchet space E_0 having for inverse the operator $S : E_0 \to E_0,\ g \mapsto S(g) = \sum_{k\geq 0} g \circ A^k$.*

(iii) *The cohomology space $H^1(A, E) = E/\delta(E)$ is of dimension 1. As a consequence, the automorphism A is cohomologically C^l-stable.*

Before starting the proofs of the two theorems above, we recall some notions - specially in functional analysis - that we will use. Most of them can be found in Walter Rudin's book [9].

2. Preliminary notions

2.1. Fréchet spaces in terms of semi-norms

A semi-norm on a $\mathbf{K}$-vector space E ($\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$) is a function $p : E \to \mathbf{R}$ satisfying the two properties: $p(x + y) \leq p(x) + p(y)$ and $p(\lambda x) = |\lambda| p(x)$ for all $(x, y) \in E \times E$ and $\lambda \in \mathbf{K}$.
We have, in particular, $p(0) = 0$ and $p(x) \geq 0$ for all $x \in E$ but the separation property "$p(x) = 0 \Rightarrow x = 0$" is missing for p to be a norm.

It often happens that one has a separating family $(p_i)_{i\in I}$ of semi-norms on E that is to say such as:

$$\forall x \in E \setminus \{0\},\ \exists i \in I,\ p_i(x) \neq 0$$

Such a family of semi-norms provides E with a Hausdorff topological vector space structure for which the functions $p_i : E \to \mathbf{R}$ are continuous. It suffices to define a neighborhood of a point $u \in E$ as a part of E containing a subset of the form:

$$B_J(u) := \{x \in E \mid \forall j \in J,\ p_j(x - u) < \varepsilon_j\} = \bigcap_{j\in J} \underbrace{\{x \in E \mid p_j(x - u) < \varepsilon_j\}}_{B_j(u,\varepsilon_j)}$$

where J is a finite part of I and $\varepsilon_j \in]0, +\infty[$ for $j \in J$. We say that E is a *locally convex topological vector space.*

In the case of a countable and separating family $(p_n)_{n\in\mathbf{N}}$ of semi-norms, we can show that the induced topology by the semi-norms coincides with that induced by the metric:

$$d(x,y) = \sum_{n=0}^{+\infty} \frac{1}{2^n} \min\left(1, p_n(x-y)\right) \quad \text{for} \quad (x,y) \in E \times E$$

This metric d is clearly invariant by the translations $T_u : E \to E,\ x \mapsto x + u$ $(u \in E)$.

A Fréchet space is a topological vector space E whose topology can be defined by a countable and separating family of semi-norms $(p_n)_{n\in\mathbf{N}}$ such that the metric space (E,d) is complete.

Let's give examples of Fréchet spaces that we will use in this article:

2.2. Examples

Let Ω be a nonempty open of a space $\mathbf{R}^d$ $(d \in \mathbf{N}^*)$. A sequence (K_n) of compact subsets of Ω is called exhaustive, if for all n, $K_n \subset \circ K_{n+1}$ and $\bigcup\limits_{n\in\mathbf{N}} K_n = \Omega$. One can obtain such a sequence by taking $K_n = \{x \in \mathbf{R}^d \mid \|x\| \leq n \text{ et } d(x, \mathbf{R}^d \setminus \Omega) \geq \frac{1}{n}\}$ for $n \in \mathbf{N}^*$.

(i) Let E be the vector space $C^l(\Omega)$ of real C^l-functions on Ω $(l \in \mathbf{N})$. We can provide this space with a countable and separating family $(p_n)_{n\in\mathbf{N}^*}$ of semi-norms by considering an exhaustive sequence $(K_n)_{n\in\mathbf{N}^*}$ of compact subsets of Ω and taking :

$$p_n(f) = \max_{|\alpha|\leq l}\left(\max_{x\in K_n} |D^\alpha f(x)|\right)$$

where $\alpha = (\alpha_1, ..., \alpha_d) \in \mathbf{N}^d$, $|\alpha| = \alpha_1 + ... + \alpha_d$ and $D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}}$. Moreover, E is a Fréchet space for the topology induced by these semi-norms.

