Proyecciones (Antofagasta. On line) | vol. 39, n. 5 (2020) | pp. 1333-1346.

REVISTAS CIENTÍFICAS de la Universidad Católica del Norte. revistas.ucn.cl

10.22199/issn.0717-6279-2020-05-0081

PROYECCIONES
Journal of Mathematics

ISSN 0717-6279 (On line)

On reformulated Narumi-Katayama index

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Received: April 2020 | Accepted: August 2020

Abstract:

A graph is a mathematical model form by set of dots for vertices some of which are connected by lines named as edges. A topological index is a numeric value obtained from a graph mathematically which characterize its topology. The reformulated Narumi-Katayama index of a graph G is defined as the product of edge degrees of all the vertices of G which is introduced in 1984, to used the carbon skeleton of a saturated hydrocarbons. The degree of an edge is given by the sum of degrees of the end vertices of the edge minus 2. In this paper, we compute the reformulated Narumi-Katayama index for different graph operations.

Keywords: Degree; Graph; Graph operations; Reformulated NK-index; Topological indices.

MSC (2020): 05C35, 05C07, 05C40.

Cite this article as (IEEE citation style):

M. Cancan, N. De, M. Alaeiyan, and M. Reza Farahani, "On reformulated Narumi-Katayama index", *Proyecciones (Antofagasta, On line)*, vol. 39, no. 5, pp. 1333-1346, Oct. 2020, doi: 10.22199/issn.0717-6279-2020-05-0081.

1. Introduction

Let G be a simple connected graph with vertex set $V(G)$ and edge set $E(G)$. We denote by $d_G(v)$ the degree of a vertex v of G, which is defined as the number of edges incident to v. The first and second Zagreb indices of a graph G are denoted by $M_1(G)$ and $M_2(G)$ respectively and defined as follows:

$$M_1(G) = \sum_{v \in V(G)} d_G(v)^2 = \sum_{uv \in E(G)} [d_G(u) + d_G(v)]$$

and

$$M_2(G) = \sum_{uv \in E(G)} d_G(u).d_G(v).$$

These indices were introduced by Gutman et al. in [1] and are among the oldest and most studied degree-based topological indices of graphs.

Narumi and Katayama in 1984 [2], introduced a multiplicative version of degree-based graph invariant for representing the carbon skeleton of a saturated hydrocarbon and named it as a "simple topological index". This invariant is defined as

$$NK(G) = \prod_{v \in V(G)} d_G(v).$$

Tomovic and Gutman [3], later renamed this index as "Narumi-Katayama index" or "NK-index". Todeschini et al. in 2010 [4], have introduced the multiplicative version of additive graph invariants and in this regard Eliasi, Iranmaresh and Gutman in 2012 [5], introduced a new multiplicative version of first Zagreb index and called it multiplicative sum Zagreb index, which is defined as

$$\Pi_1^*(G) = \prod_{uv \in E(G)} [d_G(u) + d_G(v)].$$

For further study about multiplicative version Zagreb indices we refer our reader to [6, 7, 8, 9, 10, 11]. We know that, the reformulated Zagreb indices are the edge version of classical Zagreb indices which are introduced by Miličević et al. [12], in 2004 and are defined as

$$EM_1(G) = \sum_{e \in E(G)} d(e)^2 = \sum_{uv \in E(G)} [d_G(u) + d_G(v) - 2]^2,$$

$$EM_2(G) = \sum_{e \sim f \in E(G)} d(e)d(f)$$

where $e \sim f$ means that the edges e and f share a common vertex in G. For further study, we refer our reader to [13, 14, 15, 16, 17, 18, 19, 20]. Analogus to reformulated Zagreb indices, in this paper, we introduced a new version of NK-index by replacing vertex degrees with the edge degrees, where degree of an edge $e = uv$ is defined as $d_G(e) = d_G(u) + d_G(v) - 2$. Thus, the reformulated NK-index of a graph G, denoted by $NK_E(G)$ is defined as

$$NK_E(G) = \prod_{e \in E(G)} d_G(e).$$

In this paper, first we present some preliminary results for the reformulated NK-index and hence derive some explicit for the reformulated NK-index of different graph operations such as join, Cartesian product, Corona product etc. and also consider some special cases.

