Proyecciones Journal of Mathematics
Vol. 42, N^o 1, pp. 1-17, February 2023.
Universidad Católica del Norte
Antofagasta - Chile

10.22199/issn.0717-6279-4488

Commuting graph of $CA-$groups

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Received : September 2020. Accepted : August 2022

Abstract

A group G is called a $CA-$group, if all the element centralizers of G are abelian and the commuting graph of G with respect to a subset A of G, denoted by $\Gamma(G, A)$, is a simple undirected graph with vertex set A and two distinct vertices a and b are adjacent if and only if $ab = ba$. The aim of this paper is to generalize results of a recently published paper of F. Ali, M. Salman and S. Huang [On the commuting graph of dihedral group, Comm. Algebra **44** *(6) (2016) 2389–2401] to the case that G is an $CA-$group.*

Keywords: *Commuting graph, $CA-$group, distance, detour distance, metric dimension.*

2010 Mathematics Subject Classification: *05C65, 05C31, 05E30.*

1. Basic Definitions

Throughout this paper all groups are assumed to be finite and graphs will be simple and undirected. Suppose u and v are vertices in a graph Γ. The distance $d(u, v)$ and detour distance $d_D(u, v)$ are defined as the length of a shortest and longest path in Γ, where shortest and longest paths are paths containing minimum and maximum number of edges, respectively. The eccentricity and detour eccentricity of $u \in V(\Gamma)$ can be defined as $ecc(u) = max\{d(u, v)|v \in V(\Gamma)\}$ and $ecc_D(u) = max\{d_D(u, v)|v \in V(\Gamma)\}$, respectively. The radius, detour radius, diameter and detour diameter of Γ are defined as the minimum eccentricity $rad(\Gamma)$, the minimum detour eccentricity $rad_D(\Gamma)$, the maximum eccentricity $diam(\Gamma)$ and maximum detour eccentricity $diam_D(\Gamma)$, respectively. A vertex $u \in V(\Gamma)$ is called central (detour central) if $ecc(u) = rad(\Gamma)$ ($ecc_D(u) = rad_D(\Gamma)$). The set of all central and detour central vertices of Γ are denoted by $Cent(\Gamma)$ and $cent_D(\Gamma)$, respectively.

In a similar way, a vertex u in Γ is called a peripheral or detour peripheral if $ecc(u) = diam(\Gamma)$ or $ecc_D(u) = diam_D(\Gamma)$, respectively. The detour degree $d_D(u)$ is defined as the size of

$$D(u) = \{v \in V(\Gamma) \mid ecc_D(u) = d_D(u, v)\}$$

and the number $D_{av} = \frac{1}{|V(\Gamma)|} \sum_{i=1}^{|\Gamma|} d_D(v_i)$ is the average detour degree of Γ. The non-increasing sequence of detour degree vertices of Γ is named the detour degree sequence and denoted by $D(\Gamma)$.

Set $D_i(u) = |\{v \in V(\Gamma) \mid d_D(u, v) = i\}|$. The sequence

$$(D_0(v), D_1(v), \cdots, D_{ecc(v)}(v))$$

is called the detour distance sequence of a vertex v in Γ and we use the notation $dds_D(\Gamma)$ for this sequence. It is easy to see that $D_0(v) = 1$ and $D_{ecc(v)}(v) = d_D(v)$.

Suppose v is a vertex in Γ. The neighborhood of v, $N(v)$, is the set of all vertices in Γ adjacent to v and the set $N[v] = N(v) \cup \{v\}$ is called the closed neighborhood of v. The vertex v is a boundary vertex if $d(u, v) \geq d(v, t)$, for each neighbors t of v. If $N(v)$ induces a complete subgraph then the vertex v is said to be complete. Note that the vertex v is a boundary vertex if and only if it is complete vertex. A vertex t on a $u - w$ path with this property that $d(u, t) = d(t, w)$ is an interior vertex and the subgraph induced by all interior vertices is the interior of Γ and denoted by $Int(\Gamma)$.

It is easy to see that a vertex v is an interior vertex if and only if it is not a boundary vertex.

Suppose $W = \{w_1, w_2, \cdots, w_k\}$ is an ordered subset of vertices of Γ and $v \in V(\Gamma)$ is an arbitrary vertex. The $k-$vector $r(v|W) = (d(v, w_1), d(v, w_2), \cdots, d(v, w_k))$ is called the representation of v with respect to W. The set W is called a resolving set for Γ if $r(v|W) \neq r(v|W)$, for distinct vertices $u, v \in V(\Gamma)$. The metric dimension $\beta(\Gamma)$ is the minimum cardinality of a resolving set for Γ. For example, it is easy to see that $\beta(K_n) = n - 1$. An $i-$element subset of $V(\Gamma)$ is also named an $i-$subset. The resolving polynomial of Γ is defined as $\beta(\Gamma, x) = \sum_{i=\beta(\Gamma)}^{n} r_i x^i$, where $R(\Gamma, i)$ is a family of resolving sets for Γ, r_i is the size of $R(\Gamma, i)$ and n is the number of vertices in Γ [1]. The sequence $(r_{\beta(\Gamma)}, \cdots, r_n)$ of coefficients of $\beta(\Gamma, x)$ is called the resolving sequence of Γ. The distinct vertices of $u, v \in V(\Gamma)$ are called twins if $N[u] = N[v]$ or $N(u) = N(v)$ depend on u and v are adjacent or non-adjacent. A subset U of $V(\Gamma)$ is called a twin-set in Γ if u, v are twins in Γ for every pair of distinct vertices in U. One can easily seen that if Γ is connected and u, v are twins in Γ then $d(u, x) = d(v, x)$, for every vertex $x \in V(\Gamma) \setminus \{u, v\}$. Every resolving set for Γ contains at least $l - 1$ vertices of U, where l is the size of U.