(ii) Similarly, the vector space $E = C^\infty(\Omega)$ of real C^∞-functions on Ω is a Fréchet space for the topology induced by the family $(p_n)_{n\in\mathbf{N}^*}$ of semi-norms:

$$p_n(f) = \max_{|\alpha|\leq n}\left(\max_{x\in K_n} |D^\alpha f(x)|\right) \ for f \in E$$

These Fréchet structures are independent of the choice of exhaustive sequence (K_n).

2.3. Basic properties

In a Fréchet space E, where the topology is defined by a countable and separating family $(p_n)_{n\in\mathbf{N}}$ of semi-norms, we have the following basic assertions:

1. A sequence $(u_k)_k$ converges to a limit u in E if and only if for any semi-norm p_n, the sequence of real numbers $(p_n(u_k - u))_k$ converges to zero.

2. A sequence $(u_k)_k$ is Cauchy sequence in E if and only if, it is for any semi-norm p_n.

3. The absolute convergence of a series $\sum_k u_k$ (i.e. the convergence of the series of real numbers $\sum_k p_n(u_k)$ for each semi-norm p_n), implies the convergence of this series in E.

4. If V is a vector subspace of E, with topological vector space structure induced by that of E, a linear map $T : V \to E$ is continuous if and only if for any semi-norm p_n, there exists an integer $N \in \mathbf{N}$ and a real constant $C > 0$ such that

$$\forall u \in V,\ p_n\left(T(u)\right) \leq C \max_{0\leq k\leq N} \left(p_k(u)\right)$$

3. proofs of the two main theorems

Let E be the Fréchet space $E = C^l(\mathbf{R}^d)$, with $0 \leq l \leq \infty$. We consider the discrete cohomological equation in E:

$$(e)\ :\ f - f\circ A = g$$

and the two linear operators: $\Delta : u \in E \longmapsto u(0) \in \mathbf{R}$ and $\delta : u \in E \longmapsto u - u\circ A \in E$.
The closed balls $K_n = \overline{B}_n(0,n)$ $(n \in \mathbf{N}^*)$ in $\mathbf{R}^d$ form an exhaustive sequence of compact subsets of $\mathbf{R}^d$ and we can therefore provide, as in Examples 2.1, the vector space E of its Fréchet structure where the topology is defined by the separating family of semi-norms $(p_n)_{n\geq 1}$ such that:

$$\forall n \in \mathbf{N}^*,\ \forall u \in E,\ p_n(u) = \max_{|\alpha|\leq l}\left(\max_{x\in K_n} |D^\alpha u(x)|\right)\ \text{ if }\ E = C^l(\mathbf{R}^d)$$

with $0 \leq l < \infty \forall n \in \mathbf{N}^*,\ \forall u \in E,\ p_n(u) = \max_{|\alpha|\leq n}\left(\max_{x\in K_n} |D^\alpha u(x)|\right)$ if $E = C^\infty(\mathbf{R}^d)$

3.1. Remarks

1. $\forall n \in \mathbf{N}^*,\ A(K_n) \subset K_n$. Indeed, $\forall x \in K_n,\ \|A(x)\| \leq \|A\|.\|x\| \leq \|x\|$ because $\|A\| < 1$.

2. The equation (e) admits a solution for the data g, if and only if $g \in \delta(E)$.

3. If the equation (e) admits a solution f for the data g, then $g(0) = f(0) - f(0) = 0$.

4. If f is a solution of the equation (e), then for all $c \in \mathbf{R},\ f + c$ is also a solution of (e).

3.2. Lemma

1. Let $B = (b_{ij})$ be a real square matrix of order d. For all $l \in \mathbf{N},\ u \in C^l(\mathbf{R}^d), \alpha \in \mathbf{N}^d$, such as $|\alpha| \leq l$, and $x \in \mathbf{R}^d$; We have :

$$|D_\alpha (u \circ B)(x)| \leq d^l \|B\|^{|\alpha|} \max_{|\beta| \leq l} |D_\beta u(Bx)|$$

2. For any function $g \in C^0(\mathbf{R}^d)$ which is differentiable at point 0 and such that $g(0) = 0$, the function series $\sum_k g \circ A^k$ converge in $C^0(\mathbf{R}^d)$.