2. Preliminary Results.

As usual, let the complete graph, cycle graph and path graph with n number of vertices are denoted by K_n, C_n and P_n respectively, where as $K_{m,n}$ denotes a complete bipartite graph with $(m+n)$ number of vertices. From the definition of reformulated NK-index, in this section first we derive reformulated NK-index of these standard classes of graph from direct calculation.

Example 1.

$$\begin{aligned}
(i)\ NK_E(K_n) &= (2n-4)^{\frac{n(n-1)}{2}}, \\
(ii)\ NK_E(C_n) &= 2^n,\ (n \geq 3), \\
(iii)\ NK_E(P_n) &= 2^{n-3},\ (n \geq 3), \\
(iv)\ NK_E(K_{m,n}) &= (m+n-2)^{mn}.
\end{aligned}$$

Now we recall the well-known A.M.-G.M. inequality, which is required in this paper.

Lemma 1. *Let $x_1,\ x_2,\ ...., x_n$ be non negative numbers, then*

$$\frac{x_1 + x_2 + ... + x_n}{n} \geq \sqrt[n]{x_1,\ x_2,\ ...., x_n}$$

with equality if and only if $x_1 = x_2 = ... = x_n$.

Now using above lemma, in the following we derive upper and lower bounds for reformulated NK-index of a graph G.

Proposition 1. *Let G ba connected graph with order n and size m. Then*

$$NK_E(G) \leq (\frac{M_1(G)}{m} - 2)^m$$

with equality if and only if G be a regular graph.

Proof. Using lemma 1, we have

$$\begin{aligned} \frac{1}{m}\sum_{e\in E(G)} d_G(e) &\geq [\prod_{e\in E(G)} d_G(e)]^{\frac{1}{m}} \\ or, \frac{1}{m}(M_1(G) - 2m) &\geq [NK_E(G)]^{\frac{1}{m}} \end{aligned}$$

with equality when G be a regular graph. Hence, the desired result follows.

Proposition 2. *Let G ba connected graph with order n and size m. Then*

$$NK_E(G) \geq \frac{1}{8^m}\Pi_1^*(G)^2,$$

with equality if and only if G is a cycle or the star graph on 4 vertices.

Proof. We have from definition of multiplicative sum Zagreb index

$$\begin{aligned} \Pi_1^*(G) &= \prod_{uv\in E(G)} [d_G(u) + d_G(v)]^2 \\ &= \prod_{uv\in E(G)} [(d_G(u) + d_G(v) - 2) + 2]^2 \end{aligned}$$

So, using lemma 1, we have

$$\begin{aligned} \Pi_1^*(G) &\geq \prod_{uv\in E(G)} [2\sqrt{(d_G(u) + d_G(v) - 2) \times 2}]^2 \\ &= 8^m \prod_{uv\in E(G)} [d_G(u) + d_G(v) - 2] \\ &= 8^m NK_E(G). \end{aligned}$$

From, lemma 1, it is clear that, in the above inequality, equality holds if and only if for every $e = uv \in E(G)$, $d_G(e) = 2$. Thus, G is either a cycle or the star graph on 4 vertices. Hence, the desired result follows.

3. Reformulated NK-index of graph operations:

In this section, we study some upper bounds of reformulated NK-index under different graph operations. Hence, we find the equality conditions for these inequalities unless stated otherwise, we consider G_i connected graph with n_i number of vertices and m_i number of edges, for $i \in \{1, 2\}$.