Our other notions are standard and can be taken from the books [2, 9]. Our calculations are done with the aid of GAP [15].

2. Commuting Graph

Suppose G is a finite group. The commuting graph of G with respect to a subset A of G, denoted by $\mathcal{C}(G, A)$, is a simple undirected graph whose vertices are all elements of A and two distinct vertices are adjacent if and only if they are commute to each other. If A is a set of involutions then $\Gamma(G, A)$ is called a commuting involution graph, and if $A = G$ then we write $\mathcal{C}(G)$ as $\mathcal{C}(G, G)$. The commuting graphs have been studied by mathematicians for about half of a century. With the best of our knowledge the first appearance of these graphs is related to the pioneering work of Fischer on $3-$transposition groups when the subset A is a conjugacy class of involutions [4].

Iranmanesh and Jafarzadeh [8, Conjecture 2.2] conjectured that there is a natural number b such that if G is a finite non-abelian group with $\mathcal{C}(G)$ connected, then $diam(\mathcal{C}(G)) \leq b$. In 2014, Giudici and Pope [6] obtained some upper and lower bounds for diameter of the commuting graphs of some classes of finite groups. They also have produced an infinite family of finite groups with trivial center and diameter six. In the same year, Giudici and Parker [5] constructed an infinite family of groups with nilpotency class two and unbounded diameter which disproved the mentioned conjecture of Iranmanesh and Jafarzadeh. Morgan and Parker [10] proved that this conjecture is true for finite groups with trivial center.

3. Commuting Graph of $CA-$Groups

Throughout this section all groups are assumes to be $CA-$group and the radius, diameter and detour distance of commuting graphs are calculated for these groups. As a consequence, all results of [1] are generalized to the $CA-$groups. We also assume that $Z(G) = \{u_1, u_2, \cdots, u_t\}$, C_{m_1}, C_{m_2}, $\cdots$, C_{m_s} are components of $\mathcal{C}(G \setminus Z(G))$ and $m_1 = |V(C_{m_1})| \geq m_2 = |V(C_{m_2})| \geq \cdots \geq m_s = |V(C_{m_s})|$. The complete subgraph of Γ induced by $Z(G)$ is denoted by C_0. Note that all components are complete subgraphs of Γ and the elements of center are adjacent to all other vertices of Γ.

Lemma 3.1. *Let G be a $CA-$group and $\Gamma = \mathcal{C}(G)$. Then the followings are hold:*

1) $rad(\Gamma) = 1$ and $diam(\Gamma) = 2$,

2) $Cent(\Gamma) = \{u \in V(\Gamma) | u \in Z(G)\}$,

3) $\Gamma = Cent(\Gamma) \vee Per(\Gamma)$,

4) $\Gamma = C_0 \vee (C_{m_1} \cup C_{m_2} \cup \cdots \cup C_{m_s})$, where C_o is a complete subgraph of Γ induced by $Z(G)$ and $C_{m_1}, \cdots, C_{m_s}$ are components of $\mathcal{C}(G \setminus Z(G))$.

Proof. (1) For each $u, v \in V(\Gamma)$ and $x \in Z(G)$, there exists a path $u - x - v$ hence $diam(\Gamma) = 2$ and hence $ecc(x) = 1$. This proves that $rad(\Gamma) = 1$. (2) and (3) are easy consequences of [1, Proposition 1.2 and 1.3]. (4) By [10, Proposition 3.1] and Case 3, the commuting graph $\mathcal{C}(G \setminus Z(G))$ is a union of complete graphs if and only if Γ is a $CA-$group. $\square$

Lemma 3.2. *If $|Z(G)| \geq s$ then for each $u, v \in V(\Gamma)$, $ecc_D(u) = ecc_D(v) = |G| - 1$.*

Proof. Choose $u, v \in V(\Gamma)$ such that $d(u, v) = |G| - 1$. Since the elements of the center is adjacent to all other vertices, there exists a Hamilton cycle connecting u and v, as desired. □

The following corollary is a direct consequence of Lemmas 3.1 and 3.2:

Corollary 3.3. *Let G be a $CA-$group and $\Gamma = \mathcal{C}(G)$ then,*

$$\begin{array}{rcl} ecc_D(u) & = & \left\{ \begin{array}{ll} |G|-1 & t \geq s \\ \sum_{i=1}^{t}(m_i) + t - 1 & t < s, u \in Z(G) \\ \sum_{i=1}^{t+1}(m_i) + t - 1 & t < s, u \notin Z(G) \end{array} \right. , \\ rad_D(\Gamma) & = & \left\{ \begin{array}{ll} |G|-1 & t > s \\ |G| - |(C_{m_s})| - 1 & t = s \\ \sum_{i=1}^{t}(m_i) + t - 1 & t < s \end{array} \right. , \\ diam_D(\Gamma) & = & \left\{ \begin{array}{ll} |G|-1 & t \geq s \\ \sum_{i=1}^{t+1}(m_i) + t - 1 & t < s \end{array} \right. . \end{array}$$