3. The operators $\Delta : E \to \mathbf{R},\ u \mapsto \Delta(u) = u(0)$ and $\delta : E \to E,\ u \mapsto \delta(u) = u - u \circ A$ are continuous.

Proof of the lemma:

1. Let's proceed by recurrence on the integer l. The property being true for $l = 0$, let's say it's true for l then show that it is still true for $l + 1$. For that consider an element $u \in C^{l+1}(\mathbf{R}^d)$ and a multi-index $\alpha = (\alpha_1, ..., \alpha_d) \in \mathbf{N}^d$ such that $|\alpha| = \alpha_1 + ... + \alpha_d \leq l + 1$. Since the property is checked for $\alpha = 0$, we can assume that $1 \leq |\alpha| \leq l + 1$. Moreover, even to swap two components of α, we can assume that

$\alpha_d \geq 1$. We then have for all $x \in \mathbf{R}^d$,

$$\begin{aligned} D_\alpha(u \circ B)(x) &= \left(\tfrac{\partial}{\partial x_1}\right)^{\alpha_1} ... \left(\tfrac{\partial}{\partial x_d}\right)^{\alpha_d} (u \circ B)(x) \\ &= D_{\alpha'}\left[\tfrac{\partial}{\partial x_d}(u \circ B)\right](x) \text{ with } \alpha' = (\alpha_1, \alpha_2, ..., \alpha_d - 1) \\ &\quad \text{and } |\alpha'| = |\alpha| - 1 = l \\ &= D_{\alpha'}\left[\textstyle\sum_{i=1}^d b_{id} \tfrac{\partial u}{\partial x_i} \circ B\right](x) \\ &= \textstyle\sum_{i=1}^d b_{id} D_{\alpha'}\left[\tfrac{\partial u}{\partial x_i} \circ B\right](x) \end{aligned}$$

So, we have:

$$\begin{aligned} |D_\alpha(u \circ B)(x)| &= \left|\textstyle\sum_{i=1}^d b_{id} D_{\alpha'}\left(\tfrac{\partial u}{\partial x_i} \circ B\right)(x)\right| \\ &\leq \textstyle\sum_{i=1}^d |b_{id}| \left|D_{\alpha'}\left(\tfrac{\partial u}{\partial x_i} \circ B\right)(x)\right| \\ &\leq \textstyle\sum_{i=1}^d |b_{id}| d^l \|B\|^{|\alpha|-1} \max_{|\beta| \leq l} \left|D_\beta\left(\tfrac{\partial u}{\partial x_i}\right)(Bx)\right| \\ &\quad \text{(recurrence hypothesis for the } \tfrac{\partial u}{\partial x_i}) \\ &\leq \textstyle\sum_{i=1}^d |b_{id}| d^l \|B\|^{|\alpha|-1} \max_{|\beta| \leq l+1} |D_\beta u(Bx)| \\ &\quad \text{with } |b_{id}| \leq \|B\| \\ &\leq d^{l+1} \|B\|^{|\alpha|} \max_{|\beta| \leq l+1} |D_\beta u(Bx)| \end{aligned}$$

Which proves that the property is still true to the rank $l+1$. So it's true for all l.

2. Let $g \in C^0(\mathbf{R}^d)$ such that g is differentiable in 0 and $g(0) = 0$. We consider the sequence (S_n) of partial sums $S_n = \sum_{k=0}^{n} g \circ A^k$.

   We have $\lim_{x \to 0} \underbrace{\dfrac{g(x) - dg(0).x}{\|x\|}}_{\varphi(x)} = 0$ where the function $\varphi : x \mapsto \varphi(x) = \dfrac{g(x) - dg(0).x}{\|x\|}$, with $\varphi(0) = 0$, is continuous on $\mathbf{R}^d$.