The join of graphs:

The join $G_1 + G_2$ of two graphs G_1 and G_2 with disjoint vertex sets $V(G_1)$ and $V(G_2)$, and edge sets $E(G_1)$ and $E(G_2)$ is the graph union $G_1 \cup G_2$ together with all the edges between $V(G_1)$ and $V(G_2)$. Thus, $|V(G_1+G_2)| = |V(G_1)|+|V(G_2)|$ and $|E(G_1+G_2)| = |E(G_1)|+|E(G_2)|+|V(G_1)||V(G_2)|$. The degree of a vertex $v \in V(G_1 + G_2)$ is given by

$$d_{G_1+G_2}(v) = \begin{cases} d_{G_1}(v) + |V(G_2)|, \ if \ v \in V(G_1) \\ d_{G_2}(v) + |V(G_1)|, \ if \ v \in V(G_2). \end{cases}$$

Now, in the following we derive the upper bound of reformulated NK-index of $G_1 + G_2$.

Theorem 1. *Let G_1 and G_2 be two vertex disjoint connected graphs. Then*

$$\begin{aligned} NK_E(G_1 + G_2) \ \leq \ & [\frac{M_1(G_1)}{m_1} + 2(n_2 - 1)]^{m_1} [\frac{M_1(G_2)}{m_2} \\ & +2(n_1 - 1)]^{m_2} \times [\frac{2m_1}{n_1} + \frac{2m_2}{n_2} \\ & +n_1 + n_2 - 2]^{n_1 n_2} \end{aligned}$$

In the above inequality, equality holds if and only if both G_1 and G_2 are regular graphs.

Proof. The edge set of G_1+G_2 can be partitioned into three subsets, namely $E_1 = E(G_1)$, $E_2 = E(G_2)$ and $E_3 = \{e = uv : u \in V(G_1), v \in V(G_2)\}$. Then from definition of reformulated NK-index and using lemma 1, the contribution of the edges of E_1 is given by

$$\begin{aligned} J_1 &= \prod_{uv \in E(G_1)} \{d_{G_1+G_2}(u) + d_{G_1+G_2}(v) - 2\} \\ &= \prod_{uv \in E(G_1)} \{d_{G_1}(u) + n_2 + d_{G_1}(v) + n_2 - 2\} \\ &\leq [\frac{1}{m_1} \sum_{uv \in E(G_1)} \{(d_{G_1}(u) + d_{G_1}(v)) + 2(n_2 - 1)\}]^{m_1} \\ &= [\frac{M_1(G_1)}{m_1} + 2(n_2 - 1)]^{m_1} \end{aligned}$$

In the above inequality, equality holds, if and only if for every $uv \in E(G_1)$ and $xy \in E(G_1)$

$$d_{G_1}(u) + d_{G_1}(v) + 2(n_2 - 1) = d_{G_1}(x) + d_{G_1}(y) + 2(n_2 - 1).$$

Similarly, the contribution of the edges of E_2 is given by

$$\begin{aligned} J_2 &= \prod_{uv \in E(G_2)} \{d_{G_1+G_2}(u) + d_{G_1+G_2}(v) - 2\} \\ &= \prod_{uv \in E(G_2)} \{(d_{G_2}(u) + d_{G_2}(v)) + 2(n_1 - 1)\} \\ &\leq [\frac{M_1(G_2)}{m_2} + 2(n_1 - 1)]^{m_2} \end{aligned}$$

In the above inequality, similarly equality holds, if and only if for every $uv \in E(G_2)$ and $xy \in E(G_2)$

$$d_{G_2}(u) + d_{G_2}(v) + 2(n_1 - 1) = d_{G_2}(x) + d_{G_2}(y) + 2(n_1 - 1).$$

Again, the contribution of the edges of E_3 to the reformulated NK-index of join of graphs is given by

$$\begin{aligned} J_3 &= \prod_{uv \in E(G_1+G_2), u \in V(G_1), v \in V(G_2)} (d_{G_1+G_2}(u) \\ &\quad + d_{G_1+G_2}(v) - 2) \\ &= \prod_{u \in V(G_1)} \prod_{v \in V(G_2)} d_{G_1}(u) + n_2 + d_{G_2}(v) + n_1 - 2 \end{aligned}$$

