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A kind of characterization of homeomorphism and homeomorphic spaces by Core fundamental groupoid: a good invariant

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Abstract

In this paper, we give a new topological invariant and a kind of characterization to homeomorphisms. We introduce this topological invariant by constructing an algebraic kind of groupoid structure to each topological space, which is an extension of the concept of fundamental groups. This construction we call it "Core fundamental groupoid" and it is different from the fundamental groupoid as in the book and articles of Ronald Brown [30, 31]. Moreover, both groupoid notions are not equivalent, but, it will be a wide subgroupoid of the fundamental groupoid in category-theoretic. In the entire paper, we consider groupoid in the algebraic sense. First, we discuss basic properties of the Core fundamental groupoid and their significant importance in topological invariants. Further, we separately present the Core fundamental groupoid as an algebraic structure and a topo-algebraic structure and investigate their properties separately. We have an explicit description of both the algebraic structure of the groupoid and a unique topological structure of Core fundamental groupoid. Besides, we give a kind of characterization of homeomorphisms in terms of an invariant that we have obtained, i.e., "$f : M \to N$ is a homeomorphism if and only if $f_{\#} : \bar{\pi}_1 M \to \bar{\pi}_1 N$ is a topological groupoid isomorphism". This result addresses the open problem stated in [1, 19, 24, 26] and also gives a kind of characterization for homeomorphic spaces, but, computationally it seems difficult and it becomes sometimes trivial when intact with topology. Induced groupoid homomorphism, induced base map, topological and smooth structure on the Core fundamental groupoid, homotopic properties of induced groupoid homomorphisms of continuous maps and topological groupoid homomorphism-related properties of induced groupoid homomorphisms are discussed. We also present the relation of homotopy type on Core fundamental groupoid.

Keywords: Groupoid, Fundamental group, Fundamental groupoid, Topological groupoid, Induced homomorphism, Bundle.

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1. Introduction

Classification plays a significant role in any branch of science. Classification of the topological spaces is an active research area and that becomes a reason for the rigorous advances in various allied branches of Mathematics and Science. If we consider the entire class of topological spaces, some mathematicians remarked that it seems impossible to obtain a useful classifying invariant, but, a considerable member of them are trying a good one (we are also in the same category of people who are searching for such invariant). However, finding invariants which classify topological spaces up to homeomorphism is a basic problem of topology, in fact, still an open problem in topology [1, 19, 24, 26]. If we see historically, existing topological invariants are not sufficient topological invariants in general, but some are known to be sufficient invariants to classify spaces when we restrict the family of spaces under study. For instance, the fundamental group is a sufficient invariant for compact orientable surfaces without boundaries. Besides, classifying the continuous maps up to homotopy and homotopy type spaces known as 'CW' complexes has been successfully resolved with the help of homotopy theory. Algebraic topology is a branch of topology in which we study of problems of topology with the help of algebra. Here, one can commonly see association of the algebraic structures to topological spaces. We can find wide applications of algebraic topology in Classification theory, Combinatorics, Computer Engineering and theoretical Physics. The stated open problem was partially solved using certain invariants, such as homotopy, homology, cohomology, Euler characteristics etc. and indeed, these invariants witnessed the advances of algebraic topology in the twentieth century [3, 22, 24]. These invariants played a significant role in the development of allied branches of Mathematics. But, they are not sufficient topological invariants, for example, homotopy equivalence is weaker than homeomorphism.

Henri Poincare introduced the term "homeomorphism" in 1895. Also, he invented homotopy, fundamental group and homology group around 1895 (Analysis Situs, 1895) [24]. The fundamental group is a heavily used invariant by many algebraic topologists and geometers and its explicit explanation of such usage with applications be found also in an article [17] of Heinrich Tietze's. Further, many studies have concentrated on interacting possible topologies on the topological invariants(on algebraic structured sets) which come from the theory of homotopy, one such development has been introduced by Hurwitz [20]. The literature says, in 1950, J. Dugundji viewed the fundamental group as a topological space by introducing topology on it [12]. By generalizing the results of J. Dugundji, Biss introduced a quotient topology inherited from $Hom((S^1, 1), (X, x))$ with the compact-open topology on the fundamental group [6]. This space is denoted as ${\pi_1}^{top}$ and it is a topological invariant finer than the usual fundamental group [20].

Classification of spaces plays a vital role not only in topology but also in geometry. Therefore, this becomes the reason for interest in finding such invariants of smooth manifolds, Lie groups and Riemannian manifolds [18, 23]. A similar study for smooth manifolds was initiated by R. Hamilton around 1980s by introducing the Ricci flow technique [28]. Such invariants of smooth manifolds also act as topological invariants since smooth manifolds are topological spaces and diffeomorphism implies homeomorphism. But, modern geometry has developed a lot of invariants, from which certain kinds of spaces are partially classified. Also, by confining to specific spaces (like Manifolds) with some weaker conditions, a complete classification in geometry took place [26, 27]. One such classification problem was resolved by Russian mathematician Grigori Perelman in 2006 and he proved Poincare's conjecture, which is one of the classification problems of geometry and topology [14, 24]. But, up to diffeomorphism, not all smooth manifolds are completely classified. Characterizing homeomorphism cannot imply characterization of diffeomorphism, but this would be a necessary condition. Such developments will gain significant progress in Physics through Topology, Knot theory and Geometry.

Let M and N be two closed hyperbolic 3-manifolds, Mostow rigidity implies that M and N are homeomorphic if and only if their fundamental groups are isomorphic [27, 34]. The authors of [34] appealed to Sela's solution of the isomorphism problem for torsion-free word hyperbolic groups, the homeomorphism problem for M and N and the mentioned can be solved [34]. Further, the initial objective of this [27] paper was to provide a more geometrical approach to the homeomorphism problem. But the authors Peter Scott and Hamish gave a more geometric approach to an algorithm for deciding whether two hyperbolic 3-manifolds are homeomorphic, besides, they also gave a more algebraic approach to some other parts of the homeomorphism problem. Therefore, the fundamental group is sufficient for the homeomorphisms in their work, but it is not sufficient in general topological spaces. We can see this fact by a simple example, i.e., the real line and a closed interval in the real line are not homeomorphic, even though their fundamental groups are isomorphic. Therefore, we have considered this problem as an interesting case in characterizing homeomorphism in terms of a good invariant. We are trying certain associations to topological space with the help of groupoid structure and we will demand it as a good topological invariant for classification.

The notion of a groupoid is an algebraic concept, this appeared for the first time in 1926 by H. Brandt, which is a generalization of the group and having certain beautiful identities [15]. C. Ehresmann generalized Brandt groupoids in which topological and differentiable structures are added [9]. Many applications of the structure of a groupoid are available in different branches of science, for instance, one can see that mainly in the name of partial actions (in the actual name of groupoid actions) in many problems. The notion of groupoid could help as an invariant as Ronald Brown mentioned in the name of "fundamental groupoid" as in the categorical sense in his book "Topology and groupoid" [31]. He has discussed the fundamental groupoid (category sense) as a topological invariant and placed a lot of its basic properties along with applications. In this context and the considered problem, we have tried this investigation to arrive at a new invariant. Moreover, defined invariant has significant applications in allied branches of both topology and geometry. Interestingly, it takes a role in the construction of covering spaces of certain topological spaces and the detailed discussion of a special quotient topology on such non-trivial construction is discussed in [7].

In this present detailed investigation, we have introduced an algebraic version of a groupoid and named it as Core fundamental groupoid, which we felt is a good topological invariant. We have discussed the algebraic structure and also a natural topological structure induced by the topology of base space on the introduced Core fundamental groupoid. And, we have presented a characterization of homeomorphism through this good invariant. This is a kind of solution to the considered open problem, and an explanation of it is in Proposition 4.30.

2. Preliminaries

In the entire paper, we denote by $(M, \mathcal{I}_M)$ or whenever there is no confusion, simply M for a topological space. Generally, $\pi_1(M, x)$ denotes the fundamental group of topological space M and for a point $x \in M$ [3, 19, 22, 24]. Throughout this paper, we denote γ_x for a loop based at x and $\overline{\gamma}$ for the reverse of the path γ and c_x to the constant loop based at x. Indeed, the path homotopy equivalence class $[c_x]$ is the identity element in the fundamental group. For each continuous map $f : M \to N$ and $x \in M$, there is an induced homomorphism and which is defined by $f_{\#x} : \pi_1(M, x) \to \pi_1(N, f(x))$ by $f_{\#x}([\gamma_x]) = [f \circ \gamma_x]$ for all equivalence classes of loops based at x, i.e., $[\gamma_x] \in \pi_1(M, x)$ [3, 19, 22, 24].

A non-empty set G associated with $*^{-1} : G \to G$ a unary operation and $* : G \times G \to G$ a partial function, but not a binary operation satisfying i) Associativity: If $a * b$ and $b * c$ are defined then $a * (b * c)$ and $(a * b) * c$ are defined and in such case $a * (b * c) = (a * b) * c$, ii) Inverse: $a^{-1} * a$ and $a * a^{-1}$ are always defined. iii) Identity: If $a * b$ is defined, then $a * b * b^{-1} = a$ and $a^{-1} * a * b = b$ are always defined, is called a groupoid [2, 4, 5, 10, 13, 21, 25, 29]. Generally, one can see that $(a^{-1})^{-1} = a$ and $(a * b)^{-1} = b^{-1} * a^{-1}$ for defined $a * b$, are the often used properties in groupoid [21]. An element $e \in G$ is called an identity if there exists $e * g$ implies $e * g = g$ and there exists $g' * e$ implies $g' * e = g'$. Commonly, G_0 denotes set of all identities of groupoid G, it is called identity set of G. There are some important definitions and results, which are to be used later.