$\forall x \in \mathbf{R}^d,\ g(x) = dg(0).x + \|x\|\varphi(x)$. So, for all strictly positive integers n and j and for any compact K_r, we have:

$$\begin{aligned}
\forall x \in K_r,\ |S_{n+j}(x) - S_n(x)| &= \left|\sum_{k=n+1}^{n+j} g(A^k x)\right| \\
&= \left|\sum_{k=n+1}^{n+j} \left[dg(0).(A^k x) + \|A^k x\|\varphi(A^k x)\right]\right| \\
&\leq \sum_{k=n+1}^{n+j} \left[\|dg(0)\| r \|A\|^k + r\|A\|^k p_r(\varphi)\right] \\
&\leq r\left(\|dg(0)\| + p_r(\varphi)\right) \sum_{k=n+1}^{n+j} \|A\|^k \\
&\leq r\left(\|dg(0)\| + p_r(\varphi)\right) \frac{\|A\|}{1-\|A\|}.\|A\|^n
\end{aligned}$$

Since we have $\|A\| < 1$, then for all $r \in \mathbf{N}^*$, the sequence (S_n) is Cauchy sequence for the semi-norm p_r of $C^0(\mathbf{R}^d)$. The series $\sum_k g \circ A^k$ is therefore convergent in $C^0(\mathbf{R}^d)$.

3. **Continuity of Δ :** $\forall u \in E,\ |\Delta(u)| = |u(0)| \leq \max_{x \in K_1} |u(x)| \leq p_1(u)$, which proves that the linear form Δ is continuous on $E = C^0(\mathbf{R}^d)$ (resp. $E = C^l(\mathbf{R}^d)$, with $1 \leq l \leq \infty$).

   Continuity of δ :

   For $n \in \mathbf{N}^*$ and $u \in E$, we have :

$$A(K_n) \subset K_n \text{ and } p_n(\delta(u)) = p_n(u - u \circ A) \leq p_n(u) + p_n(u \circ A)$$

   • In the case $E = C^0(\mathbf{R}^d)$, we have :

$$p_n(\delta(u)) \leq \max_{x \in K_n} |u(x)| + \max_{x \in K_n} |u(A(x))| \leq 2p_n(u)$$

   • In the case $E = C^l(\mathbf{R}^d)$, with $1 \leq l < \infty$, we have for all $x \in K_n$ and for all multi-index α, such as $|\alpha| \leq l$,

$$\begin{aligned}
|D_\alpha(u \circ A)(x)| &\leq d^l \|A\|^{|\alpha|} \max_{|\beta| \leq l} |D_\beta u(Ax)| \\
&\quad \text{according to point (i) above} \\
&\leq d^l \max_{|\beta| \leq l} \left(\max_{t \in K_n} |D_\beta u(t)|\right) \\
&\leq d^l p_n(u)
\end{aligned}$$

which proves that $p_n(u \circ A) \leq d^l p_n(u)$ and that $p_n(\delta(u)) \mid \leq \left(1 + d^l\right) p_n(u)$.

• In the case $E = C^\infty(\mathbf{R}^d)$, we have for all $x \in K_n$ and for all multi-index α, such as $|\alpha| \leq n$,

$$\begin{aligned} |D_\alpha(u \circ A)(x)| &\leq d^n \|A\|^{|\alpha|} \max_{|\beta| \leq n} |D_\beta u(Ax)| \\ & \text{according to point (i) above} \\ &\leq d^n \max_{|\beta| \leq n} \left(\max_{t \in K_n} |D_\beta u(t)|\right) \\ &\leq d^n p_n(u) \end{aligned}$$

which proves that $p_n(u \circ A) \leq d^n p_n(u)$ and that $p_n(\delta(u)) \mid \leq (1 + d^n) p_n(u)$.

In any case, we have proved that for all $n \in \mathbf{N}^*$, there is a constant C_n such as

$$\forall u \in E,\ p_n(\delta(u)) \leq C_n p_n(u)$$

The linear operator δ is therefore continuous in all cases.

3.3. Proof of Theorem 1 (the case $E = C^0(\mathbf{R}^d)$)

1. Suppose that the equation (e) admits a solution f for the data g. Then $f - f \circ A = g$, $g(0) = 0$ and $\forall k \in \mathbf{Z},\ f \circ A^k - f \circ A^{k+1} = g \circ A^k$.

