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The Holder continuity of the solutions to quasi-linear system of elliptic partial differential equations with singular coefficients

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Abstract

This article establishes the Holder continuity of the solutions to a quasi-linear system of elliptic partial differential equations with singular coefficients under the assumption of its form-boundary.

Keywords: *Holder continuity, partial differential equation, singular coefficients, Sobolev space.*

Subject classification: *35J15, 35J60, 35K62.*

1. Introduction

This article is dedicated to the Holder regularity conditions for the solutions to the quasi-linear system of elliptic partial differential equations with singular slow-growing coefficients [1-6]. In our previous works was shown that such a system has the solutions in Sobolev space [1]; in the present article, the Holder properties of these solutions are studied [47-49], [7-10, 36, 37]. The main goal of this article to establish conditions under which the solutions to a quasi-linear system of elliptic partial differential equations belong to the functional Holder space.

Let us consider the quasi-linear system of elliptic partial differential equations

$$\lambda u^k - \sum_{i,j=1,...,l} \frac{\partial}{\partial x_i}\left(a_{ij}(x,\vec{u})\frac{\partial}{\partial x_j}u^k\right) + b^k(x,\vec{u},\nabla\vec{u}) = f^k, \quad k = 1,...,N \tag{1.1}$$

where $\lambda > 0$ is a real number; the $\vec{u}$ is an unknown vector-function of vector-argument $x \in R^l$, $\ l > 2$ and $\vec{f} = f(x)$ is given vector-function $f \in L^p \cap L^\infty$. The $\vec{b}(x,u,\nabla u)$ is given vector-function. 1. $b(x,y,z)$ is a real measurable function of its arguments and $b \in L^1_{loc}(R^l)$; 2. Function $\vec{b}(x,y,z)$ almost everywhere satisfies an inequality

$$\left|\vec{b}\,(x,\vec{u},\nabla\vec{u})\right| \le \mu_1(x)\,|\nabla\vec{u}| + \mu_2(x)\,|\vec{u}| + \mu_3(x) \tag{1.2}$$

where $\mu_1^2 \in PK_\beta(A)$, $\mu_2 \in PK_\beta(A)$, $\mu_3 \in L^p(R^l)$; 3. Growth of the function $\vec{b}(x,y,z)$ almost everywhere satisfies a condition

$$\left|\vec{b}\,(x,\vec{u},\nabla\vec{u}) - \vec{b}\,(x,\vec{v},\nabla\vec{v})\right| \le \mu_4(x)\,|\nabla\,(\vec{u}-\vec{v})| + \mu_5(x)\,|\vec{u}-\vec{v}|\,, \tag{1.3}$$

where $\mu_4^2 \in PK_\beta(A)$, $\mu_5 \in PK_\beta(A)$.

The $a_{ij}(x)$ is a measurable matrix of $l \times l$ size and satisfies the condition $\exists\nu,\ \mu: \quad 0 < \nu < \mu < \infty$ such that

$$\nu\sum_{i=1}^{l}\xi_i^2 \le \sum_{ij=1,...,l} a_{ij}\xi_i\xi_j \le \sum_{i=1}^{l}\xi_i^2 \quad \forall\xi \in R^l. \tag{1.4}$$

The functional class of form-bounded functionsPK_β can be defined as

$$PK_\beta(A) = \left\{f \in L^1_{loc}(R^l, d^l x): \quad \left|\left\langle f\,|h|^2\right\rangle\right| \le \beta\left\langle A^{\frac{1}{2}}h, A^{\frac{1}{2}}h\right\rangle + c\,(\beta)\,||h||_2^2\right\},$$

where a function $h \in D(A^{\frac{1}{2}})$ and a number $\beta > 0$ is a form-boundary and constant$c(\beta) \in R^1$ [6].

Let us denote

$$\|\vec{u}\|_{L^p(R^l)} = \left\langle \sum_{i=1,\dots,N} \left|u^i\right|^p \right\rangle^{\frac{1}{p}} = \left(\sum_{i=1,\dots,N} \left\langle \left|u^i\right|^p \right\rangle \right)^{\frac{1}{p}},$$

$$\langle \vec{u}, \vec{v} \rangle = \sum_{i=1,\dots,N} \left\langle u^i, v^i \right\rangle \qquad \forall u \in L^p(R^l)\ \forall v \in L^q(R^l),$$

then we have equality

$$\|\vec{u}\|^{p-1}_{L^p(R^l)} = \left\langle \sum_{i=1,\dots,N} \left|u^i\right|^p \right\rangle^{\frac{p-1}{p}} = \left\langle \sum_{i=1,\dots,N} \left(\left|u^i\right|^{\frac{p}{q}} \right)^q \right\rangle^{\frac{1}{q}} = \left\| |\vec{u}|^{p-1} \right\|_{L^q(R^l)}.$$

Next, we denote

$$|\nabla \vec{u}|^p = \sum_{i=1,\dots,N} \sum_{k=1,\dots l} \left| \frac{\partial}{\partial x_k} u^i \right|^p$$

and

$$\|u\|_p^p = \left\langle \sum_{i=1}^{N} u^i u^i \left|u^i\right|^{p-2} \right\rangle \equiv \sum_{i=1}^{N} \left\langle u^i u^i \, |u|^{p-2} \right\rangle.$$

Definition (*of weak solution*). *A vector-function $\vec{u} \in W_1^p(R^l, d^l x)$ is called a weak solution to a quasilinear system of elliptic partial differential equations if the integral identity*

$$\lambda \langle \vec{u}, \vec{\nu} \rangle + \left\langle \sum_{i,j=1,\dots,l} a_{ij} \frac{\partial}{\partial x_j} \vec{u}, \frac{\partial}{\partial x_i} \vec{\nu} \right\rangle + \left\langle \vec{b}, \vec{\nu} \right\rangle = \left\langle \vec{f}, \vec{\nu} \right\rangle$$

is valid for all vector-functions $v \in W_{1,0}^q(R^l, d^l x)$.

The main result of this article can be formulated follows.

Theorem 1. *The weak solution $\vec{u} \in W_1^p(R^l, d^l x)$ to the quasilinear system (1) under the assumptions 1-4 belongs to Holder space of continuous functions.*

2. The estimation of the main part of the elliptic differential operator

Let us consider a simpler elliptic system

$$\frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}\vec{u}\right)=0,$$

let us compose the integral identity as

$$\left\langle \sum_{i,j=1,\dots,l} a_{ij}\frac{\partial}{\partial x_j}\vec{u},\frac{\partial}{\partial x_i}\vec{v}\right\rangle=0$$

where $\vec{v}\in W^2_{1,0}(R^l,d^lx)$ and $\vec{u}\in W^2_1(R^l,d^lx)$ under the ellipticity condition and the condition $vrai\max|\vec{u}|<\infty$.

Let us assume that function $\vec{u}\,(x)$ measurable in a ball K_{ρ_0} and there are N_1 functions $\vec{w}^1(x),...,\vec{w}^{N_1}(x)$ such that for the arbitrary ball $K_{b\rho}$, concentric with K_{ρ_0}, there is at least one function $\vec{w}^r(x)$ such that

$$osc\{|\vec{w}^r|\,,\ \Omega_{b\rho}\}\geq\delta_1\ \max\ osc\ \{|\vec{u}|\,,\ \Omega_{b\rho}\},$$

for the function $\vec{u}\,(x)$, we are obtaining that holds at least one of the following inequalities

$$osc\{|\vec{w}^r|\,,\ \Omega_\rho\}\leq c_1\rho^\delta,$$

Or

$$osc\{|\vec{w}^r|\,,\ \Omega_\rho\}\leq\vartheta\, osc\{|\vec{w}^r|\,,\ \Omega_{b\rho}\},$$

where the balls K_{ρ_0}, K_ρ and $K_{b\rho}$ have the same center, and constant b is a fix; others satisfy the following conditions $b\rho\leq\rho_0,\ b>1,\ c_1\leq1,\ \delta\leq 1,\ \vartheta<1\Omega_\rho=\Omega\cap K_\rho$.