Definition 2.1. *[2, 11, 13, 21] A subgroupoid H of G is called wide if $H_0 = G_0$.*

Definition 2.2. *[5, 11, 13, 21] Let G, G' be groupoids under partial functions $*$ and $*'$ respectively, a map $T : G \to G'$ is called a groupoid homomorphism if $\forall a, b \in G$ and $a * b$ defined implies $T(a) *' T(b)$ defined, in such case $T(a * b) = T(a) *' T(b)$.*

Definition 2.3. *[13, 21] Let G, G' be groupoids a map $T : G \to G'$ is called groupoid isomorphism if it is bijective and both T and T^{-1} are groupoid homomorphisms.*

Definition 2.4. *[21] Let G, G' be groupoids, $T : G \to G'$ be a groupoid homomorphism and $H \subset G$ and $F \subset G'$ then $T(H) = \{T(a) \in G' : \text{for } a \in G\}$, $T^{-1}(F) = \{a \in G : T(a) \in F\}$, also $Ker(T) = \{a \in G : T(a) \in {G'}_0\}$.*

Proposition 2.5. *[11] Let G, G' be groupoids and $T : G \to G'$ be a groupoid homomorphism, then $T(a^{-1}) = (T(a))^{-1}, \forall a \in G$.*

Corollary 2.6. *[11] Let G, G' be groupoids and $T : G \to G'$ be a groupoid homomorphism, then $T(a * a^{-1}) = T(a) *' (T(a))^{-1}, \forall a \in G$.*

Proposition 2.7. *[5, 21] Let G, G' be groupoids and $T : G \to G'$ be a groupoid homomorphism and* $\mathrm{Ker}(T) \subset G_0$ *then T is a groupoid monomorphism.*

Proposition 2.8. *The composition of two groupoid homomorphisms is a groupoid homomorphism.*

Definition 2.9. *[10, 25, 29, 32] A topological groupoid is a groupoid $(G, *)$ together with a topology on G such that unary operation and its partial function are continuous.*

Definition 2.10. *[33] Let G, G' be two topological groupoids and a groupoid homomorphism $T : G \to G'$ is said to be a topological groupoid homomorphism if T is continuous. And if both T and T^{-1} are topological groupoid homomorphism, then it is called a topological groupoid isomorphism.*

In this paper, we call a groupoid homomorphism $T : G \to G'$ is a trivial whenever the $T(G) \subseteq {G'}_0$. There is an interesting result in groupoid theory that, the disjoint union of two groups forms a groupoid under the operation of restriction operations on elements from the same group and a unary operation is just an inverse operation on the respective groups [11]. Besides, another result is that two group homomorphisms yield a groupoid homomorphism from the disjoint union of groups of domains to disjoint union of groups of the codomains of those group homomorphisms and which is restricted under respective group homomorphisms [11].

3. Core Fundamental groupoid: a topological invariant

All topological invariants are not necessarily algebraic structures. But, some of them are indeed such structures and they enabled us to partially settle the homeomorphism problem and lead to key points in classification theory. We define an algebraic structure associated with the topological spaces that we have claimed and have shown a good topological invariant. The theory of homotopy and fundamental groups are well-known for a long back, which has prominent applications in topology and geometry [3, 19, 22, 24]. The fundamental group is one, that acts as a notable invariant in topology.

Now, we define Core fundamental groupoid of a given topological space M. Define Core fundamental groupoid as the disjoint union of the fundamental groups at points of M and will be denoted by $\bar{\pi}_1 M$ and mathematically $\bar{\pi}_1 M = \bigcup_{x \in M} \pi_1(M, x)$. Most of the topological properties of space can indeed be absorbed with only the help of loops and their class (path homotopy). In fact, for instance, we are seeing this in the fundamental group. Studying topological spaces employing all paths and their class is found in the fundamental groupoid. But, the same kind of information can yield only by collecting core parts out of fundamental groupoid. That is, either only a set of all loops or only their classes contain the most of all information about space, which indeed is the core part of data associated with the space as well as the core part of them out of the association "fundamental groupoid". Therefore, construction $\bar{\pi}_1 M = \bigcup_{x\in M} \pi_1(M,x)$ contains the said one and it is a minimum data to study a topological space for many good reasons and applications. Hence the name is natural to have on $\bar{\pi}_1 M$ as Core fundamental groupoid and also its notation.

We consider groupoid in the algebraic sense, not as a category (Groupoid is a small category in which every morphism is invertible [31, 32]). In addition, the defined algebraic sense of this Core fundamental groupoid is different from the existing fundamental groupoid that contains path homotopy equivalence classes of paths in M, which forms a groupoid in the category sense [30, 31, 32, 33]. But in the category point of view, this Core fundamental groupoid becomes a wide subgroupoid of fundamental groupoid.

The defined Core fundamental groupoid structure will be an extension of the fundamental group. This new notion seems like a bundle structure as in differential geometry as well as topology (tangent bundles, cotangent bundle, fibre bundle, etc.), but intuitively provides better applications for many areas of Mathematics. As we have, the defined Core fundamental groupoid is a set that contains path homotopy classes of loops based at all points of the topological space M. It is an arbitrary union of the fundamental groups of a topological space. Using the induced homomorphism on the respective fundamental groups of a continuous map, we extend such induced maps to Core fundamental groupoids. We define that as, if $f: M \to N$ is a continuous map, then define a map $f_\# : \bar{\pi}_1 M \to \bar{\pi}_1 N$ by $f_\#([\gamma_x]) = f_{\# x}([\gamma_x])$ or $[f \circ \gamma_x]$ for all $[\gamma_x] \in \bar{\pi}_1 M$, it will be called as induced groupoid homomorphism.

In addition to this, we define standard projection $p : \bar{\pi}_1 M \to M$, by $p([\gamma_x]) = x$. It is clear that the triple $(\bar{\pi}_1 M, p, M)$ becomes a bundle but not necessarily a fibre bundle. In some cases, one can see $\bar{\pi}_1 M$ lead to a fibre bundle structure, which we will discuss in future research work. The composition of path homotopy equivalence classes $*$ is a well-defined operation in fundamental groups, but cannot be defined on the set $\bar{\pi}_1 M$, for that i.e., to define under this $*$ operation between two elements, they have to be path homotopy classes of loops based at the same point [3, 19, 22, 24]. Thus, this operation cannot be defined between any two arbitrary elements of $\bar{\pi}_1 M$. But, we can see with the same definition of $*$, it has a certain algebraic structure on $\bar{\pi}_1 M$ as in the following proposition.

Proposition 3.1. *i) Let M be a topological space then $\bar{\pi}_1 M$ forms a groupoid, under a unary operation $*^{-1} : \bar{\pi}_1 M \to \bar{\pi}_1 M$ defined by $*^{-1}([\gamma]) = [\gamma]^{-1} = [\overline{\gamma}]$ for all $[\gamma] \in \bar{\pi}_1 M$ and a partial function $* : \bar{\pi}_1 M \times \bar{\pi}_1 M \to \bar{\pi}_1 M$ defined by $[\gamma] * [\delta] = [\gamma * \delta]$, whenever both loops γ and δ have the same base point for all $[\gamma], [\delta] \in \bar{\pi}_1 M$.*
ii) Let $\bar{\pi}_1 M$ be the Core fundamental groupoid of a space M then $(\bar{\pi}_1 M)_0$ the set of identities of $\bar{\pi}_1 M$ is a wide subgroupoid.
iii) Let $\bar{\pi}_1 M$ be the Core fundamental groupoid of a space M and $N \subset M$ then $\bigcup_{x \in N} \pi_1(M, x)$ is a subgroupoid of $\bar{\pi}_1 M$.

Proof. All are trivial to see and the second is true because $(\bar{\pi}_1 M)_0 = ((\bar{\pi}_1 M)_0)_0$. □

Proposition 3.2. *Let $f : M \to N$ be a continuous map then $f_\# : \bar{\pi}_1 M \to \bar{\pi}_1 N$ is a groupoid homomorphism.*

Proof. This is due to the induced homomorphisms between fundamental groups. □

Proposition 3.3. *Let $f : M \to N$ be a continuous map then*
i) The induced groupoid homomorphism $f_\# : \bar{\pi}_1 M \to \bar{\pi}_1 N$ satisfy $f_\#([c_x]) = [c_{f(x)}]$, for every $[c_x] \in \bar{\pi}_1 M$.
ii) If N is a simply connected space then $f_\# : \bar{\pi}_1 M \to \bar{\pi}_1 N$ is a trivial groupoid homomorphism.
iii) If $f : M \to N$ be a constant map, then $f_\# : \bar{\pi}_1 M \to \bar{\pi}_1 N$ is a constant trivial groupoid homomorphism.

Proof. i) For the continuous map $f : M \to N$ the $f_\# : \bar{\pi}_1 M \to \bar{\pi}_1 N$ is a groupoid homomorphism. Therefore, for every $[c_x] \in \bar{\pi}_1 M$, choose one of $[\gamma_x] \in \pi_1(M, x)$ and we can see $f_\#([c_x]) = f_\#([\gamma_x * \overline{\gamma}_x]) = f_\#([\gamma_x]) * f_\#([\overline{\gamma}_x]) = f_{\#x}([\gamma_x]) * (f_{\#x}([\gamma_x]))^{-1} = [f \circ \gamma_x] * [\overline{f \circ \gamma_x}] = [c_{f(x)}]$. Hence the proof.
ii) It follows from Proposition 3.2 and simply connectedness of N.
iii) Trivial. □

Proposition 3.4. *Let $f : M \to N$ be a continuous map and $p_M : \bar{\pi}_1 M \to M, p_N : \bar{\pi}_1 N \to N$ be the standard projections of $\bar{\pi}_1 M$ and $\bar{\pi}_1 N$ respectively, then diagram commutes.*

$$\begin{tikzcd} M \arrow[r, "f"] & N \\ \bar{\pi}_1 M \arrow[u, "p_M"] \arrow[r, "f_{\#}"'] & \bar{\pi}_1 N \arrow[u, "p_N"'] \end{tikzcd}$$

i.e., $f \circ p_M = p_N \circ f_{\#}$.

Proof. It is quite clear that, for every $[\gamma_x] \in \bar{\pi}_1 M$, we can have $f \circ p_M([\gamma_x]) = f(x)$ and also $p_N \circ f_{\#}([\gamma_x]) = p_N([f \circ \gamma_x]) = f(x)$. In addition, Proposition 4.19 shows that both $f \circ p_M$ and $p_N \circ f_{\#}$ are continuous. □

Proposition 3.5. *i) Let $f : M \to N$ be a homeomorphism then $f^{-1}_{\#} = (f_{\#})^{-1}$.*
ii) Let $Id : M \to M$ be the identity map then $Id_{\#} = Id_{\bar{\pi}_1 M}$.

Proof. i) Let $[\beta_y] \in \bar{\pi}_1 N$ be an arbitrary element, then $f^{-1}_{\#}([\beta_y]) = [f^{-1} \circ \beta_y] = (f_{\#})^{-1}([\beta_y])$. Because $f_{\#}((f_{\#})^{-1}([\beta_y])) = [f \circ f^{-1} \circ \beta_y] = [\beta_y]$.
ii) Let $[\gamma_x] \in \bar{\pi}_1 M$ be an arbitrary element, then we can see $Id_{\#}([\gamma_x]) = [Id \circ \gamma_x] = [\gamma_x] = Id_{\bar{\pi}_1 M}([\gamma_x])$. □

Proposition 3.6. *Let $f : M \to N$ and $g : N \to R$ be continuous maps then*
i) $(g \circ f)_{\#} = g_{\#} \circ f_{\#}$.
ii) If both f, g are homeomorphisms then $(g \circ f)^{-1}{}_{\#} = f^{-1}_{\#} \circ g^{-1}_{\#}$.