Proposition 3.4. *Suppose G is a $CA-$group and $\Gamma = \mathcal{C}(G)$ then,*

$$\begin{array}{rcl} dds(\Gamma) & = & \{(1, |G|-1)^t, (1, t+m_1-1, |G|-t-m_1)^{m_1}, \cdots, \\ & & (1, t+m_s-1, |G|-t-m_s)^{m_s}\}. \end{array}$$

Proof. By Lemma 3.1, $\Gamma = C_0 \vee (C_{m_1} \cup C_{m_2} \cup \cdots \cup C_{m_s})$. If $u \in Z(G)$ then $d_0(u) = 1, d_1(u) = |G|-1$ and $ecc(u) \leq 1$ and so $dds(u) = (1, |G|-1)$. Choose $u \in \mathcal{C}(G, V(C_{m_i}))$. If $v \in \mathcal{C}(G, Z(G) \cup V(C_{m_i}))$ then $d(u, v) = 1$ and hence $d_0(u) = 1$ and $d_1(u) = t + m_i - 1$. If $v \in \mathcal{C}(G, G \setminus (Z(G) \cup V(C_{m_i})))$ then $d(u, v) = 2$, $d_2(u) = |G| - t - m_i$ and so $dds(u) = (1, t + m_i - 1, |G| - t - m_i)^{m_i}$. Since $ecc(u) \leq 2$, $dds(\Gamma) = \{(1, |G-1|)^t, (1, t+m_1-1, |G|-t-m_1)^{m_1}, \cdots, (1, t+m_s-1, |G|-t-m_s)^{m_s}\}$. □

Theorem 3.5. *Suppose G is a $CA-$group and,*

$$\begin{array}{rcl} \Gamma & = & \mathcal{C}(G) = C_o \vee (C_{m_1} \cup C_{m_2} \cup \cdots \cup C_{m_s}), \\ Q & = & (1, 0^{|G|-|V(C_{m_s})|-2}, |Z(G)|-1, 0^{|V(C_{m_s})|}, |G|-|Z(G)|)^{|Z(G)|}. \end{array}$$

Then,

$$dds_D(\Gamma) = \left\{ \begin{array}{ll} (1, 0^{|G|-2}, |G|-1)^{|G|-1} & if\ t > s \\ Q & if\ t = s, v \in Z(G) \\ (1, 0^{|G-2|}, |G|-1)^{|G|-|Z(G)|}) & if\ t = s, v \in G \setminus Z(G) \end{array} \right. .$$

If $t < s$ then we define $\beta_{j,s} = \sum_{j=t+1}^{s} m_j$, $\gamma_1 = \sum_{i=1}^{t-1} m_i$ and $\gamma_2 = \sum_{i=1}^{t} m_i$. We have three cases as follows:

1. *If* $u \in Z(G)$ *then*

$$dds_D(\Gamma) = (1, 0^{\gamma_1+t-2}, t-1, 0^{m_j-1}, \beta_{j,s}, 0^{m_t-m_j-1}, \gamma_2)^{|Z(G)|}.$$

2. *If* $u \in V(C_{m_i}), 1 \leq i \leq t$ *then*

$$dds_D(\Gamma) = (1, 0^{\gamma_2+t-2}, t+m_i-1, 0^{m_j-1}, 0^{m_{t+1}-m_j-1}, \gamma_2-m_i)^{m_i}.$$

3. *If* $u \in V(C_{m_j}(\Gamma)), t+1 \leq j \leq s$ *then*

$$dds_D(\Gamma) = (1, 0^{\gamma_1+m_j+t-2}, t+m_j-1, 0^{m_q-1}, \beta_{j,s}-m_j, 0^{m_t-m_q-1}, \gamma_2)^{m_j}.$$

Proof. We first assume that $t > s$ and $u \in V(\Gamma)$. Then $D_0(u) = 1$ and by Lemma 3.2, $D_i(u) = 0$ in which $1 \leq i \leq |G|-2$. Furthermore, $D_{ecc(u)}(u) = |G|-1$ and so $dds_D(\Gamma) = (1, 0^{|G|-2}, |G|-1)^{|G|-1}$.

Suppose $t = s$ and choose an arbitrary vertex $u \in Z(G)$. If $v \in Z(G)$ then $d_D(u,v) = |G| - |V(C_{m_s})| - 1$, otherwise $d_D(u,v) = |G|-1$. If $1 \leq i \leq |G|-2$ and $i \neq |G|-m_s-2$ then we have $D_i(u) = 0$. So, $dds_D(u) = (1, 0^{|G|-|V(C_{m_s})|-2}, |Z(G)|-1, 0^{|V(C_{m_s})|}, |G|-|Z(G)|)^{|Z(G)|}$. Next we assume that $u \notin Z(G)$. For all $v \in V(\Gamma)$, $d_D(u,v) = |G|-1$ and hence $dds_D(u) = (1, 0^{|G|-2}, |G|-1)^{|G|-|Z(G)|}$. Consider the cases that $t < s$ and assume that $1 \leq i \leq t, t+1 \leq j \leq s$.