Then for $\rho\leq\rho_0$ there is an estimation

$$osc\{u,\ \Omega_\rho\}\leq A\left(\frac{\rho}{\rho_0}\right)^\alpha,$$

where we denote

$\alpha=\frac{1}{N_1}\min\{-\log_b\vartheta,\ \delta\}$, $c=\frac{b^{\alpha(N_1+1)}}{\delta_1}\max\{b^{\alpha N_1}\max\limits_{i=1,\dots,N_1} osc\{|\vec{w}^i|\,,\ \Omega_{\rho_0}\},\ c_1\rho_0^\delta\}$ and $u=|\vec{u}|$.

To assert that the function $\vec{u}$ belongs to Holder space enough to show that

$$osc\{|\vec{u}|\,,\ K_R\}\leq\vartheta\, osc\{|\vec{u}|\,,\ K_{2R}\}.$$

For a positive number $\varepsilon > 0$, let us consider the function $\psi(\vec{u}) = -\ln 2(1 - |\vec{u}| + \varepsilon)$, presuppose that in the ball K_R holds the estimation $-\ln 2(1 - |\vec{u}| + \varepsilon) < L$, then we obtain that $\frac{\exp(-L)}{2} < 1 - |\vec{u}| + \varepsilon$, or $|\vec{u}(x)| < 1 - \frac{\exp(-L)}{2} + \varepsilon$ if we put $\vartheta = 1 - \frac{\exp(-L)}{2} + \varepsilon$ and the number $\varepsilon > 0$ converges to zero we obtain that $osc\{\vec{u},\ K_R\} \leq \vartheta\, osc\{\vec{u},\ K_{2R}\}$. So, we have to show that the function $w(x) = \psi(\vec{u})(x) = -\ln 2(1 - |\vec{u}(x)| + \varepsilon)$ is bounded.

It can be assumed that the oscillation of function $\vec{u}\,(x)$ in the ball K_{2R} equals one, that is $0 \leq \vec{u}\,(x) \leq 1$, then one of the properties is always executed:

$$mes\left\{x \in\ K_R,\ |\vec{u}(x)| \leq \frac{1}{2}\right\} \geq \frac{1}{2} mes K_R,$$

$$mes\left\{x \in\ K_R, 1 -\ |\vec{u}(x)| \leq \frac{1}{2}\right\} \geq \frac{1}{2} mes K_R,$$

if holds the first property we consider the function $\vec{u}\,(x)$, if the second is true we consider $1 - \vec{u}\,(x)$. Assume that the first property executes then for arbitrary function $\vec{v} \in W^2_{1,0}(K_{2R})$, we have

$$\left\langle \sum_{i,j=1,\ldots,l} a_{ij} \frac{\partial}{\partial x_j}\vec{u}, \frac{\partial}{\partial x_i}\vec{v} \right\rangle_{K_{2R}} = 0.$$

To show that the function $w(x) = \psi(\vec{u})(x) = -\ln 2(1 - |\vec{u}(x)| + \varepsilon)$ is bounded above, we denote

$$\vec{v}(x) = \frac{\vec{\xi}(x)}{1 - |\vec{u}(x)| + \varepsilon} = \psi'(\vec{u})(x)\vec{\xi}(x)$$

since

$$\frac{\partial}{\partial x_i}\vec{v} = \psi''\vec{\xi}\frac{\partial}{\partial x_i}|\vec{u}| + \psi'\frac{\partial}{\partial x_i}\vec{\xi}$$

we are obtaining

$$\left\langle \sum_{i,j=1,\ldots,l} a_{ij} \frac{\partial}{\partial x_j}\vec{u}, \psi''\vec{\xi}\frac{\partial}{\partial x_i}|\vec{u}| \right\rangle_{K_{2R}} + \left\langle \sum_{i,j=1,\ldots,l} a_{ij} \frac{\partial}{\partial x_j}\vec{u}, \psi'\frac{\partial}{\partial x_i}\vec{\xi} \right\rangle_{K_{2R}} = 0$$

$$\left\langle \sum_{i,j=1,\ldots,l} a_{ij} \frac{\partial}{\partial x_j}\vec{w}, \vec{\xi}\frac{\partial}{\partial x_i}|\vec{w}| \right\rangle_{K_{2R}} + \left\langle \sum_{i,j=1,\ldots,l} a_{ij} \frac{\partial}{\partial x_j}\vec{w}, \frac{\partial}{\partial x_i}\vec{\xi} \right\rangle_{K_{2R}} = 0.$$

Denote function $\xi = \varphi^2\left(\frac{|x-x_0|}{R}\right)$, where the function $\varphi(t)$ equals one if $\frac{|x-x_0|}{R} \in \left[0,\ \frac{3}{2}\right]$ and if argument equals two $\varphi(t)$ linearly decreases to zero

$|x - x_0| = 2R$. Applying elliptic condition $\nu \sum_{i=1}^{l} \xi_i^2 \leq \sum_{ij=1,\dots,l} a_{ij}(x)\xi_i\xi_j \leq \mu \sum_{i=1}^{l} \xi_i^2 \quad \forall \xi \in R^l$, we obtain

$$\left\langle |\nabla \vec{w}|^2 \right\rangle_{K_{\frac{3}{2}R}} \leq \frac{16\mu}{\nu R^2} mesK_{2R}.$$

Next step, we will apply Moser's idea: that from the boundedness of the convex function

$$w(x) = \psi(\vec{u})(x) = -\ln 2(1 - |\vec{u}(x)| + \varepsilon)$$

follows that the function $\vec{u}$ is Holder continuous.

Since $0 \leq \vec{u}\,(x) \leq 1$ and

$$w(x) = \psi(\vec{u})(x) = -\ln 2(1 - |\vec{u}(x)| + \varepsilon)$$

we have

$$\inf w(x) = -\ln 2(1 + \varepsilon).$$

The value $\left\langle |w|^2 \right\rangle_{K_{\frac{3}{2}R}}$ can be estimated by applying the De Giorgi method or Nash estimation, we will use the De Giorgi lemma [23, 24].