Proof. i) It is clear that, $f_{\#} : \bar{\pi}_1 M \to \bar{\pi}_1 N$ and $g_{\#} : \bar{\pi}_1 N \to \bar{\pi}_1 R$ are groupoid homomorphisms, hence composition $g_{\#} \circ f_{\#}$ exists and is a groupoid homomorphism. Let $[\gamma_x] \in \bar{\pi}_1 M$ be an arbitrary element then $(g \circ f)_{\#}([\gamma_x]) = [g \circ f \circ \gamma_x] = g_{\#}([f \circ \gamma_x]) = g_{\#}(f_{\#}([\gamma_x])) = g_{\#} \circ f_{\#}([\gamma_x])$.
ii) This follows from Propositions 3.5(i) and 3.6(i). □

Proposition 3.7. *Let $T : \bar{\pi}_1 M \to \bar{\pi}_1 N$ be a groupoid homomorphism then*

i) If $[c_x] \in \bar{\pi}_1 M$ then $T([c_x]) = [c_y]$, for some $y \in N$.
ii) If $T([c_x]) = [c_y]$, for some $y \in N$ then $T([\gamma_x]) = [\delta_y]$, for some loop δ_y based at y for all $[\gamma_x] \in \pi_1(M, x)$ (or simply $T([\gamma_x]) \in \pi_1(N, y)$ for all $[\gamma_x] \in \pi_1(M, x)$).

Proposition 3.8. *Let $f, g : M \to N$ be two continuous maps then $f_\# = g_\#$ if and only if $f = g$.*

Proof. If $f = g$ then it is obvious that $f_\# = g_\#$. Conversely, if $f_\# = g_\#$, let us see the result by contrary assumption, i.e., suppose $f \neq g$ this implies there exists at least one $x_0 \in M$ such that $f(x_0) = y_0$ (say) is not equal to $g(x_0) = y_1$ (say). But, this implies that, $f_\#([c_{x_0}]) = [c_{y_0}] \neq [c_{y_1}] = g_\#([c_{x_0}])$, which is a contradiction to the hypothesis. Therefore $f = g$. □

Proposition 3.9. *Let $T : \bar{\pi}_1 M \to \bar{\pi}_1 N$ be a groupoid homomorphism then there exists a unique map $b_T : M \to N$ such that $p_N \circ T = b_T \circ p_M$ (b_T is induced by T, so, we will call it as induced base map).*

Proof. Let $T : \bar{\pi}_1 M \to \bar{\pi}_1 N$ be a groupoid homomorphism. This implies $\forall [c_x] \in \bar{\pi}_1 M$, $T([c_x]) = [c_y]$ for some $y \in N$ by Proposition 3.7. Utilising this result and keeping the base point correspondence for our construction, we define a map $b_T : M \to N$ by setting $b_T(x) = y$ for every respective $x \in M$. It is a well-defined map, because, for each $x \in M$ it is uniquely assigned. In addition, we can see that $p_N \circ T = b_T \circ p_M$ trivially. Because $\forall [\gamma_x] \in \bar{\pi}_1 M$, consider $p_N \circ T([\gamma_x]) = p_N(T([\gamma_x])) = y = b_T(x)$ (whenever $T([c_x]) = [c_y]$, for some $y \in N$) and also $b_T \circ p_M([\gamma_x]) = b_T(x)$. Hence, it satisfies $p_N \circ T = b_T \circ p_M$. □

Proposition 3.10. *i) Let $T, S : \bar{\pi}_1 M \to \bar{\pi}_1 N$ be two groupoid homomorphisms such that for some $[\gamma_x] \in \bar{\pi}_1 M$, the $T([\gamma_x])$ and $S([\gamma_x])$ are in the different fundamental groups then $b_T \neq b_S$.*
That is if $T([\gamma_x]) \notin \pi_1(N, y) \ni S([\gamma_x])$ for some y in N then $b_T \neq b_S$.
ii) Let $T : (\bar{\pi}_1 M, \mathcal{I}_{p_M}) \to (\bar{\pi}_1 N, \mathcal{I}_{p_N})$ be a constant topological groupoid homomorphism then b_T is a constant.

Proof. Both results follow Propositions 3.7 and 3.9. □

Proposition 3.11. *i) Let $T : \bar{\pi}_1 M \to \bar{\pi}_1 N$ be a groupoid isomorphism and $b_T : M \to N$ be the induced base map of T, then $(b_T)^{-1} = b_{T^{-1}}$, hence the b_T is bijective.*

ii) Let $Id_{\bar{\pi}_1 M} : \bar{\pi}_1 M \to \bar{\pi}_1 M$ be the identity groupoid isomorphism then $b_{Id_{\bar{\pi}_1 M}} = Id_M$.

Proof. i) Here, $T : \bar{\pi}_1 M \to \bar{\pi}_1 N$ is a groupoid isomorphism. So, by Proposition 3.7 $\forall [c_x] \in \bar{\pi}_1 M$ the $T([c_x]) = [c_y]$, for some $y \in N$. Under the map $T^{-1} : \bar{\pi}_1 N \to \bar{\pi}_1 M$, it is true that $\forall [c_y] \in \bar{\pi}_1 M$ the $T^{-1}([c_y]) = [c_x]$. Therefore, for every $y \in N$, the $b_{(T^{-1})}(y) = x = (b_T)^{-1}(y)$. Hence the proof.
ii) It follows from Propositions 3.7 and 3.9. □

Proposition 3.12. *i) Let $T : \bar{\pi}_1 M \to \bar{\pi}_1 N$ and $S : \bar{\pi}_1 N \to \bar{\pi}_1 R$ be two groupoid homomorphisms then $b_{S \circ T} = b_S \circ b_T$.*
ii) Let $T : \bar{\pi}_1 M \to \bar{\pi}_1 N$ and $S : \bar{\pi}_1 N \to \bar{\pi}_1 R$ be two groupoid isomorphisms then $b_{(S \circ T)^{-1}} = b_{(T^{-1})} \circ b_{(S^{-1})} = (b_T)^{-1} \circ (b_S)^{-1}$.

Proof. i) The composition of two groupoid homomorphisms is a groupoid homomorphism, hence, $SoT : \bar{\pi}_1 M \to \bar{\pi}_1 R$ is so. Therefore, induced base map of SoT is $b_{S \circ T}$ and this is well-defined for all $x \in M$ by $b_{S \circ T}(x) = z$, where $S \circ T([c_x]) = S(T([c_x]) = [c_y]) = [c_z]$ for some $z \in R$. Since b_S and b_T are composable, we can have it and also $b_S \circ b_T(x) = b_S(y) = z$. This involves the mapping of the base point by the respective groupoid homomorphism when the point of respective identity elements of fundamental groups come into the picture, hence the proof.
ii) Propositions 3.11(i) and 3.12(i) combine to give proof. □

Proposition 3.13. *Let $T : \bar{\pi}_1 M \to \bar{\pi}_1 N$ be a groupoid homomorphism and define $O_T : \bar{\pi}_1 M \to \bar{\pi}_1 N$ by $O_T([\gamma_x]) = [c_{b_T(x)}], \forall [\gamma_x] \in \bar{\pi}_1 M$ then O_T is the unique trivial groupoid homomorphism such that $b_T = b_{O_T}$.*

Proof. Let $T : \bar{\pi}_1 M \to \bar{\pi}_1 N$ be a groupoid homomorphism, we can see that the defined map O_T is preserved under groupoid operations and also maps all elements to identity elements of the respective fundamental groups. Hence $O_T : \bar{\pi}_1 M \to \bar{\pi}_1 N$ is a trivial groupoid homomorphism. Moreover, $\forall [c_x] \in \bar{\pi}_1 M$ the $O_T([c_x]) = [c_{b_T(x)}]$, and $T([c_x]) = [c_{b_T(x)}]$, so this gives that $b_T = b_{O_T}$. Suppose O_T is not unique, there is another trivial groupoid homomorphism O'_T. This implies $O_T([\gamma_{(x_0)}]) = [c_{b_T(x_0)}] \neq [c_z] = O'_T([\gamma_{x_0}])$, for some $[\gamma_{x_0}] \in \bar{\pi}_1 M$, but, this further gives $b_T \neq b_{O'_T}$. Therefore O_T is the unique trivial groupoid homomorphism such that $b_T = b_{O_T}$. □

Note. i) If T is trivial groupoid homomorphism then $O_T = T$.
ii) Let $\rho \in Bijection(M, N)$ and $\forall[\gamma_x] \in \bar{\pi}_1 M$, define $T_\rho([\gamma_x]) = [c_{\rho(x)}]$ then it is a trivial groupoid homomorphism such that $b_{T_\rho} = \rho$ and $O_{T_\rho} = T_\rho$.

Proposition 3.14. *Let $T : \bar{\pi}_1 M \to \bar{\pi}_1 N$ be a groupoid homomorphism then for each $x \in M$, restricted map the $T_{\pi_1(M,x)} : \pi_1(M, x) \to \pi_1(N, b_T(x))$ is a group homomorphism.*

Proof. Proposition 3.7 guarantees that, if $T([c_x]) = [c_y]$, for some $y \in N$ then $\forall[\gamma_x] \in \bar{\pi}_1 M, T([\gamma_x]) = [\delta_y]$ for some $[\delta_y]$. Further, we can see $T([\gamma_x]) = [\delta_y] = [\delta_{b_T(x)=y}]$ from Proposition 3.3. So the restriction map $T_{\pi_1(M,x)}$ defined from $\pi_1(M, x)$ to $\pi_1(N, b_T(x) = y)$ is well-defined. Since groupoid homomorphism is a weaker form of group homomorphism, the restriction on the respective fundamental group of groupoid homomorphism, i.e., $T_{\pi_1(M,x)}$ is a group homomorphism. $\square$

Remark 3.15. *i) Let $f : M \to N$ be a map and for every loop γ_x in M $f \circ \gamma_x$ is continuous then f need not be a continuous map. For this, we have a counterexample, that is, for the rational number set $\mathbf{Q}$ under subspace topology induced by the usual topology of the real line, choose a (or every discontinuous map) map $f : \mathbf{Q} \to \mathbf{Q}$ defined by*

$$f(x) = \begin{cases} 2 & \text{if } x = 1 \\ 1 & \text{if } x = 2 \\ x & \text{otherwise} \end{cases}$$

is a discontinuous map. But, since connected components of $\mathbf{Q}$ are singleton, every loop γ_x in $\mathbf{Q}$ is a constant loop and also $f \circ \gamma_x$ becomes constant and hence continuous. Whereas the choice that f is not continuous.
ii) If $f : M \to N$ is a continuous map then $f_\# : \bar{\pi}_1 M \to \bar{\pi}_1 N$ by $f_\#([\gamma_x]) = [f \circ \gamma_x]$ for all $[\gamma_x] \in \bar{\pi}_1 M$ is well-defined. But not converse.