1. If $u \in Z(G)$ then

$$d_D(u,v) = \begin{cases} \gamma_1 + t - 1 & v \in Z(G) - \{u\} \\ \gamma_2 + t - 1 & v \in V(C_{m_i}) \\ \gamma + m_j + t - 1 & v \in V(C_{m_j}) \end{cases}.$$

Therefore,

$$dds_D(\Gamma) = (1, 0^{\gamma_1+t-2}, t-1, 0^{m_j-1}, \beta j, s, 0^{m_t-m_j-1}, \gamma_2)^{|Z(G)|}.$$

2. If $u \in V(C_{m_i})$ then

$$d_D(u,v) = \begin{cases} \gamma_2 + t - 1 & v \in Z(G) \text{ or } v \in V(C_{m_i}) - \{u\} \\ \gamma_2 + m_{t+1} + t - 1 & v \in \cup_{i=1}^{t}(V(C_{m_i})) \setminus V(C_{m_i}) \\ \gamma_2 + m_j + t - 1 & v \in V(C_{m_j}) \end{cases}.$$

Therefore,

$$dds_D(\Gamma) = (1, 0^{\gamma_2+t-2}, t+m_i-1, 0^{m_j-1}, 0^{m_{t+1}-m_j-1}, \gamma_2-m_i)^{m_i}.$$

3. If $u \in V(C_{m_j})$ then

$$d_D(u,v) = \begin{cases} \gamma_1 + m_j + t - 1 & v \in Z(G) \text{ or } v \in V(C_{m_j}) - \{u\} \\ \gamma_1 + m_q + t - 1 & v \in V(C_{m_q}) \subseteq \cup_{j=t+1}^{s}(V(C_{m_i})) \\ \gamma_2 + m_j + t - 1 & v \in \cup_{i=1}^{t} V(C_{m_i}) \end{cases}.$$

Therefore,

$$dds_D(\Gamma) = (1, 0^{\gamma_1+m_j+t-2}, t+m_j-1, 0^{m_q-1}, \beta j, s-m_j, 0^{m_t-m_q-1}, m_t)^{m_j}.$$

Hence the result. □

Theorem 3.6. *Let G be a non-abelian $CA-$group and $\Gamma = \mathcal{C}(G)$. Then $Int(\Gamma) = Cent(\Gamma)$.*

Proof. Suppose $v \in V(C_{m_1} \cup C_{m_2} \cup \cdots \cup C_{m_s})$. The subgraph of Γ induced by $N(v)$ is complete and therefore v is a boundary vertex. This means that $v \notin Int(\Gamma)$ and hence $Int(\Gamma) \subseteq Cent(\Gamma)$. We prove that $Int(\Gamma) \supseteq Cent(\Gamma)$. Since G is non-abelian, $s > 1$. Consider the elements $v \in cent(\Gamma), u \in V(C_{m_i})$ and $w \in V(C_{m_j}), i \neq j$. Thus, $u, w \in N(v)$. But v is not adjacent with u and so $N(v)$ is not a complete subgraph of Γ. Hence v is not a boundary vertex which implies that $v \in Int(\Gamma)$. This proves the result. □

Remark 3.7. *In Theorem 3.6, if G is a abelian group then Γ is a complete graph and $N[v] = V(\Gamma)$. Therefore, each vertex of Γ is a boundary vertex and so $Int(\Gamma) = \emptyset$.*

Fix a vertex v in a graph Γ. Each vertex u such that $d(u,v) = ecc(v)$ is called an eccentric vertex of v and in such a case the vertex v is said to be an eccentric vertex of Γ. The subgraph induced by all eccentric vertices of Γ, $Ecc(\Gamma)$, is the eccentric subgraph of Γ and the graph Γ is an eccentric graph if $\Gamma = Ecc(\Gamma)$ [3].

Theorem 3.8. *Let G be a $CA-$group and $\Gamma = \mathcal{C}(G)$. Then,*

$$Ecc(\Gamma) = \begin{cases} \Gamma & |Z(G)| > 1 \\ Per(\Gamma) & |Z(G)| = 1 \end{cases}.$$

Proof. If $\mathcal{C}(G \setminus Z(G)$ is connected then Γ is a complete graph and clearly $Ecc(\Gamma) = \Gamma$, as desired. Suppose Γ has at least two components U and V. for any $u \in U$ and $v \in V$, $d(u, v) = 2$ and so $ecc(u) = ecc(v) = 2$. Therefore, each vertex of U is an eccentric vertex for all vertices of V. Assume that $z \in Z(G)$. Our main proof will consider the following two cases:

1. If $Z(G) > 1$ then z is an eccentric vertex for any vertices of central elements and therefore each vertex of Γ is an eccentric vertex for other vertices of Γ. Hence $Ecc(\Gamma) = \Gamma$.
2. If $Z(G) = \{e\}$ then for every vertex $u \in \Gamma$, $d(z, u) = 1$ and so e can not be an eccentric vertex. So $Ecc(\Gamma) = Per(\Gamma)$.

This completes our argument □

By Theorem 3.8, we have the following immediate consequence:

Corollary 3.9. *For all $n > 1$, the commuting graph of the groups T_{4n}, SD_{8n} and $U_{n,m}$ are an eccentric graph.*

Suppose Γ is a graph of order n. The closure $Cl(\Gamma)$ is the graph constructed from Γ by recursively joining pairs of non-adjacent vertices whose sum of degrees is at least n. A graph Γ is said to be a closed graph if $Cl(\Gamma) = \Gamma$.