Now using lemma 1, we get

$$\begin{aligned} J_3 &\leq [\frac{1}{n_1 n_2} \sum_{u \in V(G_1)} \sum_{v \in V(G_2)} (d_{G_1}(u) + d_{G_2}(v) + n_1 + n_2 \\ &\quad -2)]^{n_1 n_2} \\ &= [\frac{1}{n_1 n_2}(2m_1 n_2 + 2m_2 n_1 + n_1 n_2(n_1 + n_2 - 2))]^{n_1 n_2} \end{aligned}$$

It is clear that, in the above inequality equality holds if and only if both G_1 and G_2 are regular graphs. Hence, combining the contributions J_1, J_2 and J_3 we get the desired result as theorem 1.

Corollary 1. *Let G_1 be a r_1-regular graph and G_2 be a r_2-regular graph, then*

$$\begin{aligned} NK_E(G_1 + G_2) &= 2^{m_1+m_2}(r_1 + n_2 - 1)^{m_1}(r_2 + n_1 - \\ &\quad 1)^{m_2} \times (n_1 + n_2 + r_1 + r_2 - 2)^{n_1 n_2}. \end{aligned}$$

Example 2.

$$(i)\ NK_E(C_n + C_m) = 2^{m+n}(n+1)^m(m+1)^n(n+m+2)^{nm}$$

Let $G = G_1 + G_2 + ... + G_k$ and $n = n_1 + n_2 + ... + n_k$ for some positive integer $k(\geq 2)$. Then, the degree of the vertices of G are given by

$$d_G(v) = d_{G_i}(v) + \bar{n_i},\ if\ v \in V(G_i)$$

where, $\bar{n_i} = n - n_i$ for $1 \leq i \leq k$. Then, using theorem 1, we get the following result using mathematical induction.

Theorem 2. *Let G_1, G_2,.....,G_k be k-number of vertex disjoint graphs. Then*

$$\begin{aligned} NK_E(G) &= \prod_{i=1}^{k} [\frac{M_1(G_i)}{m_i} + 2(\bar{n_i} - 1)]^{m_i} \prod_{1 \leq i \leq j \leq k} [\frac{2m_i}{n_j} \\ &\quad + \frac{2m_j}{n_i} + (\bar{n_i} + \bar{n_j} - 2)]^{n_i n_j}. \end{aligned}$$

The Cartesian Product of Graphs:

The Cartesian product $G_1 \times G_2$ of two graphs G_1 and G_2 is the graph with vertex set $V(G_1 \times G_2) = V(G_1) \times V(G_2)$ and $(u,x)(v,y)$ is an edge of $G_1 \times G_2$ if $uv \in E(G_1)$ and $x = y$, or $u = v$ and $xy \in E(G_2)$. Clearly, $|E(G_1 \times G_2)| = |E(G_1)||V(G_2)| + |E(G_2)||V(G_1)|$. In the following theorem, we obtain reformulated NK-index of the Cartesian product of two graphs G_1 and G_2. The degree of the vertices of $G_1 \times G_2$ are given by $d_{G_1 \times G_2}(u,v) = d_{G_1}(u) + d_{G_2}(v)$.

Theorem 3. *let G_1 and G_2 be two vertex-disjoint connected graphs. Then $[NK_E(G_1 \times G_2) \leq [\frac{M_1(G_1)}{n_2} + \frac{4m_2}{n_2} - 2]^{n_2 m_1}[\frac{M_1(G_2)}{n_1} + \frac{4m_1}{n_1} - 2]^{n_1 m_2}]$ with equality if and only if both G_1 and G_2 are regular graphs.*