Let vector $\vec{u} \in W_1^p(\Omega)$, for all positive number k, we denote by A_k the set $A_k \equiv \{x \in \Omega : \ |\vec{u}(x)| > k\}$ and the sets $A_k^0 \equiv \{x \in \Omega : \ |\vec{u}(x)| = k\}$, $A_{k,\rho} \equiv \{x \in K_\rho : \ |\vec{u}(x)| > k\}$ and function $u_k(x) = \max(|\vec{u}(x)| - k, \, 0)$. Form definition we deduce the following properties:

$$1. A_k = \bigcup_{\varepsilon > 0} A_{k+\varepsilon}$$

$$2. mes\,(A_k \backslash A_{k+\varepsilon}) \xrightarrow{\varepsilon \to 0} 0$$

$$3. mes\left(A_{k-\varepsilon} \backslash A_k \bigcup A_k^0\right) \xrightarrow{\varepsilon \to 0} 0$$

$$4. u_k \in W_1^p(\Omega).$$

Lemma 1. *Let $\vec{u} \in W_1^2(K_\rho)$ and A is an arbitrary subset of a set K_ρ, denote set $A_0 \equiv \{x \in K_\rho : \ |\vec{u}(x)| = 0\}$ and positive number $\beta > 0$ that only depended on the dimension of space then for all $k \geq n$ the following inequalities hold*

$$\langle |\vec{u}| \rangle_A \leq \beta \frac{\rho^l}{mesA_0} (mesA)^{\frac{1}{l}} \langle |\nabla \vec{u}| \rangle_{K_\rho},$$

$$(n-k)\left(mesA_{n,\rho}\right)^{1-\frac{1}{l}} \leq \beta \frac{\rho^l}{mes\left(K_\rho \backslash A_{n,\rho}\right)A_0} \left\langle |\nabla \vec{u}| \right\rangle_{A_{k,\rho} \backslash A_{n,\rho}},$$

$$\left\langle \left|\nabla\left(|\vec{u}_k|^m\right)\right| \right\rangle_{K_\rho} \leq m \left\langle |\nabla \vec{u}|^m \right\rangle_{A_{k,\rho}}^{\frac{1}{m}} \left\langle \left||\vec{u}|-k\right|^m \right\rangle_{A_{k,\rho}}^{1-\frac{1}{m}}.$$

There is a constant that c that depends only on the dimension of the space and the ellipticity of the matrix, such that

$$\left\langle |\nabla w|^2 \right\rangle_{K_{\frac{3}{2}R}} \leq cR^l.$$

Let us denote

$$\xi(x) = \varphi^2(x) \max\left(w(x)-k,\ 0\right) \quad \forall k$$

where the function φ is a cutoff function for the ball K_ρ, $\quad \rho \in \left[R,\ \frac{3}{2}R\right]$ then we obtain estimation (here we denote $A_{k,\rho} \equiv \{x \in K_\rho :\ w(x) > k\}$)

$$\left\langle \sum_{i,j=1,\ldots,l} a_{ij} \frac{\partial}{\partial x_j} w, \varphi^2 \frac{\partial}{\partial x_i} w \right\rangle_{A_{k,\rho}} + \left\langle \sum_{i,j=1,\ldots,l} a_{ij} \frac{\partial}{\partial x_j} w, 2\varphi\left(w-k\right) \frac{\partial}{\partial x_i} \varphi \right\rangle_{A_{k,\rho}} \leq 0,$$

that is

$$\left\langle |\nabla w|^2 \varphi^2 \right\rangle_{A_{k,\rho}} \leq ! \left\langle (w-k)^2 |\nabla \varphi|^2 \right\rangle_{A_{k,\rho}},$$

thus, we have obtained that there is a constant M that depends only on the ellipticity constants and dimension l of the space such that holds inequality

$$vrai \max_{K_R}\ w(x) < M,$$

so, the function-solution u belongs to the Holder's functional class.

Lemma 2. *Let $\vec{u}\,(x)$ is a given measurable in a ball K_1 function and balls $K_1,\ K_\rho$and $K_{\rho-\sigma\rho}$ with a common center, and constants $1 > \sigma_0 > 0,\ \gamma > 0,\ \alpha > 0,\ \varepsilon > 0,$ and $\varepsilon \leq \frac{m}{l},\ m \leq \alpha < \varepsilon m + m,$ and for arbitrary natural number $k > k_0$ holds the inequality*

$$\left\langle |\nabla \vec{u}|^m \right\rangle_{A_{k,\rho-\sigma\rho}} \leq \gamma \sigma^{-m} \left\langle \left(|u|-k\right)^m \right\rangle_{A_{k,\rho}} + \gamma k^\alpha \left(mesA_{k,\rho}\right)^{1-\frac{m}{l}+\varepsilon}.$$

Then in the ball $K_{1-\sigma}$, the value $vrai \max\limits_{K_{1-\sigma}}\ |\vec{u}(x)|$ can be estimated by a constant that only depends on $\sigma_0,\ \gamma,\ \alpha,\ \varepsilon,\ k_0,\ m,\ l$ and magnitude $a = \left\langle \left||\vec{u}\,(\cdot)|-k\right|^m \right\rangle_{A_{k_0,1}}$.

Proof. Let $\tilde{k} > k_0$ be the natural number and consider the sequence $K_{\rho_i}$of balls having a common center and with radii $\rho_i = 1 - \sigma_0 + \frac{\sigma_0}{2^i},\ i = 0,\, 1,\, 2, ...$, and the sequence of the planes $k_i = 2\tilde{k} - \frac{\tilde{k}}{2^i},\ i = 0,\, 1,\, 2, ...$ Let us denote $\xi\,(t)$ the continuous differentiable non-increase function of the argument$t \in (-\infty,\, \infty)$, which equals one when $t \leq \sigma_0$ and zero when $t \geq \frac{3}{2}\sigma_0$ next we denote the sequence of functions

$$\xi_i(t) = \xi\left(2^{i+1}(|x| - 1 + \sigma_0)\right),\ i = 0,\, 1,\, 2, ...$$

and sequence of numbers

$$J_i = \left\langle (u - k_i)^m \right\rangle_{A_{k_i,\rho_i}},\ i = 0,\, 1,\, 2, ...$$

We estimate

$$J_{i+1} \leq \left\langle (u - k_{i+1})^m\, \xi_i^m \right\rangle_{A_{k_{i+1},\frac{\rho_i + \rho_{i+1}}{2}}},\ i = 0,\, 1,\, 2, ...$$

and applying Holder estimation, we are obtaining recurrent inequality

$$\begin{aligned}
&J_{i+1} \leq \left\langle (u - k_{i+1})^m\, \xi_i^m \right\rangle_{A_{k_{i+1},\frac{\rho_i + \rho_{i+1}}{2}}} \leq \\
&\leq C\left(mesA_{k_{i+1},\frac{\rho_i + \rho_{i+1}}{2}}\right)^{\frac{m}{l}} \left\langle |\nabla u|^m \right\rangle_{A_{k_{i+1},\frac{\rho_i + \rho_{i+1}}{2}}} + \\
&+C\left(mesA_{k_{i+1},\frac{\rho_i + \rho_{i+1}}{2}}\right)^{\frac{m}{l}} \max_{t \in \left[\sigma_0,\, \frac{3}{2}\sigma_0\right]} |\xi'(t)|^m\, 2^{mi} \left\langle (u - k_i)^m \right\rangle_{A_{k_i,\rho_i}} = \\
&= C\left(mesA_{k_{i+1},\frac{\rho_i + \rho_{i+1}}{2}}\right)^{\frac{m}{l}} \left\langle |\nabla u|^m \right\rangle_{A_{k_{i+1},\frac{\rho_i + \rho_{i+1}}{2}}} + \\
&+C\left(mesA_{k_{i+1},\frac{\rho_i + \rho_{i+1}}{2}}\right)^{\frac{m}{l}} \max_{t \in \left[\sigma_0,\, \frac{3}{2}\sigma_0\right]} |\xi'(t)|^m\, 2^{mi} J_i,\ i = 0,\, 1,\, 2, ...
\end{aligned}$$

We put $k = k_{i+1}$, $\rho = \rho_i$, $\rho - \sigma\rho = \frac{\rho_i + \rho_{i+1}}{2}$, then

$$\begin{aligned}
&\left\langle |\nabla u|^m \right\rangle_{A_{k_{i+1},\frac{\rho_i + \rho_{i+1}}{2}}} \leq \gamma 2^{im+3m} \left\langle |u - k_{i+1}|^m \right\rangle_{A_{k_{i+1},\rho_i}} + \\
&+\gamma k_{i+1}^{\alpha}\, \left(mesA_{k_{i+1},\rho_i}\right)^{1-\frac{m}{l}+\varepsilon} \leq \\
&\leq \gamma\left(2^{3m} + 2^{\alpha}\right) 2^{mi} J_i + \gamma\left(2^{3m} + 2^{\alpha}\right) \tilde{k}^{\alpha}\, \left(mesA_{k_{i+1},\rho_i}\right)^{1-\frac{m}{l}+\varepsilon}.
\end{aligned}$$