   In particular, by adding member to member these equalities from rank $k = 0$ up to rank $k = n$, for each $n \in \mathbf{N}^*$, we get:

$$\forall n \in \mathbf{N}^*,\ \sum_{k=0}^{n} g \circ A^k = f - f \circ A^{n+1}$$

   or:

$$\forall n \in \mathbf{N}^*,\ \forall x \in \mathbf{R}^d,\ \sum_{k=0}^{n} g(A^k x) = f(x) - f(A^{n+1} x).$$

   Since $\|A\| < 1$ and f is a continuous function, we have:

$$\forall x \in \mathbf{R}^d,\ \lim_{n \to +\infty} A^{n+1} x = 0 \ \text{ and so } \lim_{n \to +\infty} f(A^{n+1} x) = f(0)$$

   from where :

$$\lim_{n \to +\infty} \sum_{k=0}^{n} g(A^k x) = f(x) - f(0).$$

This implies that the series $\sum_{k\geq 0} g\circ A^k$ converges simply to the solution $f-f(0)$ of (e) taking the value 0 in 0. Since the space E is complete, it suffices to prove that the sequence of partial sums $S_n = \sum_{k=0}^{n} g\circ A^k$ is Cauchy sequence in E for all semi-norm p_r $(r\in \mathbf{N}^*)$.

Let r be a fixed positive integer and $\varepsilon > 0$ a real number. Since the function f is uniformly continuous on the compact ball $K_r = \overline{B}(0,r)$, there exists a real number $\alpha > 0$ such that:

$$\forall (x,y)\in K_r^2,\ \|x-y\| < \alpha \Rightarrow |f(x)-f(y)| < \varepsilon$$

On the other hand, there is a rank $N\in\mathbf{N}$ such that $\|A\|^n r < \frac{\alpha}{2}$ as soon as $n\geq N$. So for two integers n and j, with $n\geq N$, we have:

$$\forall x\in K_r,\ |S_{n+j}(x)-S_n(x)| = |f(A^{n+1}x)-f(A^{n+j+1}x)|$$

with:

$$|A^{n+1}x - A^{n+j+1}x| \leq 2\|A\|^n r < \alpha.$$

This implies $p_r\left(S_{n+j}-S_n\right) < \varepsilon$.

We deduce that: for all $r\in\mathbf{N}^*$ and for all real number $\varepsilon > 0$, there is a rank $N\in\mathbf{N}$ such as $p_r\left(S_{n+j}-S_n\right) < \varepsilon$ for $n\geq N$ and j arbitrary. This shows that the sequence (S_n) is a Cauchy sequence in E and therefore convergent in E, from which we deduce the convergence of the series $\sum_k g\circ A^k$.

Conversely, if the series $\sum_k g\circ A^k$ converges in E, the sequence (S_n) converges in E to a limit f and then $\lim_{n\to+\infty} g\circ A^n(0) = 0$ which implies $g(0)=0$.

On the other hand, the linear operator $\delta : E\to E,\ h\mapsto \delta(h) = h - h\circ A$ being continuous, we have:

$$\lim_{n\to+\infty} \delta\left(S_n\right) = \delta(f) \ \text{ or else } \ \lim_{n\to+\infty}\left(g - g\circ A^{n+1}\right) = f - f\circ A$$

Now for a fixed integer $r\in\mathbf{N}^*$, we have :

$$p_r(g\circ A^{n+1}) = \sup_{x\in K_r} |g\circ A^{n+1}(x)| = |g\circ A^{n+1}(x_n)| = |g(A^{n+1}x_n)|$$

for a certain point $x_n \in K_r$ with $\|A^{n+1}x_n\| \leq \|A\|^{n+1}r$. Then:

$$\lim_{n\to+\infty} p_r(g \circ A^{n+1}) = \lim_{n\to+\infty} |g(A^{n+1}x_n)| = |g(0)| = 0.$$

We deduce that $\lim\limits_{n\to+\infty} g \circ A^{n+1} = 0$ in E and the function f satisfies the relation $g = f - f \circ A$, that is the equation (e) admits f as solution. We have proved above that for $g \in \delta(E)$, the series $\sum\limits_k g \circ A^k$ converges in E and its sum $f = S(g)$ is a solution of (e).