Theorem 3.10. *Suppose G is a $CA-$group, $\Gamma = \mathcal{C}(G)$ and $C_1, C_2, \cdots, C_n$ are connected components of $\mathcal{C}(G \setminus Z(G))$. If the summation of the size of each choice of $n-2$ components is greater than $|Z(G)|-2$ then $Cl(\Gamma) = \Gamma$.*

Proof. Suppose u and v are vertices of distinct components U_i and U_j of Γ. By [1, Lemma 2.15], it is sufficient to prove that $d(u)+d(v) < n$. By our assumption, $\sum_{k=1,k\neq i,j}^{n} |C_k| > |Z(G)| - 2$ and so $\sum_{k=1}^{n} |C_k| > |C_i| + |C_j| + |Z(G)|-2$ which implies that $\sum_{k=1}^{n} |C_k|+|Z(G)| > |C_i|+|C_j|+2|Z(G)|-2$. Hence $n > d(u) + d(v)$, proving the result. □

If the previous theorem we substitute $n = 1$ to deduce that $\Gamma \cong K_{|G|}$ and so $Cl(\Gamma) = \Gamma$.

Corollary 3.11. *Suppose G is an CA-group. Then the commuting graph of G is a closed graph.*

Proof. It is clear that the summation of the size of each choice of $n-2$ components is greater than $|Z(G)|-2$. By the Theorem 3.10 we have result. □

4. Metric Dimension of some Commuting Graphs

In this section, we continue the interesting work of Ali et al. [1] in computing the resolving set and metric dimension of dihedral groups. Some results of the mentioned paper will be generalized.

Lemma 4.1. *Suppose* $\Gamma_1 = K_{m_1} \cup \cdots \cup K_{m_s}$ *and* $\Gamma = K_n \vee \Gamma_1$, *then* $\beta(\Gamma) = |V(\Gamma)| - s - 1$ *if* Γ_1 *has no isolated vertex and* $\beta(\Gamma) = |V(\Gamma)| - (s-k) - 2$ *if* Γ_1 *has* k *isolated vertices.*

Proof. If Γ_1 has no isolated vertex then the sets $V(K_n), V(K_{m_1}), \cdots,$ $V(K_{m_s})$ are twin sets and therefore $\beta(\Gamma) \geq n-1+\sum_{j=1}^{s} m_i - 1 = |V(\Gamma)| - s - 1$ and the bound is sharp, since by omitting one vertex from each set we obtain a resolving set for Γ and so $\beta(\Gamma) = |V(\Gamma)| - s - 1$. If Γ_1 has k isolated vertices then the sets $V(K_n), V(K_{m_1}), \cdots, V(K_{m_{s-k}})$ and the set of all isolated vertices are twin sets. Thus, $\beta(\Gamma) \geq n-1 + \sum_{j=1}^{s-k}(m_i - 1) + k - 1$ $= |V(\Gamma)| - (s-k) - 2$ and similar to the last case, $\beta(\Gamma) = |V(\Gamma)| - (s-k) - 2$. □

Lemma 4.2. *Let* $\Gamma = K_n \vee (mK_s \cup K_d)$ *then,*

$$\beta(\Gamma) = \begin{cases} |V(\Gamma)| - m - 2 & if\ mK_s \cup K_d\ has\ no\ isolated\ vertex \\ |V(\Gamma)| - (m-k) - 3 & if\ mK_s \cup K_d\ has\ k\ isolated\ vertices. \end{cases}$$

Proof. Apply Lemma 4.1. □

Theorem 4.3. *Suppose* $mK_s \cup K_d$ *has no isolated vertex and* $\Gamma = K_n \vee (mK_s \cup K_d)$. *Then,*

$$\begin{aligned} \beta(\Gamma, x) &= nds^m x^{|V(\Gamma)|-m-2} + (nd + (n+d)s)s^{m-1}x^{|V(\Gamma)|-m-1} + x^{|V(\Gamma)|} \\ &+ |V(\Gamma)|x^{|V(\Gamma)|-1} + \sum_{i=|V(\Gamma)|-m}^{|V(\Gamma)|-2} r_i x^i, \end{aligned}$$

where $r_i = s^{|V(\Gamma)|-i-2}(nd\binom{m}{|V(\Gamma)|-i-2} + s(n+d)\binom{m}{|V(\Gamma)|-i-1} + s^2\binom{m}{|V(\Gamma)|-i})$.

Proof. To compute the sequence $(r_{|V(\Gamma)|-m-2}, ..., r_{|V(\Gamma)|})$, we first notice that