That is, even function f is not a continuous map, but $f_\# : \bar{\pi}_1 M \to \bar{\pi}_1 N$ by $f_\#([\gamma_x]) = [f \circ \gamma_x]$ for all $[\gamma_x] \in \bar{\pi}_1 M$ is well-defined. We have a counterexample, that is due to Remark 3.15(i). The function defined in 3.15(i) is not continuous, but, since the connected components of $\mathbf{Q}$ are singleton, the fundamental group at each point of the rational set is trivial thus $|\bar{\pi}_1 \mathbf{Q}| = |\mathbf{Q}|$ and also obviously, $f_\# : \bar{\pi}_1 \mathbf{Q} \to \bar{\pi}_1 \mathbf{Q}$ by $f_\#([\gamma_x]) = [f \circ \gamma_x]$ for all $[\gamma_x] \in \bar{\pi}_1 \mathbf{Q}$ is well-defined due to the details of Remark 3.15(i).

Theorem 3.16. *(Main result) If $f : M \to N$ is a homeomorphism then $f_\# : \bar{\pi}_1 M \to \bar{\pi}_1 N$ is a groupoid isomorphism.*

Proof. If f is a homeomorphism then $f_\#$ and $(f^{-1})_\# = (f_\#)^{-1}$ are groupoid homomorphisms from Propositions 3.2 and 3.5(i). Hence $f_\#$ is a groupoid isomorphism. $\square$

Generally, the converse need not hold, as we can see in the following example.

Example 3.17. *The function considered in Remark 3.15(ii) becomes a counter example. Here, the induced map $f_\#([\gamma_x]) = [f \circ \gamma_x]$ for all $([\gamma_x]) \in \bar{\pi}_1\mathbf{Q}$ and is a well-defined groupoid isomorphism, but f is not a continuous map.*

Remark 3.18. *Theorem 3.16 is not a characterization of the homeomorphisms, but, it shows that the Core fundamental groupoid is a topological invariant. Homeomorphism of f implies induced groupoid homomorphism $f_\#$ is a groupoid isomorphism, thus M homeomorphic N implies $\bar{\pi}_1 M$ is groupoid isomorphic $\bar{\pi}_1 N$. But, with this structure, we cannot coin this result "M is homeomorphic to N if and only if $\bar{\pi}_1 M$ is groupoid isomorphic to $\bar{\pi}_1 M$". This is because sometimes two Core fundamental groupoids are groupoid isomorphic, but, may or may not the spaces are homeomorphic. The following example clarifies us. For the topological spaces, $M = [0,1]$ and $N = \{-1, 2\} \cup (0,1)$ under subspace topology induced by the usual topology of the real line, it is true that the map $T : \bar{\pi}_1 M \to \bar{\pi}_1 N$ defined by*

$$T([\gamma_x]) = \begin{cases} [c_{-1}] & \textit{if } [\gamma_x] = [c_0] \\ [c_2] & \textit{if } [\gamma_x] = [c_1] \\ [\gamma_x] & \textit{Otherwise} \end{cases}$$

$\forall [\gamma_x] \in \bar{\pi}_1 M$ is a groupoid isomorphism, but M is not homeomorphic to N.

To overcome this problem or to build sufficient conditions, we are introducing a richer structure on the Core fundamental groupoid with the help of topology, which will at least help us in characterizing the homeomorphism. We will give a kind of explicit solution for the homeomorphic spaces problem in Proposition 4.30 and 4.31 in the next section.

We denote by $GpdHom(\bar{\pi}_1 M, \bar{\pi}_1 N)$ for the set of all groupoid homomorphisms from $\bar{\pi}_1 M$ to $\bar{\pi}_1 N$, if there is no confusion use simply $Hom(\bar{\pi}_1 M, \bar{\pi}_1 N)$. For different homomorphisms $T, S \in Hom(\bar{\pi}_1 M, \bar{\pi}_1 N)$ the induced base map may be the same. That is even $T \neq S$ of $Hom(\bar{\pi}_1 M, \bar{\pi}_1 N)$ then $b_T = b_S$ may be possible, see Remark 4.28. This idea builds an intuition of partitioning the $Hom(\bar{\pi}_1 M, \bar{\pi}_1 N)$ by means of the following equivalence relation.

Proposition 3.19. *Let M, N be two topological spaces and define μ : $Hom(\bar{\pi}_1 M, \bar{\pi}_1 N) \to Hom(\bar{\pi}_1 M, \bar{\pi}_1 N)$ by, $T\mu S$ if and only if $b_T = b_S$, then it is an equivalence relation.*

Under this equivalence relation, we can partition the set $Hom(\bar{\pi}_1 M, \bar{\pi}_1 N)$. Denote $[T]_\mu$ for μ equivalence class containing T. If the fundamental groups at each point of space N is abelian then we can obtain groupoid structures on $Hom(\bar{\pi}_1 M, \bar{\pi}_1 N)$ as follows. Define a unary operation $\otimes^{-1}$: $Hom(\bar{\pi}_1 M, \bar{\pi}_1 N) \to Hom(\bar{\pi}_1 M, \bar{\pi}_1 N)$ by $\otimes^{-1}(T) = \overline{T} = T^{-1}$ for all $T \in Hom(\bar{\pi}_1 M, \bar{\pi}_1 N)$ and a partial function
$\otimes : Hom(\bar{\pi}_1 M, \bar{\pi}_1 N) \times Hom(\bar{\pi}_1 M, \bar{\pi}_1 N) \to Hom(\bar{\pi}_1 M, \bar{\pi}_1 N)$ by $T \otimes S = T * S$, where $\overline{T} = T^{-1} : \bar{\pi}_1 M \to \bar{\pi}_1 N$ by $\overline{T}([\gamma]) = (T([\gamma]))^{-1}$ and $T * S$: $\bar{\pi}_1 M \to \bar{\pi}_1 N$ by $(T * S)([\gamma]) = T([\gamma]) * S([\gamma])$, whenever both $T([\gamma])$ and $S([\gamma])$ are in the same fundamental group, for every $[\gamma] \in \bar{\pi}_1 M$ then $Hom(\bar{\pi}_1 M, \bar{\pi}_1 N)$ forms a groupoid.

Proposition 3.20. *Let fundamental groups at each point of space N be abelian and $[T]_\mu$ be a μ equivalence class containing T for a $T \in Hom(\bar{\pi}_1 M, \bar{\pi}_1 N)$ and define $\otimes : [T]_\mu \times [T]_\mu \to [T]_\mu$ by $S \otimes L = S * L$ for all $S, L \in [T]_\mu$, where $S * L : \bar{\pi}_1 M \to \bar{\pi}_1 N$ by $(S * L)([\gamma]) = S([\gamma]]) * L([\gamma]])$ for all $[\gamma] \in \bar{\pi}_1 M$ then $([T]_\mu, \otimes)$ is an abelian group.*

Proof. For all $S, L \in [T]_\mu$, the $S \otimes L = S * L : \bar{\pi}_1 M \to \bar{\pi}_1 N$ defined by $(S * L)([\gamma]) = S([\gamma]) * L([\gamma])$, for all $[\gamma] \in \bar{\pi}_1 M$ is a well-defined map. Moreover, for all $[\gamma], [\delta] \in \bar{\pi}_1 M$, such that $[\gamma] * [\delta]$ is defined, then $(S \otimes L)([\gamma] * [\delta]) = S([\gamma] * [\delta]) * L([\gamma] * [\delta]) = S([\gamma]) * S([\delta]) * L([\gamma]) * L([\delta]) = S([\gamma]) * L([\gamma]) * S([\delta]) * L([\delta]) = (S \otimes L)([\delta]) * (S \otimes L)([\delta])$, hence, $S \otimes L$ is a groupoid homomorphism. Since for every $[c_x] \in \bar{\pi}_1 M$ the $S \otimes L([c_x]) = [c_{(b_T(x)}]$ or $[c_{(b_S(x))}]$, one can see $b_{(S \otimes L)} \in [T]_\mu$. Associativity is followed by the associativity of fundamental groups. The map $O([\gamma_x]) = [c_{(b_T(x))}]$ is a trivial groupoid homomorphism and also $O \in [T]_\mu$ such that $S \otimes O = S = O \otimes S$ vacuously. Finally for every $S \in [T]_\mu$, there is a groupoid homomorphism $S^{-1} : \bar{\pi}_1 M \to \bar{\pi}_1 N$ defined by $S^{-1}([\gamma]) = (S([\gamma]))^{-1}$ and also we can see $S \otimes S^{-1} = O = S^{-1} \otimes S$. Lastly, commutativity of each fundamental groups at each point of space N guarantees that $\otimes$ is commutative. Thus $([T]_\mu, \otimes)$ is an abelian group. $\square$

Proposition 3.21. *Let $Gpdiso(\bar{\pi}_1 M, \bar{\pi}_1 M)$ be set of all groupoid isomorphisms from $\bar{\pi}_1 M$ to $\bar{\pi}_1 M$ then it is a group under the composition of function.*

Proof. It is well-known that the composition of groupoid isomorphism is a groupoid isomorphism. The associativity is trivial and the element $Id_{\bar{\pi}_1 M}$ is an identity element here because it is true that $T \circ Id_{\bar{\pi}_1 M} = T = Id_{\bar{\pi}_1 M} \circ T$, for all $T \in Gpdiso(\bar{\pi}_1 M, \bar{\pi}_1 M)$. Finally, for every $T \in Gpdiso(\bar{\pi}_1 M, \bar{\pi}_1 M)$ there is $S \in Gpdiso(\bar{\pi}_1 M, \bar{\pi}_1 M)$ defined by $S([\gamma]) = (T^{-1}([\gamma]))$ this satisfy $S \circ T = Id_{\bar{\pi}_1 M} = T \circ S$. Thus $Gpdiso(\bar{\pi}_1 M, \bar{\pi}_1 M)$ forms a group. □

4. A natural Topology on Core fundamental groupoid yields a topological groupoid: A good invariant

We establish a topology on an algebraic structure defined in section 3 called Core fundamental groupoid, which makes it richer under an extra structure. Moreover, this becomes a good topological invariant and gives a kind of characterization to homeomorphism. Let $(M, \mathcal{I}_M)$ be a topological space and the standard projection induces a unique topology on Core fundamental groupoid $\bar{\pi}_1 M$ from the topology of $(M, \mathcal{I}_M)$, that topology is defined explicitly by $\mathcal{I}_p = \{p^{-1}(U) : U \in \mathcal{I}_M\}$, which guarantees that the map p is a quotient map. Hence, the topology acquired by the base space is unique and in fact, it is the induced topology from the base space by the standard projection p on the Core fundamental groupoid. Since under this topology the map standard projection p is a quotient map, we are calling this topology on $\bar{\pi}_1 M$ by quotient topology instead of induced topology(one may use name induced topology). Moreover, the standard projection becomes an open map, because, for every open set O in $\bar{\pi}_1 M$, we can see $O = p^{-1}(U)$ for some open set U in M and also $p(O) = p(p^{-1}(U)) = U$ is an open set in M. If the space M is simply connected space then $|M| = |\bar{\pi}_1 M|$ and moreover, those two spaces behave same.