Proof. We have, from definition of reformulated NK-index

$$\begin{aligned} NK_E(G_1 \times G_2) &= \prod_{(a,x)(a,y) \in E(G_1 \times G_2)} [d_{G_1 \times G_2}(a,x) + \\ &\quad d_{G_1 \times G_2}(b,y) - 2] \\ &= \prod_{(a,x)(a,y), xy \in E(G_2)} [d_{G_1 \times G_2}(a,x) \\ &\quad + d_{G_1 \times G_2}(a,y) - 2] \\ &\quad \times \prod_{(a,x)(b,x), ab \in E(G_1)} [d_{G_1 \times G_2}(a,x) \\ &\quad + d_{G_1 \times G_2}(b,x) - 2] \\ &= C_1 \times C_2. \end{aligned}$$

Where C_1 and C_2 denote the products of the above terms respectively. Now,

$$\begin{aligned} C_1 &= \prod_{(a,x)(a,y), xy \in E(G_2)} [d_{G_1 \times G_2}(a,x) + d_{G_1 \times G_2}(a,y) - 2] \\ &= \prod_{a \in V(G_1)} \prod_{xy \in E(G_2)} [2d_{G_1}(a) + d_{G_2}(x) + d_{G_2}(y) - 2]. \end{aligned}$$

Now using lemma 1, we have

$$\begin{aligned} C_1 &\leq [\frac{1}{n_1 m_2} \sum_{a \in V(G_1)} \sum_{xy \in E(G_2)} \{2d_{G_1}(a) + (d_{G_2}(x) + \\ &\quad d_{G_2}(y)) - 2\}]^{n_1 m_2} \\ &= [\frac{1}{n_1 m_2} \{4m_1 m_2 + n_1 M_1(G_2) - 2n_1 m_2\}]^{n_1 m_2} \\ &= [\frac{M_1(G_2)}{m_2} + \frac{4m_1}{n_1} - 2]^{n_1 m_2}. \end{aligned}$$

Similarly, we have

$$C_2 \leq [\frac{M_1(G_1)}{m_1} + \frac{4m_2}{n_2} - 2]^{n_2 m_1}.$$

From the equality condition of arithmatic-geometric inequality, it is clear that, in the above results equality holds, if and only if both G_1 and G_2 are regular graphs. Thus, combining the contribution of C_1 and C_2, we get the desired result.

Note that, if G_1 and G_2 are γ_1 and γ_2-regular graphs, then $G_1 \times G_2$ be a $(r_1 + r_2)$-regular graph. Hence, the following corollary follows:

Corollary 2. *If G_1 be a r_1-regular graph and G_2 be a r_2-regular graph, then*

$$NK_E(G_1 \times G_2) = \{2(r_1 + r_2 - 1)\}^{\frac{n_1 n_2}{2}(r_1 + r_2)}.$$

Using above results, the following example follows immediately.

Example 3.

$$\begin{aligned} (i)\ NK_E(C_n \times C_m) &= 36^{nm} \\ (ii)\ NK_E(K_n \times K_m) &= \{2(n + m - 3)\}^{\frac{nm}{2}(n+m-3)}. \end{aligned}$$

Corona Product:

The corona product of two graphs G_1 and G_2, denoted by $G_1 \circ G_2$, is the graph obtained by taking one copy of G_1 and $|V(G_1)|$ copies of G_2; and by joining each vertex of i-th copy of G_2 to the i-th vertex of G_1, for $1 \leq i \leq |V(G_1)|$. Thus $|V(G_1 \circ G_2)| = |V(G_1)| + |V(G_1)||V(G_2)|$ and

$|E(G_1 \circ G_2)| = |E(G_1)| + |V(G_1)||E(G_2)| + |V(G_1)||V(G_2)|$. The degree of the vertices of $G_1 \circ G_2$ is given by

$$d_{G_1 \circ G_2}(v) = \begin{cases} d_{G_1}(v) + n_2, \ if \ v \in V(G_1) \\ d_{G_2}(v) + 1, \ if \ v \in V(G_2). \end{cases}$$

Now, in the following theorem, we derive reformulated NK-index of $G_1 \circ G_2$.