$$\begin{aligned}
r_{|V(\Gamma)|-m-2} &= \binom{n}{n-1}\binom{d}{d-1}\binom{s}{s-1}^m, \\
r_{|V(\Gamma)|-m-1} &= \binom{n}{n-1}\binom{d}{d}\binom{s}{s-1}^m + \binom{n}{n}\binom{d}{d-1}\binom{s}{s-1}^m \\
&+ \binom{n}{n-1}\binom{d}{d-1}\binom{m}{1}\binom{s}{s-1}^{m-1} \\
&= ns^m + ds^m + ndms^{m-1} \\
&= ((n+d)s + ndm)s^{m-1}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
r_i &= \binom{n}{n-1}\binom{d}{d-1}\binom{m}{|V(\Gamma)|-i-2}\binom{s}{s-1}^{|V(\Gamma)|-i-2} \\
&+ \binom{n}{n-1}\binom{d}{d}\binom{m}{|V(\Gamma)|-i-1}\binom{s}{s-1}^{|V(\Gamma)|-i-1} \\
&+ \binom{n}{n}\binom{d}{d-1}\binom{m}{|V(\Gamma)|-i-1}\binom{s}{s-1}^{|V(\Gamma)|-i-1} \\
&+ \binom{n}{n}\binom{d}{d}\binom{m}{|V(\Gamma)|-i}\binom{s}{s-1}^{|V(\Gamma)|-i} \\
&= s^{|V(\Gamma)|-i-2}nd\binom{m}{|V(\Gamma)|-i-2} + s^{|V(\Gamma)|-i-1}n\binom{m}{|V(\Gamma)|-i-1} \\
&+ s^{|V(\Gamma)|-i-1}d\binom{m}{|V(\Gamma)|-i-1} + s^{|V(\Gamma)|-i}\binom{m}{|V(\Gamma)|-i-2} \\
&= s^{|V(\Gamma)|-i-2}(nd\binom{m}{|V(\Gamma)|-i-2} + s(n+d)\binom{m}{|V(\Gamma)|-i-1} \\
&+ s^2\binom{m}{|V(\Gamma)|-i}),
\end{aligned}$$

It is easy to see that $r_{|V(\Gamma)|} = 1$ and $r_{|V(\Gamma)|-1} = |V(\Gamma)|$. This completes the proof. □

5. Applications

In this section, we apply our results given last section to calculate some parameters of the commuting graph of the finite groups T_{4n}, SD_{8n} and $U_{n,m}$.

Example 5.1. *Set $G = T_{4n}, n > 1$ and $\Gamma = \mathcal{C}(G)$. Then by applying [14, Lemma 2.7] and Lemmas 3.3 and 3.1, we have $C_0 = K_2, C_{m_1} = K_{2n-2}$ and $C_{m_l} = K_2$, where $2 \le l \le n+1$. Since $n > 1$, the quantity t in Theorem 3.5 will be less than s and so the following holds:*

1. *If $z \in C_0$, $v \in C_1$ and $u \in C_l$ then z is adjacent with u and v but u and v are not adjacent. Therefore, $d_D(z, v) = 2n+1$ and $d_D(u, v) = 2n+3$. Thus, $ecc_D(u) = \begin{cases} 2n+1 & u \in C_0 \\ 2n+3 & u \notin C_0 \end{cases}$, $rad_D(\Gamma) = 2n-1$ and $diam(\Gamma) = 2n+3$.*
2. *The following properties are direct consequences of Theorem 3.5:*
   1. *The detour distance degree sequence of T_{4n} can be computed as follows:*
   $$dds_D(u) = \begin{cases} (1, 0^{2n-2}, 1, 4n-2)^2 & u \in C_0 \\ (1, 0^{2n}, 2n-1, 0, 2n)^{2n-2} & u \in C_1 \\ (1, 0^{2n}, 2n+1, 0, 2n-2)^{2n-2}) & \\ or\ (1, 0^{2n}, 2, 0, 4n-3)^2 & u \in C_l \end{cases};$$
   2. *$D(\Gamma) = ((4n-2)^2, (4n-3)^2, (2n)^{2n-2}, (2n-2)^{2n-2})$;*
   3. *$D_{av}(\Gamma) = \frac{4n^2+2n-3}{2n}$.*

Example 5.2. *Set $G = SD_{8n}, n > 3$ and $\Gamma = \mathcal{C}(G)$. By [14, Lemma 2.10] and Lemmas 3.3 and 3.1, if n is even then $C_0 = K_2, C_{m_1} = K_{4n-2}, C_{m_l} = K_2$, where $2 \le l \le 2n+1$. If n is odd, then $C_0 = K_4, C_{m_1} = K_{4n-4}, C_{m_l} = K_4$, where $2 \le l \le n+1$. We note that in Theorem 3.5, if $n > 3$ then $t < s$. Apply Theorem 3.5(3), then we have:*

1. *By Lemma 3.3 and a similar argument as Example 5.1,*
   1. *Suppose u is a vertex in Γ. The detour eccentricity of u is computed as follows:*
   $$ecc_D(u) = \begin{cases} \begin{cases} 4n+1 & u \in C_0 \\ 4n+3 & u \notin C_0 \end{cases} & \&\ \ 2 \mid n \\ \begin{cases} 4n+11 & u \in C_0 \\ 4n+15 & u \notin C_0 \end{cases} & \&\ \ 2n \end{cases},$$

2. *The radius of* Γ *is:* $rad_D(\Gamma) = \begin{cases} 4n-1 & 2 \mid n \\ 4n+7 & 2n \end{cases}$,
3. *The diameter of* Γ *can be computed by the following formula:* $diam(\Gamma) = \begin{cases} 4n+3 & 2 \mid n \\ 4n+15 & 2n \end{cases}$.