On the way of topological groupoid, we can see Core fundamental groupoid is one such object with a unique topology on it. Including such a result, the following outcomes are some of the topological properties of the Core fundamental groupoid.

Proposition 4.1. *Let $\mathcal{I}_p$ be the induced topology on $\bar{\pi}_1 M$ by the standard projection, then $\bar{\pi}_1 M$ is a topological groupoid.*

Proof. Proposition 3.1 says $\bar{\pi}_1 M$ is a groupoid. One can see $*^{-1} : \bar{\pi}_1 M \to \bar{\pi}_1 M$ is a continuous map because for every open set O in $\bar{\pi}_1 M$ then $O = p^{-1}(U) = \bigcup_{x \in U} \pi_1(M, x)$ for some open set U in M. If $[\gamma] \in O$ implies $*^{-1}([\gamma]) = [\overline{\gamma}] \in O$, therefore the inverse image of O under map

$*^{-1}$ is O itself. And also, for every open set O in $\bar{\pi}_1 M$ there is an open set $O \times O$ in product space $\bar{\pi}_1 M \times \bar{\pi}_1 M$ such that $*(O \times O) = \{*([\gamma], [\delta]) : [\gamma], [\delta] \in O\} = O$, so that partial function $*$ is also continuous. Thus $\bar{\pi}_1 M$ is a topological groupoid. □

From now onwards, the Core fundamental groupoid $\bar{\pi}_1 M$ will be used as a topological groupoid.

Proposition 4.2. *Let $\mathcal{I}_p$ be the induced topology on $\bar{\pi}_1 M$ by the standard projection, then $\bar{\pi}_1 M$ is Hausdorff if and only if M is Hausdorff and has the trivial fundamental group at each point of M.*

Proof. Choose arbitrary distinct points x, y from M and we can see that $[c_x], [c_y]$ are distinct elements of the Core fundamental groupoid $\bar{\pi}_1 M$. Since $\bar{\pi}_1 M$ is a Hausdorff space, there exist two disjoint open sets $O_1 \ni [c_x]$ and $O_2 \ni [c_y]$. By definition of $\mathcal{I}_p$ the $O_1 = p^{-1}(U_1)$ and $O_2 = p^{-1}(U_2)$ for some open sets U_1 and U_2 in M. The definition of the open sets guarantees that, we can see that U_1 and U_2 are containing x and y respectively and also $U_1 \cap U_2 = p(p^{-1}(U_1)) \cap p(p^{-1}(U_2)) \subseteq p(p^{-1}(U_1) \cap p^{-1}(U_2)) = \emptyset$.

And on the contrary, the Core fundamental groupoid $\bar{\pi}_1 M$ be non-trivial at some x_0, hence cordiality for the fundamental group satisfies, $|\pi_1(M, x_0)| > 1$. Therefore, there exist at least two distinct elements $[\gamma_{x_0}]$ and $[\alpha_{x_0}]$ in the same fundamental group based at x_0. There are indeed no two disjoint open sets containing $[\gamma_{x_0}]$ and $[\alpha_{x_0}]$ respectively. Suppose there exist open sets O_1 and O_2 in $\bar{\pi}_1 M$ such that $O_1 \ni [\gamma_{x_0}]$ and $O_2 \ni [\alpha_{x_0}]$. By definition of $\mathcal{I}_p$ the $O_1 = p^{-1}(U_1)$ and $O_2 = p^{-1}(U_2)$ for some open sets U_1 and U_2 in M. Moreover, one can see that both U_1 and U_2 are containing x_0 therefore $O_1 \cap O_2 = p^{-1}(U_1) \cap p^{-1}(U_2) = p^{-1}(U_1 \cap U_2)$ is non-empty, which contradicts to Hausdorffness of $\bar{\pi}_1 M$. Hence the proof.

Conversely, for the trivial group $\pi_1(M, x)$ for each $x \in M$, we can pick only a single element from each fundamental group based at x, but not more elements. For arbitrary distinct points $[\gamma_x], [\alpha_y]$ from $\bar{\pi}_1 M$, implies $x \neq y$. Property Hausdorff of M gives there exist disjoint open sets U_1 and U_2 in M, respectively containing x and y. This implies $p^{-1}(U_1)$ and $p^{-1}(U_2)$ are open sets in $\bar{\pi}_1 M$ and also contain $[\gamma_x]$ and $[\alpha_y]$ respectively, such that $p^{-1}(U_1) \cap p^{-1}(U_2) = p^{-1}(U_1 \cap U_2) = \emptyset$. □

In general, concerning almost all the topological spaces, their Core fundamental groupoid is not necessarily a Hausdorff space. To attain Hausdorffness in the Core fundamental groupoid, necessarily the base space has trivial fundamental groups at each point. A Contrapositive version of this is "If $\pi_1(M, x)$ is non-trivial for some $x_0 \in M$ then $\bar{\pi}_1 M$ is non-Hausdorff".

Proposition 4.3. *Let $\mathcal{I}_p$ be the induced topology on $\bar{\pi}_1 M$ by the standard projection then M is compact if and only if $\bar{\pi}_1 M$ is compact.*

Proof. Let us prove that $\bar{\pi}_1 M$ is a compact space, taking arbitrary open cover $G = \{O_\lambda\}$ for some index set $\triangle$. The set $H = \{U_\lambda\}$ where $p^{-1}(U_\lambda) = O_\lambda$ for all $\lambda \in \triangle$, forms an open cover for M. Since M is compact, it possess a finite subcover $H_0 = \{U_i : i = 1, 2, ..., n\}$, which guarantee that the $G_0 = \{p^{-1}(U_i) : i = 1, 2, ..., n\}$ forms finite subcover for $\bar{\pi}_1 M$ due to $\bigcup\limits_{i=1}^{n} p^{-1}(U_i) = p^{-1}(\bigcup\limits_{i=1}^{n} U_i) = p^{-1}(M) = \bar{\pi}_1 M$.

On the other hand, it is well-known that the continuous image of a compact space is compact. It is therefore obvious that M is compact, due to the standard projection p is a continuous map. □

Proposition 4.4. *Let $\mathcal{I}_p$ be the induced topology on $\bar{\pi}_1 M$ by the standard projection then M is connected if and only if $\bar{\pi}_1 M$ is connected.*

Proof. Suppose $\bar{\pi}_1 M$ is not connected, this implies that there are two non-empty disjoint open sets O_1 and O_2 such that $O_1 \cup O_2 = \bar{\pi}_1 M$. By definition of topology on $\bar{\pi}_1 M$, the $O_1 = p^{-1}(U_1)$ and $O_2 = p^{-1}(U_2)$ for some open sets U_1 and U_2 in M. One can see $U_1 \cup U_2 = p(p^{-1}(U_1)) \cup p(p^{-1}(U_2)) = p(O_1) \cup p(O_2) = p(O_1 \cup O_2) = M$. And also $U_1 \cap U_2 = p(p^{-1}(U_1)) \cap p(p^{-1}(U_2)) \subseteq p(p^{-1}(U_1) \cap p^{-1}(U_2)) = \emptyset$. It is a contradiction to the connectedness of M. Hence $\bar{\pi}_1 M$ is connected.

On the other hand, it is well-known that the continuous image of a connected space is connected. It is therefore obvious that M is connected, due to the standard projection p is a continuous map. □

Proposition 4.5. *Let $\mathcal{I}_p$ be the induced topology on $\bar{\pi}_1 M$ by the standard projection then M is second countable if and only if $\bar{\pi}_1 M$ is second countable.*

Proof. Let us prove M is second countable. Since $\bar{\pi}_1 M$ is a second countable space, there exists a countable basis set $\beta = \{B_\lambda : \lambda \in \mathbf{N}\}$. It is well-known that every element of β is an element of the topology of $\bar{\pi}_1 M$, so every $B_\lambda \in \beta$ implies $B_\lambda \in \mathcal{I}_p$. Under the topology $\mathcal{I}_p$, it is true that the $B_\lambda = p^{-1}(U_\lambda)$ for some open set U_λ in M for each $\lambda \in \mathbf{N}$. Therefore, we can construct a set $\beta' = \{U_\lambda : \lambda \in \mathbf{N}\}$, which becomes a countable basis for M. Because $\bigcup\limits_{\lambda \in \mathbf{N}} U_\lambda = \bigcup\limits_{\lambda \in \mathbf{N}} p(p^{-1}(U_\lambda)) = p(\bigcup\limits_{\lambda \in \mathbf{N}} p^{-1}(U_\lambda)) = M$.

Also, if $U_\lambda \cap U_\delta \neq \emptyset$ and for every x in $U_\lambda \cap U_\delta$, there is a $p^{-1}(x) \in p^{-1}(U_\lambda) \cup p^{-1}(U_\delta) = B_\lambda \cap B_\delta$ and the definition of the basis of β gives there exists $B_\sigma \ni p^{-1}(x)$ in β, such that $p^{-1}(x) \in B_\sigma \subset B_\lambda \cap B_\delta$. This implies $x \in U_\sigma \subset U_\lambda \cap U_\delta$ for element U_σ in β'. Therefore M is second countable.