Further, we assess

$$J_i \geq \left\langle |u - k_i|^m \right\rangle_{A_{k_{i+1},\rho_i}} \geq (k_{i+1} - k_i)^m\, \left(mesA_{k_{i+1},\rho_i}\right) = 2^{-m(i+1)}\tilde{k}^m\, mesA_{k_{i+1},\rho_i}.$$

Then, when $\frac{m^2}{l} \geq m + m\varepsilon - \alpha > 0$, we have

$$J_{i+1} \leq C_1 2^{i\left(m+\frac{m^2}{l}\right)} \left(\tilde{k}^{-\frac{m^2}{l}} J_i^{1+\frac{m}{l}} + \tilde{k}^{-m-m\varepsilon+\alpha} J_i^{1+\varepsilon}\right) .$$

However, the estimations

$$J_i = \langle (u-k_i)^m \rangle_{A_{k_i,\rho_i}} \leq \left\langle \left(u-\tilde{k}\right)^m \right\rangle_{A_{k_0,1}} ,\ i = 0,\, 1,\, 2, ...$$

are holding and so for $\tilde{k} \geq 1$ we are obtaining the recurrent inequalities

$$J_{i+1} \leq C_1 \left(1 + \left\langle \left(u-\tilde{k}\right)^m \right\rangle_{A_{k_0,1}}^{\frac{m}{l}-\varepsilon}\right) 2^{i\left(m+\frac{m^2}{l}\right)} \tilde{k}^{-m-m\varepsilon+\alpha} J_i^{1+\varepsilon},\ \ i = 0,\, 1,\, 2, ...$$

Let us choose the value $\tilde{k} \geq 1$ such that the inequality

$$\tilde{k} \geq \max \left(k_0,\, 1,\, \left(C_1 1 + C_1 \left\langle \left(u-\tilde{k}\right)^m \right\rangle_{A_{k_0,1}}^{\frac{m}{l}-\varepsilon}\right)^{\frac{1}{m+m\varepsilon-\alpha}}\right.$$

$$\left. 2^{\left(m+\frac{m^2}{l}\right)\frac{1}{\varepsilon(m+m\varepsilon-\alpha)(1+\varepsilon)}} \left\langle \left(u-\tilde{k}\right)^m \right\rangle_{A_{k_0,1}}^{\frac{\varepsilon}{m+m\varepsilon-\alpha}}\right)$$

holds. So, we have

$$J_1 \leq \left(C_1 1 + C_1 \left\langle \left(u-\tilde{k}\right)^m \right\rangle_{A_{k_0,1}}^{\frac{m}{l}-\varepsilon}\right) \tilde{k}^{-m-m\varepsilon+\alpha} \left\langle \left(u-\tilde{k}\right)^m \right\rangle_{A_{k_0,1}}^{1+\varepsilon} \leq$$
$$\leq \tilde{k}^{\frac{m+m\varepsilon-\alpha}{\varepsilon}} \left(C_1 1 + C_1 \left\langle \left(u-\tilde{k}\right)^m \right\rangle_{A_{k_0,1}}^{\frac{m}{l}-\varepsilon}\right)^{-\frac{1}{\varepsilon}} 2^{-\left(m+\frac{m^2}{l}\right)\frac{1}{\varepsilon^2}},$$

applying the recursivity of the last estimation we obtain

$$J_{i+1} \leq Const 2^{-\left(m+\frac{m^2}{l}\right)\frac{i}{\varepsilon}},\ i = 0,\, 1,\, 2, ...,$$

$$J_{i+1} \overset{i\to\infty}{\longrightarrow} 0.$$

Thus, we have obtained $vrai \max\limits_{K_{1-\sigma_0}} |u(x)| = 2\tilde{k},$ the lemma 2 has been proven.

Lemma 3. Let function $\vec{u} \in W_1^1(\Omega)$, $\quad l > 2$ and $B(r)$ is a ball radius r. Then there is an estimation

$$mes\,(\Theta) \left\langle |\vec{u}| \right\rangle_{\Xi} \leq \beta r^l \left(mes\,(\Xi)\right)^{\frac{1}{l}} \left\langle |\nabla \vec{u}\,(\cdot)| \right\rangle_{B(r)} ,$$

here Θ is a set of points of the ball $B\left(r\right)$ such that $\vec{u}\left(x\right)=0$, and constant β is a function of the dimension of Euclid space.

Proof. For almost every $x\in B\left(r\right)$ and $y\in\Xi$, there is a representation

$$\vec{u}\left(y\right)-\vec{u}\left(x\right)=\int_0^{|x-y|}\frac{\partial\vec{u}\left(x+\omega\rho\right)}{\partial\rho}d\rho$$

where $(\rho,\quad\omega)$ are spherical coordinates. Next, we integrate this with respect to $y\in\Xi$ and obtain a iquality

$$-\vec{u}\left(x\right)mes\left(\Theta\right)=\left\langle\int_0^{|x-y|}\frac{\partial\vec{u}\left(x+\omega\rho\right)}{\partial\rho}d\rho\right\rangle_{\Theta}.$$

We can estimate

$$\begin{aligned}&\left\langle\int_0^{|x-y|}\frac{\partial\vec{u}(x+\omega\rho)}{\partial\rho}d\rho\right\rangle_{\Theta}\leq\\&\leq\int_{B(r)}|x-y|^{l-1}\,d\,|x-y|\int_0^{|x-y|}\frac{\partial\vec{u}(x+\omega\rho)}{\partial\rho}d\rho\leq\\&\leq\int_0^{2r}|x-y|^{l-1}\,d\,|x-y|\left\langle\frac{|\nabla\vec{u}(\cdot)|}{|\cdot-\xi|^{l-1}}\right\rangle_{B(r)}=\frac{(2r)^l}{l}\left\langle\frac{|\nabla\vec{u}(\cdot)|}{|\cdot-\xi|^{l-1}}\right\rangle_{B(r)}\end{aligned}$$

so, we have an inequality

$$|\vec{u}\left(x\right)|\,mes\left(\Theta\right)\leq\frac{(2r)^l}{l}\left\langle\frac{|\nabla\vec{u}\left(\cdot\right)|}{|\cdot-\xi|^{l-1}}\right\rangle_{B(r)}.$$

We integrate over Ξ

$$\langle|\vec{u}|\rangle_{\Theta}\,mes\left(\Theta\right)\leq\frac{(2r)^l}{l}\int_{B(r)}|\nabla\vec{u}\left(y\right)|\,dy\int_{\Xi}\frac{d\xi}{|y-\xi|^{l-1}}.$$

It is easy to see that

$$\int_{|y-\xi|\leq\varepsilon}\frac{d\xi}{|y-\xi|^{l-1}}=\varepsilon\cdot mes\left(S\right)$$

and

$$\int_{|y-\xi|\geq\varepsilon}\frac{d\xi}{|y-\xi|^{l-1}}\leq\varepsilon^{1-l}\cdot mes\left(\Xi\right)$$

so, we obtain an estimation

$$\int_{\Xi}\frac{d\xi}{|y-\xi|^{l-1}}\leq\varepsilon\cdot mes\left(S\right)+\varepsilon^{1-l}\cdot mes\left(\Xi\right).$$

Lemma 3 is proven.