2. *Apply Theorem 3.5.*

   1. *n is odd. Then,*

$$dds_D(u) = \begin{cases} (1, 0^{4n-2}, 1, 0, 8n-2)^2 & u \in C_0 \\ (1, 0^{4n}, 4n-1, 0, 4n)^{2n-2} & u \in C_1 \\ (1, 0^{4n}, 3, 0, 4n-4)^{2n}, (1, 0^{4n}, 2, 8n-3)^{2n} & u \in C_l \end{cases}.$$

   2. *n is even. Thus,*

$$dds_D(u) = \begin{cases} (1, 0^{4n-6}, 3, 0^6, 8n-4)^4 & u \in C_0 \\ (1, 0^{4n-10}, 4n-5, 0^3, 4n+4)^{4n-4} & u \in C_1 \\ (1, 0^{4n-10}, 7, 0^3, 8n-8)^4, & \\ (1, 0^{4n-10}, 4, 0^3, 8n-5)^{4n-4} & u \in C_l \end{cases}.$$

   3. *The degree sequence of* Γ *is as follows:*

$$D(\Gamma) = \begin{cases} ((8n-4)^4, (8n-5)^{4n-4}, (8n-8)^4, (4n+4)^{4n-4}) & 2 \mid n \\ ((8n-2)^2, (8n-3)^{2n}, (4n)^{2n-2}, (4n-4)^{2n}) & 2n \end{cases}.$$

   4. *The average detour degree of* Γ *can be computed by the following formula:*

$$D_{av}(\Gamma) = \begin{cases} \frac{12n^2+3n-11}{2n} & 2 \mid n \\ \frac{16n^2-3n-2}{4n} & 2n \end{cases}.$$

Example 5.3. *Set* $G = U_{n,m}$ *and* $\Gamma = \mathcal{C}(G)$. *We first assume that* m *is even. Then by considering [14, Theorem 2.3] and Lemmas 3.3 and 3.1, we have* $C_0 = K_{2n}, C_{m_1} = K_{(m-2)n}, C_{m_l} = K_{2n}$ *in which* $2 \leq l \leq \frac{m}{2}+1$. *Suppose* n *is odd. Hence* $C_0 = K_n, C_{m_1} = K_{(m-1)n}, C_{m_l} = K_n$, *for* $2 \leq l \leq m+1$. *In Theorem 3.5, if* m *is even then* $t = 2n$ *and* $s = \frac{m}{2}+1$ *and if* m *is odd then* $t = n$ *and* $s = m+1$. *In what follows, some properties of detour distance of the commuting graph of* $G = U_{n,m}$ *are given.*

1. *The following properties are immediate consequences of Theorem 3.5.*

1. *If m is even, then the detour eccentricity of vertex u in Γ is as follows:*

$$ecc_D(u) = \begin{cases} 2nm-1 & m \leq 4n-2 \\ 4n^2+mn-2n-1 & m > 4n-2, \quad u \in C_0 \\ 4n^2+mn-1 & m > 4n-2, \quad u \notin C_0 \end{cases}.$$

2. *If m is odd, then,*

$$ecc_D(u) = \begin{cases} 2nm-1 & m \leq n-1 \\ n^2+mn-n-1 & m > n-1, \quad u \in C_0 \\ n^2+mn-1 & m > n-1, \quad u \notin C_0 \end{cases}.$$

3. *The radius of Γ is as follows:*

$$rad_D(\Gamma) = \begin{cases} |G|-1 & m < 4n-2 \\ |G|-2n-1 & m = 4n-2, \quad 2 \mid m \\ |G|-n-1 & m = n-1, \quad 2m \\ 4n^2+mn-4n-1 & m > 4n-2, \quad 2 \mid m \\ n^2+mn-2n-1 & m > n-1, \quad 2m \end{cases}.$$

4. *The diameter of Γ can be computed as:*

$$diam(\Gamma) = \begin{cases} |G|-1 & m \leq 4n-2 \\ 4n^2+mn-1 & m > 4n-2, \quad 2 \mid m \\ n^2+mn-1 & m > n-1, \quad 2m \end{cases}.$$

2. *To compute the distance degree sequence of an arbitrary vertex $v \in \Gamma$ four cases are considered as follows:*

1. *$m \leq 4n-2$ and $2 \mid m$. Then,*

$$dds_D(\Gamma) = \begin{cases} (1, 0^{2nm-2}, 2nm-1)^{2nm-1} & m < 4n-2 \\ (1, 0^{2nm-2n-2}, 2n-1, 0^{2n}, 2nm-2n)^{2n} & m = 4n-2, \\ & v \in C_0 \\ (1, 0^{2nm-2}, 2nm-1)^{2nm-2n} & m = 4n-2, \\ & v \notin C_0 \end{cases}.$$

2. *$m \leq n-1$ and $2m$. Then,*

$$dds_D(\Gamma) = \begin{cases} (1, 0^{2nm-2}, 2nm-1)^{2nm-1} & m < n-1 \\ (1, 0^{2nm-n-2}, n-1, 0^{n}, 2nm-n)^{n} & m = n-1, v \in C_0 \\ (1, 0^{2nm-2}, 2nm-1)^{2nm-n} & m = n-1, v \notin C_0 \end{cases}.$$

3. $m > 4n-2$ and m is even. Thus,

$$dds_D(u) = \begin{cases} (1, 0^{4n^2+mn-4n-2}, 2n-1, & 0^2n-1, 2nm-2n)^{2n} \\ & u \in C_0 \\ (1, 0^{4n^2+mn-2n-2}, mn-1, & 0^2n-1, nm)^{mn-2n} \\ & u \in C_1 \\ (1, 0^{4n^2+mn-2n-2}, 4n-1, & 0^2n-1, 2nm-4n)^{2n}, \\ (1, 0^{4n^2+mn-2n-2}, 2n, 0^2n-1, & 2nm-2n-1)^{mn-2n} \\ & u \in C_l \end{cases}.$$