On the other hand, second the countable of M implies that there exists a countable basis set $H = \{D_\lambda : \lambda \in \mathbf{N}\}$. It is well-known that every element of H is an element of the topology of M, so for every $D_\lambda \in H$ implies D_λ is an open set in M, so every $p^{-1}(D_\lambda)$ is an open set in $\bar{\pi}_1 M$. We can construct set $H' = \{p^{-1}(D_\lambda) : \lambda \in \mathbf{N}\}$, which becomes a countable basis for $\bar{\pi}_1 M$. Because $\bigcup_{\lambda \in \mathbf{N}} p^{-1}(D_\lambda) = p^{-1}(\bigcup_{\lambda \in \mathbf{N}} D_\lambda) = \bar{\pi}_1 M$. Also if $p^{-1}(D_\lambda) \cap p^{-1}(D_\delta) \neq \emptyset$ and for every q in $p^{-1}(D_\lambda) \cap p^{-1}(D_\delta)$, implies there exists some $x \in D_\lambda \cap D_\delta$ such that $p^{-1}(x) = q$. Since H is a basis set, there exists D_σ containing x such that $x \in D_\sigma \subset D_\lambda \cap D_\delta$. This implies $p^{-1}(x) \in p^{-1}(D_\lambda) \subset p^{-1}(U_\lambda) \cap p^{-1}(U_\delta)$, therefore $\bar{\pi}_1 M$ is second countable. $\square$

Proposition 4.6. *Let $q : M \to \bar{\pi}_1 M$ be defined by $q(x) = [c_x]$ then it is a continuous map such that $p \circ q = Id_M$.(This is an important map which is actually a section of $\bar{\pi}_1 M$, we will study in-details of sections of it in a future paper).*

Proof. It is clear to see that, the defined $q : M \to \bar{\pi}_1 M$ by $q(x) = [c_x]$ is a well-defined map and also $p \circ q(x) = p([c_x]) = x = Id_M(x)$. Choose arbitrary open set O of $\bar{\pi}_1 M$, by definition of topology on $\bar{\pi}_1 M$ we can see, $O = p^{-1}(V)$ for some open set V in M. Consider $q^{-1}(O) = q^{-1}(p^{-1}(V)) = (poq)^{-1}(V) = (Id_M)^{-1}(V) = V$, this is an open set in M. Thus q is a continuous map. $\square$

The section is an important notion that has a lot of applications in both geometry and topology. The relatedness of a map on sections is indeed responsible for the Lie algebra of a Lie group in the theory of manifold, and its detailed discussion and generalization are available in [8]. We take motivation from this idea to construct sections on the Core fundamental groupoid.

Proposition 4.7. *Let M be a simply connected space then the standard projection map $p : \bar{\pi}_1 M \to M$ is a homeomorphism.*

Proof. The defined standard projection map $p : \bar{\pi}_1 M \to M$ by $p([\gamma_x]) = x$ becomes a bijection due to the simply connectedness of M. In the introductory part of the standard projection, we have discussed that the map p

is a continuous map as well as an open map. Therefore p is a homeomorphism. □

Corollary 4.8. *Let M be a simply connected space then M is homeomorphic to $\bar{\pi}_1 M$.*

Proposition 4.9. *Let M be a topological space then M is a simply connected space if and only if $\bar{\pi}_1 M$ is a simply connected space.*

Proof. If M is a simply connected space, Proposition 4.7 guarantees the standard projection $p : \bar{\pi}_1 M \to M$ is a homeomorphism. The property of simply connectedness is preserved under a homeomorphism, therefore $\bar{\pi}_1 M$ is a simply connected space.

Conversely, first about path connectedness of M, let x, y be arbitrary elements of M, then certainly $[c_x], [c_y]$ are elements of $\bar{\pi}_1 M$. Since $\bar{\pi}_1 M$ is simply connected, it is a path connected. Therefore, there is a path θ from $[c_x]$ to $[c_y]$. The composition of this θ with the standard projection p gives a path say $p \circ \theta$ such that $p \circ \theta(0) = x$ and $p \circ \theta(1) = y$. Thus M is a path-connected space.

Finally, it is enough to see that, every loop in M is contractible to a base point. Choose arbitrary loop γ_x in M then we can see $q \circ \gamma_x$ is a loop based at $[c_x]$ in $\bar{\pi}_1 M$ by Proposition 4.7. Simply-connectedness of $\bar{\pi}_1 M$ implies $q \circ \gamma_x \simeq_p c_{[c_x]}$, and standard projection $p : \bar{\pi}_1 M \to M$ is continuous and post-composition theorem [19, 24] combinedly gives $p \circ q \circ \gamma_x \simeq_p p \circ c_{[c_x]}$, this implies $Id_M \circ \gamma_x \simeq_p p \circ c_{[c_x]}$ also $\gamma_x \simeq_p c_x$. Thus M is a simply connected space. □

Corollary 4.10. *Let M be a topological space*
i) M is a contractible space if and only if $\bar{\pi}_1 M$ is a contractible space.
ii) M is a star convex space if and only if $\bar{\pi}_1 M$ is a star convex space (there is an addition and scalar([0,1]) multiplication on $\bar{\pi}_1 M$ such that it satisfy convex property).

Proof. Both follow Proposition 4.9. □

Proposition 4.11. *[22, 23] Let M be a simply connected space then M is a topological manifold if and only if $\bar{\pi}_1 M$ is a topological manifold.*

Proof. Hausdorff property, second countability and local Euclideanness of a fixed dimension-n are all preserved under a homeomorphism. Since M is a simply connected space, Proposition 4.7 implies standard projection map p is a homeomorphism, this is enough for the proof. □

Proposition 4.12. *[18, 23] Let M be a simply connected space then M is a smooth (respectively c^k differentiable) manifold if and only if $\bar{\pi}_1 M$ is a smooth (respectively c^k differentiable) manifold.*

Proof. If M is a smooth (respectively c^k differentiable) manifold then there is a smooth (respectively c^k differentiable) n-atlas $\mathcal{A}_M = \{(U_i, \phi_i) : i \in I\}$. We can construct $\mathcal{A}_{\bar{\pi}_1 M} = \{(p^{-1}(U_i), \phi_i \circ p = \psi_i) : i \in I\}$ which a smooth (respectively c^k differentiable) n-atlas on $\bar{\pi}_1 M$. Because, the first two conditions for atlas are obvious by homeomorphism standard projection map p. Lastly, about transition maps, choose arbitrary ψ_i, ψ_j in $\mathcal{A}_{\bar{\pi}_1 M}$, we can see $\psi_i \circ {\psi_j}^{-1} = (\phi_i \circ p) \circ (\phi_j \circ p)^{-1} = \phi_i \circ p \circ p^{-1} \circ {\phi_j}^{-1} = \phi_i \circ {\phi_j}^{-1}$ which is a smooth (respectively c^k differentiable) map due to $\mathcal{A}_M$ is a smooth (respectively c^k differentiable) atlas.

Conversely, If $\bar{\pi}_1 M$ is a smooth (respectively c^k differentiable) manifold then there is a smooth (respectively c^k differentiable) n-atlas $\mathcal{A}_{\bar{\pi}_1 M} = \{(O_i, \psi_i) : i \in I\}$. We can construct $\mathcal{A}_M = \{(p(O_i), \psi_i \circ p^{-1}) : i \in I\}$ which forms smooth (respectively c^k differentiable) n-atlas on M. □

Proposition 4.13. *Let M be a simply connected smooth (respectively c^k differentiable) manifold then the standard projection map $p : \bar{\pi}_1 M \to M$ is smooth (respectively c^k) diffeomorphism under the smooth structure defined on $\bar{\pi}_1 M$ in Proposition 4.12.*

Proof. The standard projection $p : \bar{\pi}_1 M \to M$ by $p([\gamma_x]) = x$ becomes a bijection due to the simply connectedness of M, and Proposition 4.7 guarantees that the p is a homeomorphism. Finally for smoothness (respectively c^k) of p, for every chart (U_i, ϕ_i) in $\mathcal{A}_M$ and for every $(p^{-1}(U_i), \phi_i \circ p = \psi_i)$ in $\mathcal{A}_{\bar{\pi}_1 M}$, we can see $\phi_i \circ p \circ {\psi_i}^{-1} = \phi_i \circ p \circ (\phi_i \circ p)^{-1} = Id$. Thus $\phi_i \circ p \circ {\psi_i}^{-1}$ is a smooth (respectively c^k) map, so p is smooth (respectively c^k) map. Similarly p^{-1} is also a smooth (respectively c^k) map, therefore p is a smooth (respectively c^k) diffeomorphism. □

Proposition 4.14. *[18] Let M be a simply connected space then M be a Riemannian manifold if and only if $\bar{\pi}_1 M$ is a Riemannian manifold.*

Proof. Riemannian metrics are preserved under diffeomorphisms. Therefore, under the diffeomorphism standard projection p, Riemannian metrics of M and $\bar{\pi}_1 M$ are encoded. $\square$

Proposition 4.15. *Each $\pi_1(M,x) = p^{-1}(\{x\})$ acquires indiscrete topology under subspace topology induced by $\mathcal{I}_p$ of $\bar{\pi}_1 M$.*

Proof. It is sufficient to show that, the non-trivial subsets of $\pi_1(M,x) = p^{-1}(\{x\})$ are not open in $\pi_1(M,x)$. It is obvious that the empty set and $\pi_1(M,x)$ are elements in subspace topology, because, for every open set U containing x, the $p^{-1}(U)$ is an open set in $\bar{\pi}_1 M$, contains $p^{-1}(x) = \pi_1(M,x)$, hence $\pi_1(M,x) = \pi_1(M,x) \cap p^{-1}(U)$. Moreover, no non-trivial subsets are open in subspace topology and we can see by contradiction. That is, suppose there is a $\emptyset \underset{\neq}{\subset} B \underset{\neq}{\subset} \pi_1(M,x)$ which is open in $\pi_1(M,x)$. Since here B is non-empty, there exists at least one $[\gamma_x] \in B$ and $\pi_1(M,x) \cap p^{-1}(V) = B$ for some open set V in M that contains x. This implies $p^{-1}(x) \subseteq p^{-1}(V)$ and also $\pi_1(M,x) = B$ which contradicts the assumption. Hence $\pi_1(M,x) = p^{-1}(\{x\})$ acquires the indiscrete topology. $\square$

Proposition 4.16. *Let M be a T_1 space (or a topological manifold) then each $\pi_1(M,x) = p^{-1}(\{x\})$ is a closed subset of $\bar{\pi}_1 M$.*

Proof. Each singleton is a closed set in the T_1 space (or a topological manifold). The set $\pi_1(M,x)$ can be viewed as the inverse image of this closed set under the continuous standard projection. $\square$

Proposition 4.17. *Let M be a topological space and $\pi_1(M,x)$ be a space under the subspace topology inherited from $\bar{\pi}_1 M$ then*
i) The $\pi_1(M,x)$ non-Hausdorff (for trivial and non-trivial of $\pi_1(M,x)$)
ii) For every $x \in M$ the group $\pi_1(M,x)$ is a topological group under subspace topology inherited from $\bar{\pi}_1 M$.