Let's now show the density of $\delta(E)$ in E_0. We have just seen that the space $\delta(E)$ is exactly the subspace of E_0 consisting of the functions h such that the series $\sum_k h \circ A^k$ converges in E. Consider the subspace G of E_0 consisting of functions which are differentiable at point 0.

According to Lemma 3.2-(ii), we have the inclusion $G \subset \delta(E)$ and according to Lemma 3.2-(iii), $E_0 = Ker(\Delta)$ is closed in E. So, we have the following inclusions:

$$G \subset \delta(E) \subset E_0 \;\text{ et }\; \overline{G} \subset \overline{\delta(E)} \subset E_0$$

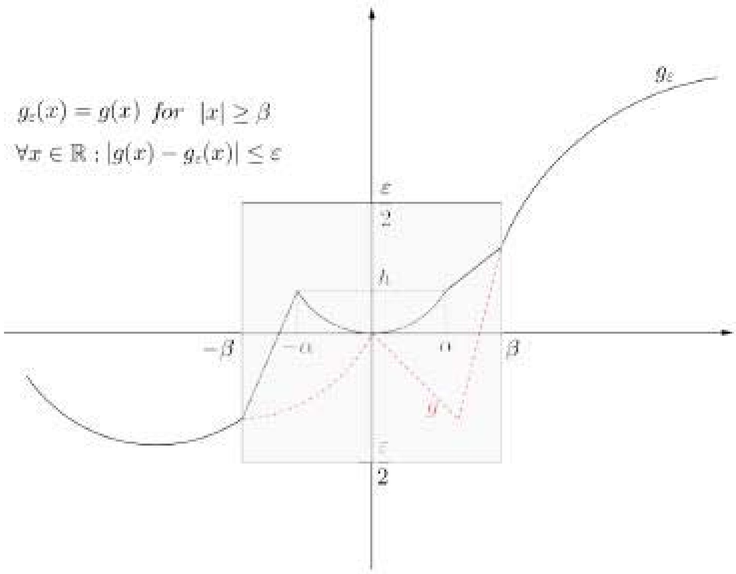
To show that $\delta(E)$ is dense in E_0, we just have to prove that G is dense in E_0.

Let $g \in E_0$, $r \in \mathbf{N}^*$ and $\varepsilon > 0$. The function g being continuous and checking $g(0) = 0$, there is a real number $\beta > 0$ such as: $\forall x \in \mathbf{R}^d,\ \|x\| < \beta \Rightarrow |g(x)| \leq \dfrac{\varepsilon}{2}$.

Consider a real number α such that $0 < \alpha < \beta$ and the function g_ε defined by:

$$g_\varepsilon : \mathbf{R}^d \to \mathbf{R},\ x \mapsto g_\varepsilon(x) = \begin{cases} \frac{\varepsilon}{2\alpha^2}\|x\|^2 & \text{if} \quad \|x\| < \alpha \\ \frac{\varepsilon(\beta - \|x\|)}{2\alpha^2(\beta-\alpha)}\|x\|^2 + \frac{\|x\|-\alpha}{\beta-\alpha} g(x) & \text{if} \quad \alpha \leq \|x\| \leq \beta \\ g(x) & \text{if} \quad \|x\| > \beta \end{cases}$$

Here is the graphic representation of the function g_ε for $d = 1$.



g_ε is clearly continuous on $\mathbf{R}^d$ and C^1 on the ball $B(0,\alpha)$. Moreover, we can easily check that we have: $\forall x \in \mathbf{R}^d,\ |g_\varepsilon(x) - g(x)| \leq \varepsilon$.

In particular, $p_r\,(g_\varepsilon - g) \leq \varepsilon$. This proves that any neighborhood of g in E_0 contains at least one element g_ε of G. Hence the space G is dense in E_0 and then so is $\delta(E)$ in E_0. The point (i) of Theorem 1 is thus proved.