4. $m > n-1$ and m is odd. In this case,

$$dds_D(u) = \begin{cases} (1, 0^{n^2+mn-2n-2}, n-1, & 0^n-1, 2nm-n)^{n} \\ & u \in C_0 \\ (1, 0^{n^2+mn-n-2}, mn-1, & 0^n-1, nm)^{mn-n} \\ & u \in C_1 \\ (1, 0^{n^2+mn-n-2}, 2n-1, & 0^n-1, 2nm-2n)^{n}, \\ (1, 0^{n^2+mn-n-2}, n, & 0^n-1, 2nm-n-1)^{mn-n} \\ & u \in C_l \end{cases}.$$

We now apply Theorem 4.3 to compute the metric dimension of the groups D_{2n}, SD_{8n}, T_{4n} and $U_{m,n}$.

Example 5.4. *For all even $n > 3$, let $\Gamma = \mathcal{C}(D_{2n})$. Then, $\beta(\Gamma, x) = 2^{\frac{n}{2}-1}(4(n-2)+n^2x)x^{\frac{3n}{2}-2} + x^{2n} + 2nx^{2n-1} + \sum_{i=\frac{3n}{2}}^{2n-2} r_i x^i$, where*

$$r_i = 2^{2n-i-1}\left((n-2)\binom{\frac{n}{2}}{2n-i-2} + n\binom{\frac{n}{2}}{2n-i-1} + 2\binom{\frac{n}{2}}{2n-i}\right).$$

Example 5.5. *Set $\Gamma = \mathcal{C}(T_{4n})$. Then by [14, Lemma 2.7], $\Gamma = K_2 \vee (nK_2 \cup K_{2n-2})$ and $\beta(\Gamma, x) = 2^{n+1}(2(n-1)+n^2x)x^{3n-2} + x^{4n} + 4nx^{4n-1} + \sum_{3n}^{4n-2} r_i x^i$, where*

$$r_i = 2^{4n-i}\left((n-1)\binom{n}{4n-i-2} + n\binom{n}{4n-i-1} + \binom{n}{4n-i}\right).$$

Example 5.6. *Set $\Gamma = \mathcal{C}(SD_{8n})$. Then, by [14, Lemma 2.10],*

$$\Gamma = \begin{cases} K_2 \vee (2nK_2 \cup K_{4n-2}) & 2 \mid n \\ K_4 \vee (nK_4 \cup K_{4n-4}) & 2n \end{cases},$$

$$\beta(\Gamma,x) = \begin{cases} 2^{2n+2}((2n-1)+2n^2x)x^{6n-2}+x^{8n}+ \\ 8nx^{8n-1}+\sum_{6n}^{6n-2} r_ix^i & 2\mid n \\ \\ 4^{n+1}(4(n-1)+n^2x)x^{7n-1}+x^{8n}+ \\ 8nx^{8n-1}+\sum_{7n}^{7n-2} r_ix^i & 2n \end{cases},$$

where

$$r_i = \begin{cases} 2^{8n-i}((2n-1)\binom{2n}{8n-i-2}+2n\binom{2n}{8n-i-1}+\binom{2n}{8n-i}) & 2\mid n \\ \\ 4^{8n-i-1}((n-1)\binom{n}{8n-i-2}+n\binom{n}{8n-i-1}+\binom{n}{8n-i}) & 2n \end{cases}.$$

Example 5.7. *Consider the commuting graph $\Gamma = \mathcal{C}(U_{m,n})$. Then by [14, Theorem 2.3],*

$$\Gamma = \begin{cases} K_{2n}\vee(\frac{m}{2}K_{2n}\cup K_{(m-2)n}) & 2\mid m \\ K_n\vee(mK_n\cup K_{(m-1)n}) & 2m \end{cases},$$

$$\beta(\Gamma,x) = \begin{cases} (2n)^{\frac{m}{2}-1}(4n^3(m-2)+(mn)^2x))x^{2nm-\frac{m}{2}-2} \\ +x^{2nm}+2nmx^{2nm-1}+\sum_{2nm-\frac{m}{2}}^{2nm-2} r_ix^i & 2\mid m \\ \\ n^{m+1}(n(m-1)+m^2x))x^{2nm-m-2} \\ +x^{2nm}+2nmx^{2nm-1}+\sum_{2nm-m}^{2nm-2} r_ix^i & 2m \end{cases},$$

where

$$r_i = \begin{cases} (2n)^{2nm-i}\left((\frac{m}{2}-1)\binom{\frac{m}{2}}{2nm-i-2}+m\binom{\frac{m}{2}}{2nm-i-1}+\binom{\frac{m}{2}}{2nm-i}\right) & 2\mid m \\ \\ n^{2nm-i}\left((m-1)\binom{m}{2nm-i-2}+m\binom{m}{2nm-i-1}+\binom{m}{2nm-i}\right) & 2m \end{cases}.$$

Acknowledgement. We are indebted to an anonymous referee for his/her suggestions and corrections that leaded us to improve this paper. The research of the first author is partially supported by the university of Kashan under grant number 890190/2.

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