Proof. i) Since $\pi_1(M,x)$ is indiscrete space, it cannot be Hausdorff.
ii) It is well-known that the $\pi_1(M,x)$ is a group, for every element x of M. It is indiscrete topological space under subspace topology of $\bar{\pi}_1 M$ by Proposition 4.15. Since every map from indiscrete topological space to indiscrete topological space is always continuous, both group operations $*$ and $*^{-1}$ are continuous. $\square$

Proposition 4.18. *Let $f : M \to N$ be a continuous map and $\pi_1(M, x), \pi_1(M, f(x))$ be subspaces of $\bar{\pi}_1 M$ and $\bar{\pi}_1 N$ respectively for each $x \in M$, then*
i) The induced map $f_{\#x} : (\pi_1(M, x), \mathcal{I}_S) \to (\pi_1(M, f(x)), \mathcal{I}_S)$ is a topological group homomorphism (basically a homomorphism [3, 19, 22, 24]).
ii) If $f : M \to N$ be a homeomorphism then $f_{\#x} : (\pi_1(M, x), \mathcal{I}_S) \to (\pi_1(M, f(x)), \mathcal{I}_S)$ is a topological group isomorphism (basically an isomorphism [3, 19, 22, 24]).

Proof. In both cases, both $\pi_1(M, x)$ and $\pi_1(M, f(x))$ are indiscrete topological spaces, so the induced map $f_{\#x}$ trivially satisfies the results. □

Note. [3, 19, 22, 24] i) Let $f : M \to N$ be a covering projection then $f_{\#x} : (\pi_1(M, x), \mathcal{I}_S) \to (\pi_1(M, f(x)), \mathcal{I}_S)$ is a topological group monomorphism.
ii) Let B be a retract of M and $r : M \to B$ be a retraction then $j_{\#x} : (\pi_1(B, x), \mathcal{I}_S) \to (\pi_1(M, x), \mathcal{I}_S)$ is a topological group monomorphism for the inclusion map j and $r_{\#x} : (\pi_1(M, x), \mathcal{I}_S) \to (\pi_1(B, r(x)), \mathcal{I}_S)$ is a topological group epimorphism.
iii) Let B be a deformation retract of M and j be the inclusion map then $j_{\#x} : (\pi_1(B, x), \mathcal{I}_S) \to (\pi_1(M, x), \mathcal{I}_S)$ is a topological group isomorphism.

Proposition 4.19. *Let $f : M \to N$ be a continuous map and $p_M : \bar{\pi}_1 M \to M$, $p_N : \bar{\pi}_1 N \to N$ be the standard projections of $\bar{\pi}_1 M$ and $\bar{\pi}_1 N$ respectively, then $f_\# : (\bar{\pi}_1 M, \mathcal{I}_{p_M}) \to (\bar{\pi}_1 N, \mathcal{I}_{p_N})$ is a topological groupoid homomorphism.*

Proof. Take an arbitrary open set $O \in \mathcal{I}_{p_N}$ and it implies $O = {p_N}^{-1}(V)$ for some open set V in N. Consider $f_\#^{-1}(O) = f_\#^{-1}({p_N}^{-1}(V)) = (p_N \circ f_\#)^{-1}(V)$, from the commute result of Proposition 3.4, we have $(f \circ p_M)^{-1}(V) = {p_M}^{-1}(f^{-1}(V))$ which is open in $\bar{\pi}_1 M$, because both f and p_M are continuous. By Proposition 3.2, $f_\#$ is a groupoid homomorphism, therefore $f_\#$ is a topological groupoid homomorphism. □

Note. i) Let $f : M \to N$ be a covering projection, then $f_\# : (\bar{\pi}_1 M, \mathcal{I}_{p_M}) \to (\bar{\pi}_1 N, \mathcal{I}_{p_N})$ need not be a topological groupoid monomorphism.
ii) Let B be a retract of M and $r : M \to B$ be a retraction, then $j_\# : (\bar{\pi}_1 B, \mathcal{I}_{p_B}) \to (\bar{\pi}_1 M, \mathcal{I}_{p_M})$ is a topological groupoid monomorphism for the

inclusion map j and $r_{\#} : (\bar{\pi}_1 M, \mathcal{I}_{p_M}) \to (\bar{\pi}_1 B, \mathcal{I}_{p_B})$ is a topological groupoid epimorphism.

Remark 4.20. *We cannot extend all properties exhibited by induced homomorphism $f_{\#x}$ to induced topological groupoid homomorphism. We support the claim by the following examples.*
i) Let $f : M \to N$ be a homotopy equivalence then $f_{\#} : (\bar{\pi}_1 M, \mathcal{I}_{p_M}) \to (\bar{\pi}_1 N, \mathcal{I}_{p_N})$ need not be a topological groupoid isomorphism.
ii) Let B be deformation retract of M then $j_{\#} : (\bar{\pi}_1 B, \mathcal{I}_{p_B}) \to (\bar{\pi}_1 M, \mathcal{I}_{p_M})$ need not be a topological groupoid isomorphism for the inclusion map j.

Proposition 4.21. *Let M, N be two topological spaces, then $\bar{\pi}_1(M \times N)$ is topological groupoid isomorphic to $\bar{\pi}_1 M \times \bar{\pi}_1 N$.*

Proof. It is well-known that first and second projection maps $p_1 : M \times N \to M$ by $p_1(x, y) = x$ and $p_2 : M \times N \to N$ by $p_2(x, y) = y$ are continuous maps and they induced $(p_1)_{\#} : \bar{\pi}_1(M \times N) \to \bar{\pi}_1 M$ and $(p_2)_{\#} : \bar{\pi}_1(M \times N) \to \bar{\pi}_1 N$ as topological groupoid homomorphisms. Define a map $Q : \bar{\pi}_1(M \times N) \to \bar{\pi}_1 M \times \bar{\pi}_1 N$ by $Q([\gamma_{(x,y)}]) = ((p_1)_{\#}([\gamma_{(x,y)}]), (p_2)_{\#}([\gamma_{(x,y)}])) = ([p_1 \circ \gamma_{(x,y)}], [p_2 \circ \gamma_{(x,y)}])$ it is a well-defined continuous map because both component functions are continuous maps, also it is a groupoid homomorphism. Let us see that this is surjective, take an arbitrary element $([\delta_x], [\beta_y]) \in \bar{\pi}_1 M \times \bar{\pi}_1 N$ this implies δ_x and β_y are loops based at x and y in the space M and N respectively. We can construct $\alpha = (\delta_x, \beta_y)$ a loop based at (x, y), so $[\alpha] \in \bar{\pi}_1(M \times N)$ such that $Q([\alpha]) = ([p_1 \circ \alpha], [p_2 \circ \alpha]) = ([\delta_x], [\beta_y])$ therefore surjective.

For injective, let us have $\mathrm{Ker}(Q) = \{[\gamma] \in \bar{\pi}_1(M \times N) : Q([\gamma]) \in (\bar{\pi}_1 M \times \bar{\pi}_1 N)_0\}$. That is, we have to collect all $[\gamma] \in \bar{\pi}_1(M \times N)$ such that $Q([\gamma]) = ([p_1 \circ \gamma], [p_2 \circ \gamma]) = ([c_x], [c_y])$ for some $([c_x], [c_y]) \in (\bar{\pi}_1 M \times \bar{\pi}_1 N)_0$. Therefore, $p_1 \circ \gamma \simeq_p c_x$ and $p_2 \circ \gamma \simeq_p c_y$, suppose $\gamma = (\gamma^1, \gamma^2)$ then $\gamma^1 \simeq_p c_x$ and $\gamma^2 \simeq_p c_y$ and also there exist two homotopies between respective loops say $G : \gamma^1 \simeq_p c_x$ and $H : \gamma^2 \simeq_p c_y$. Now Define $F : I \times I \to M \times N$ by $F(s, t) = (G(s, t), H(s, t))$, which satisfy $F(s, 0) = (G(s, 0), H(s, 0)) = (\gamma^1, \gamma^2) = \gamma$ and $F(s, 1) = (G(s, 1), H(s, 1)) = (c_x, c_y) = c_{(x,y)}$, for all $s \in I$ and $F(0, t) = (G(0, t), H(0, t)) = (x, y) = (G(1, t), H(1, t)) = F(1, t)$, for all $t \in I$. One can check it is a continuous map, hence F is a path homotopy between γ and a constant loop $c_{(x,y)}$, so $[\gamma] \in (\bar{\pi}_1(M \times N))_0$ and implies $\mathrm{Ker}(Q) \subset (\pi_1(M \times N))_0$. Hence Q is an injection. Moreover, Q is indeed an open map, thus $\bar{\pi}_1(M \times N)$ is topological groupoid isomorphic to $\bar{\pi}_1 M \times \bar{\pi}_1 N$. □

Proposition 4.22. *Let M and N be two topological spaces then $f, g : M \to N$ are homotopic if and only if induced groupoid homomorphisms $f_{\#}, g_{\#}$ are homotopic.*

Proof. First, let $f, g : M \to N$ be homotopic, this implies there is a homotopy $H : M \times I \to N$ by $(x, t) \to H(x, t)$ satisfying $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$ for all $x \in M$. Define a map $F : \bar{\pi}_1 M \times I \to \bar{\pi}_1 N$ by $F([\gamma], t) = (H_t)_{\#}([\gamma])$, where $H_t : M \to N$ by $H_t(x) = H(x, t)$ for each $t \in I$ and $(H_t)_{\#}$ is its induced groupoid homomorphism. This gives $F([\gamma], 0) = (H_0)_{\#}([\gamma]) = f_{\#}([\gamma]), F([\gamma], 1) = (H_1)_{\#}([\gamma]) = g_{\#}([\gamma])$ for all $[\gamma] \in \bar{\pi}_1 M$ and also F continuous, so a homotopy between $f_{\#}$ and $g_{\#}$.