2. The reduced cohomology space $\overline{H}^1(A,E) = E/\overline{\delta(E)} = E/E_0$ is of dimension 1. The cohomology space $H^1(A,E) = E/\delta(E)$ is not Hausdorff (because $\delta(E)$ is not closed in E) and this space is infinite dimensional since the infinite family $(h_p)_{p\geq 1}$ of continuous functions defined by:

$$h_p : \mathbf{R}^d \to \mathbf{R},\ x \mapsto h_p(x) = \begin{cases} \left[\ell_n\left(\frac{1+\|x\|}{\|x\|}\right)\right]^{-\frac{1}{p}} & \text{if} \quad x \neq 0 \\ 0 & \text{if} \quad x = 0 \end{cases}$$

forms a system of linearly independent vectors in the quotient vector space $E/\delta(E)$. Let us prove this fact.

• There is a unit vector ξ such as for all $p \geq 1$, the series $\sum_{k\geq 0} h_p \circ A^k(\xi)$ diverges. Indeed, if $z \in \mathbf{C}^d$ is an eigenvector of A associated to an eigenvalue $\lambda \in \mathbf{C}$, then the two vectors $x = \frac{z+\bar{z}}{2}$ and $y = \frac{z-\bar{z}}{2i}$ of $\mathbf{R}^d$ are not all zero and are such that $\|A^k x\| = |\lambda|^k \|x\|$ and $\|A^k y\| = |\lambda|^k \|y\|$. Assuming, for example, that $x \neq 0$ and setting $\xi = \frac{x}{\|x\|}$, we will have, for all $n \in \mathbf{N}^*$ and $k \in \mathbf{N}$,

$$h_p(A^k\xi) = \left[\ell_n\left(\frac{1+|\lambda|^k}{|\lambda|^k}\right)\right]^{-\frac{1}{p}}$$

$$= \left(-\ell_n|\lambda|^k\right)^{-\frac{1}{p}}\left[\frac{\ell_n(1+|\lambda|^k)}{-\ell_n|\lambda|^k}+1\right]^{-\frac{1}{p}} \sim \left(\ell_n\frac{1}{|\lambda|}\right)^{-\frac{1}{p}}\frac{1}{k^{\frac{1}{p}}}$$

where the Riemann series $\displaystyle\sum_k \frac{1}{k^{\frac{1}{p}}}$ diverges. Hence the series $\displaystyle\sum_k h_p(A^k\xi)$ diverges also.

• If α and β are real numbers such as $h = \alpha h_p + \beta h_q \in \delta(E)$, with $p < q$, then for the vector ξ above,

$$\lim_{k\to+\infty}\frac{h\circ A^k(\xi)}{h_q\circ A^k(\xi)} = \lim_{k\to+\infty}\left(\alpha\,\frac{h_p\circ A^k(\xi)}{h_q\circ A^k(\xi)}+\beta\right)$$

$$= \lim_{k\to+\infty}\left[\alpha\,\ell_n\left(\frac{1+|\lambda|^k}{|\lambda|^k}\right)^{-\frac{q-p}{pq}}+\beta\right] = \beta$$

If $\beta \neq 0$, the sequences $\left(h_q\circ A^k(\xi)\right)_k$ and $\left(\frac{1}{\beta}h\circ A^k(\xi)\right)_k$ would be equivalent and the associated series would therefore be of the same nature, which is absurd. We deduce that $\beta = 0$ and then $\alpha = 0$.

It can thus be shown by induction that a finite linear combination $\sum_p \alpha_p h_p$ is a cobord if and only if all the scalars α_p are zero.

3.4. Proof of Theorem 2 (the case $E = C^l(\mathbf{R}^d)$, with $1 \leq l \leq \infty$)

1. If the equation (e) admits a solution f for the data g,

$$g(0) = f(0) - f\circ A(0) = f(0) - f(0) = 0 \quad \text{and so} \quad g \in E_0$$

Conversely for $g \in E_0$, we show that the series $\displaystyle\sum_k g\circ A^k$ converges in E and its sum f is a solution of equation (e).

Let (S_n) be the sequence of partial sums of the series $\sum_k g \circ A^k$. Let's show that (S_n) is cauchy sequence in E.

Let r be a positive integer and $\alpha = (\alpha_1, \cdots, \alpha_d) \in \mathbf{N}^d$ a multi-index such that $1 \leq |\alpha| \leq l$.