3. Quasilinear system of elliptic partial differential equations with nonlinear perturbation

Let us consider a more general case of a quasilinear system of elliptic partial differential equations with nonlinear perturbation $\vec{b}$

$$\lambda \vec{u} - \frac{\partial}{\partial x_i} \left(a_{ij}(x, \vec{u}) \frac{\partial}{\partial x_j} \vec{u} \right) + \vec{b}(x, \vec{u}, \nabla \vec{u}) = 0,$$

The investigation will be carried out according to the scheme: we study the solution $\vec{u} \in W_1^p(R^l, d^l x)$ of the quasi-linear partial differential system of elliptic type, establish certain a priori estimations of this solution and its derivatives (applying the definition of a weak solution and assuming that element $\vec{v} \in W_{1,0}^q(R^l, d^l x)$, we are obtaining the theorems about this solution); study the properties of some functions of this solution $\vec{u} \in W_1^p(R^l, d^l x)$ (in the simplest case $\psi(\vec{u}) = -\ln 2(1 - |\vec{u}| + \varepsilon)$).

Applying this definition of a weak solution, we compile the following differential form $h_\lambda^p : W_1^p \times W_1^q \to R$ as

$$h_\lambda^p(\vec{u}, \vec{\nu}) \equiv \lambda \langle \vec{u}, \vec{\nu} \rangle + \langle \nabla \vec{\nu} \circ a \circ \nabla \vec{u} \rangle + \left\langle \vec{b}(x, \vec{u}, \nabla \vec{u}), \vec{\nu} \right\rangle$$

which is well defined over the functional space $W_1^p(R^l, d^l x) \times W_1^q(R^l, d^l x)$.

Let us assume that function $\vec{u} \in W_1^p(R^l, d^l x)$ is the solution of (1) that means that for an arbitrary function $\vec{v} \in W_{1,0}^q(R^l, d^l x)$ holds an integral tautology

$$h_\lambda^p(\vec{u}, \vec{\nu}) \equiv \lambda \langle \vec{u}, \vec{v} \rangle + \left\langle \sum_{i,j=1,\ldots,l} a_{ij} \frac{\partial}{\partial x_j} \vec{u}, \frac{\partial}{\partial x_i} \vec{v} \right\rangle + \left\langle \vec{b}, \vec{v} \right\rangle = 0.$$

To prove that function $\vec{u}$ is Holder continuous let us introduce a function of the solution of (1) as

$$w(x) = \psi(\vec{u})(x) = -\ln \left(\frac{osc\{u,\ K_{2R}\} - |\vec{u}(x)| + \varepsilon}{\delta_2\, osc\{u,\ K_{2R}\}} \right)$$

then we are going to show that

$$-\ln \left(\frac{osc\{u,\ K_{2R}\} - |\vec{u}(x)| + \varepsilon}{\delta_2\, osc\{u,\ K_{2R}\}} \right) \le M$$

and

$$\frac{\delta_2\, osc\{u,\ K_{2R}\}}{osc\{u,\ K_{2R}\} - |\vec{u}(x)| + \varepsilon} \le \exp(M)$$

and its conclusion

$$|\vec{u}(x)| \leq (1-\exp(-M)\delta_2)\ osc\{u,\ K_{2R}\}+\varepsilon,$$

where $u=|\vec{u}|$

Let us assume that in the integral tautology of weak solution $\vec{u}\in W_1^p(R^l,d^lx)$ the function $v\in W_{1,0}^q(R^l,d^lx)$ is $\vec{u}\,|\vec{u}|^{p-1}$ then we obtain

$$\lambda\left\langle \vec{u},\vec{u}\,|\vec{u}|^{p-1}\right\rangle+\left\langle \sum_{i,j=1,\dots l} a_{ij}\frac{\partial}{\partial x_j}\vec{u},\frac{\partial}{\partial x_i}\left(\vec{u}\,|\vec{u}|^{p-1}\right)\right\rangle+\left\langle \vec{b},\vec{u}\,|\vec{u}|^{p-1}\right\rangle=0,$$

and

$$\lambda\left\|\vec{u}\right\|^p+\frac{4\,(p-1)}{p^2}\left\langle \sum_{i,j=1,\dots l} a_{ij}\frac{p}{2}\,|\vec{u}|^{\frac{p-2}{2}}\,\nabla_j\vec{u},\frac{p}{2}\,|\vec{u}|^{\frac{p-2}{2}}\,\nabla_i\vec{u}\right\rangle+\left\langle \vec{b},\vec{u}\,|\vec{u}|^{p-1}\right\rangle=0,$$

let denote $\vec{w}=\vec{u}\,|\vec{u}|^{\frac{p-2}{2}}$ and respectively $\nabla\vec{w}=\frac{p}{2}\,|\vec{u}|^{\frac{p-2}{2}}\,\nabla\vec{u}$, the in all R^l, applying Holder and Young inequalities to Lebesgue's norms, we have

$$\begin{array}{l}\left|\left\langle \vec{b},\vec{u}\,|\vec{u}|^{p-1}\right\rangle\right|\leq\left(\left(\frac{\varepsilon^2}{p}+1\right)c\,(\beta)+\frac{1}{\sigma^q q}\right)\|\vec{w}\|^2+\\+\left(\frac{\beta\varepsilon^2}{p}+\beta+\frac{1}{p}\frac{1}{\varepsilon^2}\right)\langle\nabla\vec{w}\circ a\circ\nabla\vec{w}\rangle+\frac{\sigma^p}{p}\left\|\mu_3\right\|^p,\end{array}$$

or

$$\begin{array}{l}\lambda\left\|\vec{u}\right\|^p+\frac{4(p-1)}{p^2}\langle\nabla\vec{w}\circ a\circ\nabla\vec{w}\rangle\leq\left(\left(\frac{\varepsilon^2}{p}+1\right)c\,(\beta)+\frac{1}{\sigma^q q}\right)\|\vec{w}\|^2+\\+\left(\frac{\beta\varepsilon^2}{p}+\beta+\frac{1}{p}\frac{1}{\varepsilon^2}\right)\langle\nabla\vec{w}\circ a\circ\nabla\vec{w}\rangle+\frac{\sigma^p}{p}\left\|\mu_3\right\|^p.\end{array}$$

Let K_{ρ_0}, K_ρ and $K_{b\rho}$ be concentric balls and constant b such that $b\rho\leq\rho_0,\ b>1,\ c_1\leq1,\ \delta\leq1,\ \vartheta<1$ and $\Omega_\rho=\Omega\cap K_\rho$. The $\varsigma(x)$ is a cutoff function in the ball K_{2R} and let us choose $v=\varsigma^p\vec{u}\,|\vec{u}|^{p-2}$, we have

$$\lambda\left\langle \vec{u},\varsigma^p\vec{u}\,|u|^{p-2}\right\rangle_{K_{2R}}+\left|\left\langle \sum_{i,j=1,\dots l} a_{ij}\frac{\partial}{\partial x_j}\vec{u},\frac{\partial}{\partial x_i}\left(\varsigma^p\vec{u}\,|\vec{u}|^{p-2}\right)\right\rangle_{K_{2R}}\right|\leq$$

$$\leq\left|\left\langle \vec{b},\varsigma^p\vec{u}\,|\vec{u}|^{p-1}\right\rangle_{K_{2R}}\right|,$$