Conversely, let $f_{\#}, g_{\#}$ be induced groupoid homomorphisms of two continuous functions f, g and $f_{\#}$ and $g_{\#}$ be homotopic. So, there exists a homotopy $G : \bar{\pi}_1 M \times I \to \bar{\pi}_1 N$ satisfying $G([\gamma], 0) = f_{\#}([\gamma])$ and $H([\gamma], 1) = g_{\#}([\gamma])$ for all $[\gamma] \in \bar{\pi}_1 M$. Define a map $L : M \times I \to N$ by $L(x, t) = p_N(G([c_x], t))$ and it satisfy $L(x, 0) = p_N(G([c_x], 0)) = p_N(f_{\#}([c_x])) = f(x)$, $L(x, 1) = p_N(G([c_x], 1)) = p_N(g_{\#}([c_x])) = g(x)$, for all $x \in M$. It is clear that L is continuous, thus a homotopy between f and g. □

Proposition 4.23. *Let M and N be two topological spaces then $f : M \to N$ and $g : M \to N$ is a homotopy equivalence if and only if induced groupoid homomorphisms $f_{\#}, g_{\#}$ is a homotopy equivalence.*

Proof. Let $f : M \to N$ and $g : N \to M$ be a homotopy equivalence, this implies $f \circ g \simeq Id_N$ and $g \circ f \simeq Id_M$. Proposition 4.22 implies that $(fog)_{\#} \simeq (Id_N)_{\#}$, on simplification, we can have $f_{\#} \circ g_{\#} \simeq Id_{\bar{\pi}_1 N}$ and also similarly $g_{\#} \circ f_{\#} \simeq Id_{\bar{\pi}_1 M}$. Thus $f_{\#}, g_{\#}$ is a homotopy equivalence.

Conversely, let $f_{\#} : \bar{\pi}_1 M \to \bar{\pi}_1 N$ and $g_{\#} : \bar{\pi}_1 N \to \bar{\pi}_1 M$ be a homotopy equivalence, this implies $f_{\#} \circ g_{\#} = (f \circ g)_{\#} \simeq Id_{(\bar{\pi}_1 N)} = (Id_N)_{\#}$, and $g_{\#} \circ f_{\#} = (g \circ f)_{\#} \simeq Id_{(\bar{\pi}_1 M)} = (Id_M)_{\#}$, Proposition 4.22 implies that $f \circ g \simeq Id_N$ and $g \circ f \simeq Id_M$. Thus f, g is a homotopy equivalence. □

Corollary 4.24. *Let M and N be two topological spaces then M be same homotopy type to N if and only if $\bar{\pi}_1 M$ is same homotopy type to $\bar{\pi}_1 N$.*

Proof. The proof follows from Proposition 4.23. □

Proposition 4.25. *Let $T : (\bar{\pi}_1 M, \mathcal{I}_{p_M}) \to (\bar{\pi}_1 N, \mathcal{I}_{p_N})$ be a topological groupoid homomorphism then induced base map $b_T : M \to N$ is a continuous map such that $(b_T)_{\#}([c_x]) = T([c_x]), \forall [c_x] \in \bar{\pi}_1 M$.*

Proof. Here $T : (\bar{\pi}_1 M, \mathcal{I}_{p_M}) \to (\bar{\pi}_1 N, \mathcal{I}_{p_N})$ is a topological groupoid homomorphism. So by Proposition 3.9, induced base map $b_T : M \to N$ is a well-defined map and also satisfies that $p_N \circ T = b_T \circ p_M$. Lastly, about the continuity of b_T, for every open set V of N, the $(p_N \circ T)^{-1}(V) = (b_T \circ p_M)^{-1}(V) = {p_M}^{-1}({b_T}^{-1}(V))$ is an open set in $\bar{\pi}_1 M$. Since standard projection p_M is an open map, the $p_M((p_M)^{-1}({b_T}^{-1}(V))) = {b_T}^{-1}(V)$ is an open set in M. Thus b_T is a continuous map. Moreover, b_T gives induced map $(b_T)_\# : (\bar{\pi}_1 M, \mathcal{I}_{p_M}) \to (\bar{\pi}_1 N, \mathcal{I}_{p_N})$ and $\forall [c_x] \in \bar{\pi}_1 M$ the $(b_T)_\#([c_x]) = [b_T \circ c_x] = [c_{b_T(x)}] = T([c_x])$. □

Remark 4.26. *i) If $T, S : (\bar{\pi}_1 M, \mathcal{I}_{p_M}) \to (\bar{\pi}_1 N, \mathcal{I}_{p_N})$ are two topological groupoid homomorphisms and even though they may be different, but, sometimes one can see $b_T = b_S$.*
For this, we have an example, consider the projective plane $P^2 = S^2/\sim$ is the quotient space of the sphere over antipodal identification. The identity map $Id : (\bar{\pi}_1 P^2, \mathcal{I}_{p_{P^2}}) \to (\bar{\pi}_1 P^2, \mathcal{I}_{p_{P^2}})$ and $S : (\bar{\pi}_1 P^2, \mathcal{I}_{p_{P^2}}) \to (\bar{\pi}_1 P^2, \mathcal{I}_{p_{P^2}})$ defined by $S([\gamma_x]) = [\overline{\gamma_x}]$ are obviously topological groupoid homomorphisms and they are different, but we can see $b_{Id} = b_S$. In fact, both of them are equal to Id_{P^2}. Therefore, we cannot coin a proposition that "$T, S : (\bar{\pi}_1 M, \mathcal{I}_{p_M}) \to (\bar{\pi}_1 N, \mathcal{I}_{p_N})$ be two topological groupoid homomorphism then $T = S$ If and only if $b_T = b_S$".
ii) Converse of Proposition 3.10 need not be true, because for the map $T : (\bar{\pi}_1 P^2, \mathcal{I}_{p_{P^2}}) \to (\bar{\pi}_1 P^2, \mathcal{I}_{p_{P^2}})$ by

$$T([\gamma_x]) = \begin{cases} Id_{\pi_1(P^2, x_0)}([\gamma_x]) & \text{if } [\gamma_x] \in \pi_1(P^2, x_0) \\ [c_{x_0}] & \text{Otherwise,} \end{cases}$$

is a non-constant topological groupoid homomorphism, but, b_T is a constant.

Proposition 4.27. *Let $g : M \to N$ be a continuous map and $g_\# : (\bar{\pi}_1 M, \mathcal{I}_{p_M}) \to (\bar{\pi}_1 N, \mathcal{I}_{p_N})$ be its induced groupoid homomorphism then $b_{(g_\#)} = g$.*

Proof. Proposition 4.25 implies $(b_{(g_\#)})_\#([c_x]) = g_\#([c_x]), \forall [c_x] \in \bar{\pi}_1 M$. We can see this result by contradiction, that is, suppose $b_{(g_\#)} \neq g$, this implies there exists at least one $x_0 \in M$ such that $b_{(g_\#)}(x_0) = y_0$ (say) is not equal to $g(x_0) = y_1$ (say). But this gives that, $g_\#([c_{x_0}]) = [c_{y_1}] \neq [c_{y_0}] = (b_{(g_\#)})_\#([c_{x_0}])$, which is a contradiction. Therefore $b_{(g_\#)} = g$. □

Proposition 4.28. *Let $T : (\bar{\pi}_1 M, \mathcal{I}_{p_M}) \to (\bar{\pi}_1 N, \mathcal{I}_{p_N})$ be a topological groupoid homomorphism then $b_T = b_{(b_T)_\#}$.*

Proof. By Proposition 4.25, $(\mathrm{b}_{(b_{T\#})})_\#([c_x]) = g_\#([c_x]), \forall [c_x] \in \bar{\pi}_1 M$. We can see this result by contradiction, that is, suppose $b_T \neq b_{((b_T)_\#)}$, this implies there exists at least one $x_0 \in M$ such that $b_T(x_0) = y_0$ (say) is not equal to $b_{((b_T)_\#)}(x_0) = y_1$ (say). But this gives that, $b_{T\#}([c_{x_0}]) = [c_{y_1}] \neq [c_{y_0}] = b_{((b_T)_\#)}([c_{x_0}])$, which is a contradiction. Therefore $b_T = b_{(b_T)_\#}$. □

Proposition 4.29. *Let $T : (\bar{\pi}_1 M, \mathcal{I}_{p_M}) \to (\bar{\pi}_1 N, \mathcal{I}_{p_N})$ be a topological groupoid isomorphism and $b_T : M \to N$ be the induced base map of T, then $(b_T)^{-1} = b_{T^{-1}}$ and the b_T is a homeomorphism.*

Proof. Here $T : (\bar{\pi}_1 M, \mathcal{I}_{p_M}) \to (\bar{\pi}_1 N, \mathcal{I}_{p_N})$ is a topological groupoid isomorphism. So b_T is bijective by Proposition 3.11(i), and by Proposition 4.27 both b_T and $(b_T)^{-1} = b_{T^{-1}}$ are continuous. Hence the proof. □

Theorem 4.30. *(Main result) Let M, N be two topological spaces and $f : M \to N$ be a map then f is a homeomorphism if and only if $f_\# : (\bar{\pi}_1 M, \mathcal{I}_{p_M}) \to (\bar{\pi}_1 N, \mathcal{I}_{p_N})$ is a topological groupoid isomorphism.*

Proof. If f is a homeomorphism then $f_\#$ and $(f_\#)^{-1} = (f^{-1})_\#$ are topological groupoid homomorphisms by Propositions 4.19. Hence $f_\#$ is a topological groupoid isomorphism.

Conversely, if $f_\# : (\bar{\pi}_1 M, \mathcal{I}_{p_M}) \to (\bar{\pi}_1 N, \mathcal{I}_{p_N})$ is a topological groupoid isomorphism, by Proposition 4.27, the $b_{(f_\#)} = f$ and $b_{(f_\#^{-1})} = f^{-1}$ and they are continuous by Proposition 4.25. Thus f is a homeomorphism. □

Corollary 4.31. *(Main result) Let M, N be two topological spaces, then M is homeomorphic to N if and only if $\bar{\pi}_1 M$ is topological groupoid homomorphic to $\bar{\pi}_1 N$.*

Proof. If M is homeomorphic to N then there exists a homeomorphism $f : M \to N$. Its induced groupoid homomorphism $f_\# : (\bar{\pi}_1 M, \mathcal{I}_{p_M}) \to (\bar{\pi}_1 N, \mathcal{I}_{p_N})$ become a topological groupoid isomorphism.

Conversely, if $\bar{\pi}_1 M$ is a topological groupoid homomorphic to $\bar{\pi}_1 M$ then there exists a topological groupoid isomorphism $T : (\bar{\pi}_1 M, \mathcal{I}_{p_M}) \to (\bar{\pi}_1 N, \mathcal{I}_{p_N})$, and also from Proposition 4.30, the induced base map $b_T : M \to N$ is a homeomorphism. □

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