For any (n, j) where n and j are positive integers and any $x \in K_r$, we have:

$$\begin{aligned}
|D_\alpha(S_{n+j} - S_n)(x)| &= \left| D_\alpha \left(\textstyle\sum_{k=n+1}^{n+j} g \circ A^k \right)(x) \right| \\
&= \left| \textstyle\sum_{k=n+1}^{n+j} D_\alpha (g \circ A^k)(x) \right| \\
&\leq \textstyle\sum_{k=n+1}^{n+j} \left| D_\alpha (g \circ A^k)(x) \right| \\
&\leq \textstyle\sum_{k=n+1}^{n+j} d^l \|A^k\|^{|\alpha|} \left| (D_\alpha g)(A^k x) \right| \\
&\quad \text{(according to Lemma 3.2-(i))} \\
&\leq \textstyle\sum_{k=n+1}^{n+j} d^l \|A\|^k \max_{|\alpha| \leq l} \left(\max_{t \in K_r} |(D_\alpha g)(t)| \right) \\
&\leq d^l \max_{|\alpha| \leq l} \left(\max_{t \in K_r} |(D_\alpha g)(t)| \right) \textstyle\sum_{k=n+1}^{n+j} \|A\|^k \\
&\leq d^l \max_{|\alpha| \leq l} \left(\max_{t \in K_r} |(D_\alpha g)(t)| \right) \|A\|^{n+1} \tfrac{1 - \|A\|^j}{1 - \|A\|} \\
&\leq d^l \max_{|\alpha| \leq l} \left(\max_{t \in K_r} |(D_\alpha g)(t)| \right) \|A\|^{n+1} \tfrac{1}{1 - \|A\|}
\end{aligned}$$

Similarly, since $g(0) = 0$ and g is differentiable at point 0, then (as in the proof of Lemma 3.2-(ii)) we have:

$$\max_{x \in K_r} |S_{n+j}(x) - S_n(x)| \leq \underbrace{\left[\|dg(0)\| + \max_{x \in K_r} (\varphi(x)) \right] \frac{r}{1 - \|A\|}}_{\eta_r} \|A\|^{n+1}$$

where φ is the continuous function: $\varphi(x) = \dfrac{g(x) - dg(0).x}{\|x\|}$ for $x \neq 0$ and $\varphi(0) = 0$.

It follows that:

- if $1 \leq l < \infty$, $p_r(S_{n+j} - S_n) \leq \left[\eta_r + \dfrac{d^l p_r(g)}{1 - \|A\|} \right] \|A\|^{n+1}$

• if $l = \infty$, $p_r(S_{n+j} - S_n) \leq \left[\eta_r + \dfrac{d^r p_r(g)}{1 - \|A\|}\right] \|A\|^{n+1}$

In both cases, there is a constant C_r such that

$$\forall r \in \mathbf{N}^*,\ \forall (n,j) \in \mathbf{N}^* \times \mathbf{N}^*,\ p_r(S_{n+j} - S_n) \leq C_r \|A\|^{n+1}$$

This proves that (S_n) is Cauchy sequence for any semi-norm p_r and so converges in E to a limit f. As in the case of the space $C^0(\mathbf{R}^d)$, the function f is a solution of (e).

2. We have $\delta(E) = E_0$ and $E_0 = Ker(\Delta)$ is a Fréchet space, as a closed subspace of the Fréchet space E. On the other hand, we have both endomorphisms:

$$\delta : E_0 \to E_0,\ h \mapsto \delta(h) = h - h \circ A \ \text{ and } \ S : E_0 \to E_0,\ h \mapsto S(h) = \sum_{k \geq 0} h \circ A^k$$

such as $\delta^{-1} = S$, δ is continuous (according to Lemma 3.2-(iii)) and S is continuous as the simple limit of continuous operators $S_n : E_0 \to E_0,\ h \mapsto S_n(h) = \sum_{k=0}^{n} h \circ A^k$ on E_0 [9].

3. $\delta(E) = E_0$ being a closed hyperplan of E, the cohomology space $H^1(A, E) = E/E_0$ has dimension 1 and the automorphism A is cohomologically C^l-stable.

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