$$\begin{array}{l}\left\langle \sum_{i,j=1,\dots l} a_{ij}\frac{\partial}{\partial x_j}\vec{u},\frac{\partial}{\partial x_i}\left(\varsigma^p\vec{u}\,|\vec{u}|^{p-2}\right)\right\rangle_{K_{2R}}=\\=(p-1)\left\langle \sum_{i,j=1,\dots l}\varsigma^p a_{ij}\,|\vec{u}|^{\frac{p-2}{2}}\frac{\partial}{\partial x_j}\vec{u},|\vec{u}|^{\frac{p-2}{2}}\frac{\partial}{\partial x_i}\vec{u}\right\rangle_{K_{2R}}+\\+p\left\langle \sum_{i,j=1,\dots l} a_{ij}\vec{u}\,|\vec{u}|^{p-2}\varsigma^{p-1}\frac{\partial}{\partial x_j}\vec{u},\frac{\partial}{\partial x_i}\varsigma\right\rangle_{K_{2R}}\end{array}$$

we denote $\vec{w} = \vec{u}\,|\vec{u}|^{\frac{p-2}{2}}$ then

$$\left\langle \sum_{i,j=1,\dots,l} \varsigma^p a_{ij} \frac{\partial}{\partial x_j}\vec{u}, \frac{\partial}{\partial x_i}\left(\vec{u}\,|\vec{u}|^{p-2}\right)\right\rangle_{K_{2R}} = \frac{4\,(p-1)}{p^2}\left\langle \varsigma^p \nabla\vec{w}\circ a\circ\nabla\vec{w}\right\rangle_{K_{2R}},$$

after the transformation of the second term on the right side

$$\left\langle \sum_{i,j=1,\dots l} a_{ij}\vec{u}\,|\vec{u}|^{p-2}\frac{\partial}{\partial x_j}\vec{u}, \frac{\partial}{\partial x_i}\left(\varsigma^p\right)\right\rangle_{K_{2R}} = p\left\langle \sum_{i,j=1,\dots l} \varsigma^{p-1} a_{ij}\vec{u}\,|\vec{u}|^{p-2}\frac{\partial}{\partial x_j}\vec{u}, \frac{\partial}{\partial x_i}\varsigma\right\rangle_{K_{2R}}$$

by Young inequality, we have had

$$\begin{aligned}&\left\langle \textstyle\sum_{i,j=1,\dots,l} \varsigma^{p-1} a_{ij}\vec{u}\,|\vec{u}|^{p-2}\frac{\partial}{\partial x_j}\vec{u}, \frac{\partial}{\partial x_i}\varsigma\right\rangle_{K_{2R}} \le \\ &\le \tfrac{1}{p}\left\langle \textstyle\sum_{i,j=1,\dots,l} \varsigma^{p-1} a_{ij}\vec{u}\,|\vec{u}|^{p-2}\left|\frac{\partial}{\partial x_j}\vec{u}\right|^p, \left|\frac{\partial}{\partial x_i}\varsigma\right|^p\right\rangle_{K_{2R}} + \\ &+\tfrac{1}{q}\left\langle \textstyle\sum_{i,j=1,\dots,l} \varsigma^{p-1} a_{ij}\vec{u}\,|\vec{u}|^{p-2}\frac{\partial}{\partial x_j}\vec{u}, \frac{\partial}{\partial x_i}\varsigma\right\rangle_{K_{2R}}.\end{aligned}$$

Theorem 2. *Let $\delta_2,\ \delta_3$ be positive constants such that*

$$mes\left\{x\in\ K_R,\ |\vec{u}(x)| \le \max_{K_{2R}}|\vec{u}(x)| - \delta_2\, osc\{u,\ K_{2R}\}\right\} \ge (1-\delta_3)\, mes K_R,$$

then there is a positive constant δ_1 that depends only on the ellipticity, smoothness of coefficients, the dimension of space, and constants $\delta_2,\ \delta_3$ such that

$$osc\{u,\ K_R\} \le \max_{K_R}|u(x)| \le (1-\delta_1)\, osc\{u,\ K_{2R}\} + R^{1-\frac{l}{m}}$$

for all $m > l$, where $u = |\vec{u}|$

Comment. From Relliha – Kondrashov theorem for Sobolev space and statement that all elements belonging Sobolev space $W_1^p(R^l, d^l x)$, $l < p \le \infty$ belong to the Holder continuous functional space of $\alpha = \frac{p-l}{p}$ degree we can conclude it is enough to consider only a case when $p \le l$.

Proof. For a positive number δ_2, let us consider the function

$$w(x) = \psi(\vec{u})(x) = -\ln\left(\frac{osc\{u,\ K_{2R}\} - |\vec{u}(x)| + \varepsilon}{\delta_2\, osc\{u,\ K_{2R}\}}\right)$$

and let us assume that in the ball K_R the estimation

$$-\ln\left(\frac{osc\{u,\ K_{2R}\}-|\vec{u}(x)|+\varepsilon}{\delta_2\, osc\{u,\ K_{2R}\}}\right)<L$$

holds then we have the estimation

$$\delta_2\, osc\{u,\ K_{2R}\}\exp(-L)<osc\{u,\ K_{2R}\}-|\vec{u}(x)|+\varepsilon$$

or

$$|\vec{u}(x)|<osc\{u,\ K_{2R}\}-\delta_2\, osc\{u,\ K_{2R}\}\exp(-L)+\varepsilon=(1-\delta_2\exp(-L))\, osc\{u,\ K_{2R}\}+\varepsilon$$

put $\delta_1=\delta_2\exp(-L)$, we obtain

$$osc\{u,\ K_R\}\le\max_{K_R}|\vec{u}(x)|\le(1-\delta_1)\, osc\{u,\ K_{2R}\}+R^{1-\frac{l}{m}},$$

where $\varepsilon=R^{1-\frac{l}{m}}$. Theorem 2 has been proven.

Assuming $\vec{v}=\vec{u}\,|\vec{u}|^{p-2}$ in

$$\lambda\left\langle\vec{u},\vec{v}\right\rangle+\left\langle\sum_{i,j=1,\dots,l}a_{ij}\frac{\partial}{\partial x_j}\vec{u},\frac{\partial}{\partial x_i}\vec{v}\right\rangle+\left\langle\vec{b},\vec{v}\right\rangle=0$$

denoting $\vec{w}=\vec{u}\,|\vec{u}|^{\frac{p-2}{2}}$ and $\nabla\vec{w}=\frac{p}{2}\,|\vec{u}|^{\frac{p-2}{2}}\,\nabla\vec{u}$, we are obtaining

$$\lambda\,\|\vec{u}\|^p+\tfrac{4(p-1)}{p^2}\left\langle\nabla\left(\vec{u}\,|\vec{u}|^{\frac{p-2}{2}}\right)\circ a\circ\nabla\left(\vec{u}\,|\vec{u}|^{\frac{p-2}{2}}\right)\right\rangle\le\left(\left(\tfrac{\varepsilon^2}{p}+1\right)c\,(\beta)+\tfrac{1}{\sigma^q q}\right)\|\vec{u}\|^p+$$
$$+\left(\tfrac{\beta\varepsilon^2}{p}+\beta+\tfrac{1}{p}\tfrac{1}{\varepsilon^2}\right)\left\langle\nabla\left(\vec{u}\,|\vec{u}|^{\frac{p-2}{2}}\right)\circ a\circ\nabla\left(\vec{u}\,|\vec{u}|^{\frac{p-2}{2}}\right)\right\rangle+\tfrac{\sigma^p}{p}\,\|\mu_3\|^p.$$

