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Abstract

After publication of the above article in Proyecciones (Antofagasta). Volume 34, (2015) No 1, pp. 85-105, it has come to the attention of the authors that an extra condition was missing in Theorem 3.2.

1 Introduction

Recently it was called to our attention by professor Lech Maligranda that Theorem 3.2 in Orlicz-Lorentz Spaces and their Composition Operators (see [1]) fails to be true when the weight $w \equiv 1$, it is necessary to additionally suppose the $\Delta_2$-condition.

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Now, in order that everything will be in a good shape, we rewrite Theorem 3.2 as follows

**Theorem 1.1.** Let $T : X \to X$ be a non-singular measurable transformation. If $C_T$ induced by $T$ is bounded, then there exists $M \geq 1$ such that $\nu(T^{-1}(A) \leq M \nu(A))$.

In addition, if $\varphi$ satisfies the $\Delta_2$-condition for all $t > 0$ and $\nu(T^{-1}(A)) \leq M \nu(A)$ for $A \in \mathcal{A}$, then $C_T$ is bounded. Moreover,

$$
\|C_T f\| = \sup_{0 < \nu(A) < \infty} \left( \frac{\nu(T^{-1}(A))}{\nu(A)} \right).
$$

(1)

**Proof.** We empathize that the first part of the proof of this Theorem goes the same as in the proof of Theorem 3.2 in [1] and it holds without the $\Delta_2$-condition. Conversely, if $\nu(T^{-1}(A)) \leq M \nu(A)$ holds for all $A \in \mathcal{A}$, then

$$
\int_0^{\mu(T^{-1}(A))} w(t) \, dt \leq M \int_0^{\mu(A)} w(t) \, dt
$$

$$
\Rightarrow \int_0^{\infty} \chi_{(0, \mu(T^{-1}(A)))}(t) w(t) \, dt \leq M \int_0^{\infty} \chi_{(0, \mu(A))}(t) w(t) \, dt
$$

$$
\Rightarrow \int_0^{\infty} \chi_{T^{-1}(A)}^*(t) w(t) \, dt \leq M \int_0^{\infty} \chi_{A}^*(t) w(t) \, dt
$$

$$
\Rightarrow \int_0^{\infty} (\chi_{E \circ T}^* (t) w(t) \, dt \leq M \int_0^{\infty} \chi_{E}^*(t) w(t) \, dt
$$

$$
\Rightarrow \int_0^{\infty} (f \circ T)^*(t) w(t) \, dt \leq M \int_0^{\infty} f^*(t) w(t) \, dt.
$$

Therefore

$$(f \circ T)^*(t) \leq M f^*(t) \quad \text{a.e.}$$

Since $\varphi$ satisfies the $\Delta_2$-condition for all $t > 0$, it follows that

$$
k = \sup_{t > 0} \frac{\varphi(at)}{\varphi(t)} < \infty,$$

and thus

$$
\varphi(at) \leq k \varphi(t),
$$

then, with $k = 1/M$, we have

$$
\int_0^{\infty} \varphi \left( \frac{(f \circ T)^*(t)}{M \|f\|_{w, \varphi}} \right) w(t) \, dt \leq \frac{1}{M} \int_0^{\infty} \varphi \left( \frac{f^*(t)}{\|f\|_{w, \varphi}} \right) w(t) \, dt
$$

$$\leq \int_0^{\infty} \varphi \left( \frac{f^*(t)}{\|f\|_{w, \varphi}} \right) w(t) \, dt \leq 1.$$
Finally
\[ \| f \circ T \|_{\varphi,w} \leq M \| f \|_{\varphi,w}, \]
that is
\[ \| C_T f \|_{\varphi,w} \leq M \| f \|_{\varphi,w}. \]

On the one hand, let us prove (1). Indeed, let
\[ N = \sup_{0 < \nu(A) < \infty} \left( \frac{\nu(T^{-1}(A))}{\nu(A)} \right), \]
then
\[ \nu(T^{-1}(A)) \leq N \nu(A) \]
and thus
\[ \| C_T f \|_{\varphi,w} \leq N \| f \|_{\varphi,w}, \quad \forall f \in L_{\varphi,w} \]
hence
\[ \frac{\| C_T f \|_{\varphi,w}}{\| f \|_{\varphi,w}} \leq N, \quad \forall 0 \neq f \in L_{\varphi,w}. \]

Therefore
\[ \| C_T \| = \sup_{f \neq 0} \frac{\| C_T f \|_{\varphi,w}}{\| f \|_{\varphi,w}} < N = \sup_{0 < \nu(A) < \infty} \left( \frac{\nu(T^{-1}(A))}{\nu(A)} \right). \]
That is
\[ \| C_T \| \leq \sup_{0 < \nu(A) < \infty} \left( \frac{\nu(T^{-1}(A))}{\nu(A)} \right). \quad (2) \]

On the other hand, let us consider
\[ M = \| C_T \| = \sup_{f \neq 0} \frac{\| C_T f \|_{\varphi,w}}{\| f \|_{\varphi,w}}, \]
then
\[ \frac{\| C_T f \|_{\varphi,w}}{\| f \|_{\varphi,w}} \leq M \quad \forall 0 \neq f \in L_{\varphi,w}. \]
In particular, if \( f = \chi_A \) such that \( 0 < \mu(A) < \infty, A \in \mathcal{A}, \) then
\[ \frac{\| C_T (\chi_A) \|_{\varphi,w}}{\| \chi_A \|_{\varphi,w}} = \left( \frac{\nu(T^{-1}(A))}{\nu(A)} \right) \leq M, \]
therefore
\[
\sup_{0<\nu(A)<\infty} \left( \frac{\nu(T^{-1}(A))}{\nu(A)} \right) \leq M = \|C_T\|. \tag{3}
\]
Combining (2) and (3) we have
\[
\|C_T\| = \sup_{0<\nu(A)<\infty} \left( \frac{\nu(T^{-1}(A))}{\nu(A)} \right). \tag{4}
\]
\textit{Remark 1.2.} We like to point it out that when the weight \( w(t) = 1 \)
we obtain the characterization for boundedness of the composition operator in Orlicz space (see Theorem 2.2. in [2]).

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The authors wish to express their gratitude to professor Lech Maligranda for called our attention on the missing condition in Theorem 3.2.

\section{References}