Let assume

$$v(x)=\frac{\vec{\xi}(x)}{osc\{u,\ K_{2R}\}-|\vec{u}(x)|+\varepsilon}\equiv\frac{\vec{\xi}}{P},$$

where the $\vec{\xi}$ countable smooth cutoff in the ball K_{2R} and notice that can be written

$$\nabla\vec{w}=\frac{\nabla\vec{u}}{osc\{u,\ K_{2R}\}-|\vec{u}|+\varepsilon},$$

then we have

$$\lambda\left\langle\vec{u},\frac{\vec{\xi}}{P}\right\rangle+\left\langle\sum_{i,j=1,\dots l}a_{ij}\frac{\partial}{\partial x_j}\vec{u},\frac{\partial}{\partial x_i}\frac{\vec{\xi}}{P}\right\rangle+\left\langle\vec{b},\frac{\vec{\xi}}{P}\right\rangle=0$$

and

$$\left\langle \vec{b}, \frac{\vec{\xi}}{P} \right\rangle \leq \left\langle \mu_1 \left|\nabla \vec{u}\right| + \mu_2 \left|\vec{u}\right| + \mu_3, \frac{\left|\vec{\xi}\right|}{P} \right\rangle = \\ = \left\langle \mu_1 \left|\nabla \vec{u}\right|, \frac{\left|\vec{\xi}\right|}{P} \right\rangle + \left\langle +\mu_2 \left|\vec{u}\right|, \frac{\left|\vec{\xi}\right|}{P} \right\rangle + \left\langle \mu_3, \frac{\left|\vec{\xi}\right|}{P} \right\rangle$$

since $\vec{\xi}$ is countable smooth cutoff in the ball K_{2R} by integrating over the ballK_{2R}, we have

$$\left\langle \vec{b}, \frac{\vec{\xi}}{P} \right\rangle_{K_{2R}} \leq \left\langle \mu_1 \left|\nabla \vec{u}\right| + \mu_2 \left|\vec{u}\right| + \mu_3, \frac{\left|\vec{\xi}\right|}{P} \right\rangle_{K_{2R}} = \\ = \left\langle \mu_1 \left|\nabla \vec{u}\right|, \frac{\left|\vec{\xi}\right|}{P} \right\rangle_{K_{2R}} + \left\langle \mu_2 \left|\vec{u}\right|, \frac{\left|\vec{\xi}\right|}{P} \right\rangle_{K_{2R}} + \left\langle \mu_3, \frac{\left|\vec{\xi}\right|}{P} \right\rangle_{K_{2R}},$$

by Holder estimation

$$\left\langle \mu_1 \left|\nabla \vec{u}\right|, \frac{\left|\vec{\xi}\right|}{P} \right\rangle_{K_{2R}} = \left\langle \mu_1 \left|\nabla \vec{w}\right|, \left|\vec{\xi}\right| \right\rangle_{K_{2R}} \leq \left\|\nabla \vec{w}\right\|_{K_{2R}} \left\|\mu_1 \left|\vec{\xi}\right|\right\|_{K_{2R}},$$

and Young inequality

$$\left\|\nabla \vec{w}\right\|_{K_{2R}} \left\|\mu_1 \left|\vec{\xi}\right|\right\|_{K_{2R}} \leq \left\|\nabla \vec{w}\right\|_{K_{2R}}^2 + \left\|\mu_1 \left|\vec{\xi}\right|\right\|_{K_{2R}}^2,$$

from form-boundary, we are obtaining

$$\left\|\mu_1 \left|\vec{\xi}\right|\right\|_{K_{2R}}^2 \leq \left\langle \mu_1^2 \left|\vec{\xi}\right|^2 \right\rangle_{K_{2R}} \leq \beta \left\|\Delta \vec{\xi}\right\|_{K_{2R}}^2 + c(\beta) \left\|\nabla \vec{\xi}\right\|_{K_{2R}}^2,$$

a similar consideration gives us

$$\left\langle \mu_2 \left|\vec{u}\right|, \frac{\left|\vec{\xi}\right|}{P} \right\rangle_{K_{2R}} \leq \left\|\vec{u}\right\|_{K_{2R}} \left\|\mu_2 \frac{\left|\vec{\xi}\right|}{P}\right\|_{K_{2R}}.$$

Since $\mu_2^2 \in PK_\beta(A)$ we are applying form-boundary and obtaining

$$\left\|\mu_2 \frac{\left|\vec{\xi}\right|}{P}\right\|_{K_{2R}}^2 \leq \varepsilon^{-2} \left(\beta \left\|\Delta \vec{\xi}\right\|_{K_{2R}}^2 + c(\beta) \left\|\nabla \vec{\xi}\right\|_{K_{2R}}^2\right).$$

If we assume that $\mu_1 \in L^\infty(K_{2R})$ we have an estimation

$$\left\|\mu_2 \frac{\left|\vec{\xi}\right|}{P}\right\|_{K_{2R}} \leq \varepsilon^{-1} vrai \max \left(\mu_2 \left|\vec{\xi}\right|\right) (mesK_{2R})^{\frac{1}{2}},$$

the assumption $\mu_1 \in L^\infty(K_{2R})$ imposes too strong conditions on the coefficients.

After reducing, we are obtaining that the function

$$\vec{v}(x) = \frac{\vec{\xi}(x)}{osc\{u,\ K_{2R}\} - |\vec{u}(x)| + \varepsilon} \equiv \frac{\vec{\xi}}{P}$$

is bounded. Theorem 1 has been proved.

References

[1] D. R. Adams and L. I. Hedberg. *Function Spaces and Potential Theory.* Berlin: Springer-Verlag, 1996.

[2] S. Agmon, A. Douglis, and L. Nirenberg, "Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions I, II", *Communications on pure and applied mathematics,* vol. 12, 1959-1964.

[3] A. G. Berlyiand and Yu. A. Semenov, "On the Lp-theory of Schrodinger semigroups", *Siberian mathematical journal,* vol. 31, pp. 16-26, 1990.

[4] H. Brezis and A. Pazy, "Semigroups of nonlinear contractions on convex sets", *Journal of functional analysis,* vol. 6, no. 2, pp. 237–281, 1970. doi: 10.1016/0022-1236(70)90060-1

[5] F. E. Browder, "Existence of periodic solutions for nonlinear equations of evolution", *Proceedings of the National Academy of sciences,* vol. 53, no. 5, pp. 1100–1103, 1965. doi: 10.1073/pnas.53.5.1100

[6] F. E. Browder, "Nonlinear equations of evolution and nonlinear accretive operators in Banach spaces", *Bulletin of the American Mathematical Society,* vol. 73, no. 6, pp. 867–874, 1967. doi: 10.1090/s0002-9904-1967-11820-2

[7] M. G. Crandall and A. Pazy, "Semi-groups of nonlinear contractions and dissipative sets", *Journal of functional analysis,* vol. 3, no. 3, pp. 376–418, 1969. doi: 10.1016/0022-1236(69)90032-9

[8] J.-M. Bony, "Calcul symbolique et propagation des singularités pour les équations aux Dérivées partielles non linéaires", *Annales scientifiques de l'École normale supérieure,* vol. 14, no. 2, pp. 209–246, 1981. doi: 10.24033/asens.1404

[9] H. Brezis, “Some variational problems with lack of compactness”, in *Nonlinear functional analysis and its applications*, F. Browder, Ed. Providence: AMS, pp. 165-202, 1986.

[10] H. Brezis and J. M. Coron, “Convergence of solutions of H-systems or how to blow bubbles”, *Archive for rational mechanics and analysis*, vol. 89, no. 1, pp. 21–56, 1985. doi: 10.1007/bf00281744

[11] H. Brezis, J.-M. Coron, and E. H. Lieb, “Harmonic maps with defects”, *Communications in mathematical physics*, vol. 107, no. 4, pp. 649–705, 1986. doi: 10.1007/bf01205490

[12] H. Brezis and L. Nirenberg, “Positive solutions of nonlinear elliptic equations involving critical sobolev exponents”, *Communications on pure and applied mathematics*, vol. 36, no. 4, pp. 437–477, 1983. doi: 10.1002/cpa.3160360405

[13] F. Browder, “Fixed point theory and nonlinear problems”, *Bulletin of the american mathematical society*, vol. 9, pp. 1-39, 1983.

[14] F. E. Browder and W. V. Petryshyn, “Approximation methods and the generalized topological degree for nonlinear mappings in Banach spaces”, *Journal of functional analysis*, vol. 3, no. 2, pp. 217–245, 1969. doi: 10.1016/0022-1236(69)90041-x

[15] L. Caffarelli, R. Kohn, and L. Nirenberg, “Partial regularity of suitable weak solutions of the navier-stokes equations”, *Communications on pure and applied mathematics*, vol. 35, no. 6, pp. 771–831, 1982. doi: 10.1002/cpa.3160350604

[16] L. Caffarelli, L. Nirenberg, and J. Spruck, “The Dirichlet problem for nonlinear second-order elliptic equations I. Monge-Ampégre equation”, *Communications on pure and applied mathematics*, vol. 37, no. 3, pp. 369–402, 1984. doi: 10.1002/cpa.3160370306

[17] M. G. Crandall and P.-L. Lions, “Viscosity solutions of Hamilton-Jacobi equations”, *Transactions of the american mathematical society*, vol. 277, no. 1, pp. 1–42, 1983. doi: 10.1090/s0002-9947-1983-0690039-8

[18] B. Dacorogna, *Weak Continuity and weak lower semi-continuity of nonlinear functionals*, Berlin: Springer-Verlag, 1987.

[19] E. B. Davies, *One parameter semigroups*, San Diego: Academic Press, 1980.

[20] E. B. Davies, “Lp Spectral Theory of Higher-Order Elliptic Differential Operators”, *Bulletin of the London mathematical society*, vol. 29, no. 5, pp. 513–546, 1997. doi: 10.1112/s002460939700324x

[21] E. DeGiorgi, “Sulla differenziabilita e l’analiticita delle estremali degli integrali multipli regolari”, *Memorie della Reale accademia delle scienze di Torino, Classe di Scienze Fisiche, Matematiche e Naturali.*, vol. 3, pp. 25-43, 1957.

[22] E. DeGiorgi, “Un esempio di estremali discontinue per un problema variazionale di tipo ellitico”, *Bollettino della Unione Matematica Italiana. Series IV,* no. 1, pp. 135-137, 1968.

[23] R. J. DiPerna, “Compensated compactness and general systems of conservation laws”, *Transactions of the American mathematical society,* vol. 292, no. 2, pp. 383–420, 1985. doi: 10.1090/s0002-9947-1985-0808729-4

[24] R. J. DiPerna and P. L. Lions, “On the cauchy problem for Boltzmann equations: Global existence and weak stability”, *The annals of mathematics,* vol. 130, no. 2, p. 321, 1989. doi: 10.2307/1971423

[25] R. J. DiPerna and A. Majda, “Reduced hausdorff dimension and concentration-cancellation for two-dimensional incompressible flow”, *Journal of the American mathematical society,* vol. 1, no. 1, pp. 59–95, 1988. doi: 10.1090/s0894-0347-1988-0924702-6

[26] L. C. Evans, *Weak convergence methods for nonlinear partial differential equations.* Providence: AMS, 1988.

[27] L. Giacomelli and H. Knüpfer, “A free boundary problem of fourth order: Classical solutions in Weighted Hölder spaces”, *Communications in partial differential equations,* vol. 35, no. 11, pp. 2059–2091, 2010. doi: 10.1080/03605302.2010.494262

[28] M. Giaquinta, *Multiple integrals in the calculus of variations and nonlinear elliptic systems.* Princeton: Princeton University Press, 1983.

[29] M. Giaquinta, *Introduction to Regularity Theory for Nonlinear Elliptic Systems.* Basel: Birkhauser, 1993.

[30] B. Gidas, W.-M. Ni, and L. Nirenberg, “Symmetry and related properties via the maximum principle”, *Communications in mathematical physics,* vol. 6, pp. 883-901, 1981.

[31] D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*, 2nd ed., Berlin: Springer, 1983.

[32] E. Giusti, *Minimal Surfaces and Functions of Bounded Variation.* Basel: Birkhauser, 1984.

[33] R. Landes and V. Mustonen, “On pseudo-monotone operators and nonlinear noncoercive variational problems on unbounded domains”, *Mathematische annalen,* vol. 248, no. 3, pp. 241–246, 1980. doi: 10.1007/bf01420527

[34] G. J. Minty, “Monotone (nonlinear) operators in Hilbert Space”, *Duke mathematical journal*, vol. 29, no. 3, pp. 341-346, 1962. doi: 10.1215/s0012-7094-62-02933-2

[35] G. J. Minty, “On the generalization of a direct method of the calculus of variations”, *Bulletin of the American mathematical society*, vol. 73, no. 3, pp. 315–321, 1967. doi: 10.1090/s0002-9904-1967-11732-4

[36] I. Miyadera, “On perturbation theory for semi-groups of operators”, *Tohoku mathematical journal*, vol. 18, no. 3, pp. 299-310, 1966. doi: 10.2748/tmj/1178243419

[37] J. Necas, *Introduction to the theory of nonlinear elliptic equations*, New York: Teubner and Wiley, 1983.

[38] E. V. Radkevich, “Equations with nonnegative characteristic form. I”, *Journal of mathematical sciences*, vol. 158, no. 3, pp. 297–452, 2009. doi: 10.1007/s10958-009-9394-2

[39] E. Stein, *Harmonic analysis: real variable methods, orthogonality, and oscillatory integrals*, Princeton: Princeton University. Press, 1993.

[40] M. Struwe, “A global compactness result for elliptic boundary value problems involving limiting nonlinearities”, *Mathematische zeitschrift*, vol. 187, no. 4, pp. 511–517, 1984. doi: 10.1007/bf01174186

[41] J. Sylvester and G. Uhlmann, “A global uniqueness theorem for an inverse boundary value problem”, *The annals of mathematics*, vol. 125, no. 1, pp. 153-169, 1987. doi: 10.2307/1971291

[42] M. Taylor, *Partial differential equations*, 3 vols. Berlin: Springer, 1996.

[43] L. Veron, *Singularities of solutions of second order quasilinear equations*, New York: Longman, 1996.

[44] F. B. Weissler, “Single point blow-up for a semilinear initial value problem”, *Journal of differential equations*, vol. 55, no. 2, pp. 204–224, 1984. doi: 10.1016/0022-0396(84)90081-0

[45] M. I. Yaremenko, “The existence of a solution of evolution and elliptic equations with singular coefficients”, *Asian journal of mathematics and computer research*, vol. 15, no. 3, pp. 172-204, 2017.

[46] M. I. Yaremenko, “Quasi-linear evolution and elliptic equations”, *Journal of progressive research in mathematics*, vol. 11, 13 pp. 1645-1669, 2017.

[47] M. I. Yaremenko, “The sequence of semigroups of nonlinear operators and their applications to study the Cauchy problem for parabolic equations”, *Scientific journal of the ternopil national technical university,* vol. 14, pp. 149-160, 2016.

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