Theorem 4. *Let G_1 and G_2 be two vertex disjoint connected graphs. Then*

$$\begin{aligned} NK_E(G_1 \circ G_2) \ \leq \ & [\frac{M_1(G_1)}{m_1} + 2(n_2 - 1)]^{m_1} \\ & [\frac{M_1(G_2)}{m_2}]^{m_2 n_1} \times [\frac{2m_1}{n_1} + \frac{2m_2}{n_2} + \\ & n_2 - 1]^{n_1 n_2} \end{aligned}$$

with equality if and only if bothG_1 and G_2 are regular graphs.

Proof. From definition of reformulated NK-index of a graph, we have

$$\begin{aligned} NK_E(G_1 \circ G_2) \ = \ & \prod_{u_i u_k \in E(G_1)} (d_{G_1}(u_i) + n_2 + d_{G_1}(u_k) \\ & + n_2 - 2) \\ & \times \prod_{u_i \in V(G_1)} \prod_{v_j \in V(G_2)} (d_{G_1}(u_i) + n_2 + \\ & d_{G_2}(v_j) + 1 - 2) \\ & \times \prod_{u_i \in V(G_1)} \prod_{v_i v_l \in E(G_2)} (d_{G_2}(v_j) + 1 + \\ & d_{G_2}(v_l) + 1 - 2) \\ = \ & P_1 \times P_2 \times P_3 \end{aligned}$$

Where P_1, P_2 and P_3 denote the above three products respectively. Now applying lemma 1, we can write

$$P_1 \ = \ \prod_{u_i u_k \in E(G_1)} (d_{G_1}(u_i) + d_{G_1}(u_k) + 2n_2 - 2)$$

$$\begin{aligned} &\leq [\frac{1}{m_1}\sum_{u_iu_k\in E(G_1)}\{(d_{G_1}(u_i)+d_{G_1}(u_k)+2(n_2- \\ &1)\}]^{m_1} \\ =\ &[\frac{M_1(G_1)}{m_1}+2(n_2-1)]^{m_1} \end{aligned}$$

Similarly, applying lemma 1, we have

$$\begin{aligned} P_2 &= \prod_{u_i\in V(G_1)}\prod_{v_j\in V(G_2)}(d_{G_1}(u_i)+d_{G_2}(v_j))+(n_2-1)) \\ &\leq [\frac{1}{n_1n_2}\sum_{u_i\in V(G_1)}\sum_{v_j\in V(G_2)}\{(d_{G_1}(u_i)+d_{G_2}(v_j)) \\ &+(n_2-1)\}]^{n_1n_2} \\ &= [\frac{1}{n_1n_2}\{2m_1n_2+2m_2n_1+n_1n_2(n_2-1)\}]^{n_1n_2} \end{aligned}$$

Again, applying lemma 1, we similarly have

$$\begin{aligned} P_3 &= \prod_{u_i\in V(G_1)}\prod_{v_jv_l\in E(G_2)}(d_{G_2}(v_j)+d_{G_2}(v_l)) \\ &= [\prod_{v_jv_l\in E(G_2)}(d_{G_2}(v_j)+d_{G_2}(v_l))]^{n_1} \\ &\leq [\frac{1}{m_2}\sum_{v_jv_l\in E(G_2)}(d_{G_2}(v_j)+d_{G_2}(v_l))]^{n_1m_2} \\ &= [\frac{M_1(G_2)}{m_2}]^{n_1m_2}. \end{aligned}$$

From the equality condition of A.M.-G.M. inequality, it is evident that, in the above inequality, equality hold if and only if both G_1 and G_2 are regular graphs. Hence, combining P_1, P_2 and P_3, we get the desired result.

Corollary 3. *Let G_1 be a r_1-regular graph and G_2 be a r_2-regular graph then* $[NK_E(G_1\circ G_2)=[2(r_1+n_2-1)]^{\frac{n_1r_1}{2}}[r_1+r_2+n_2-1]^{n_1n_2}(2r_2)^{\frac{1}{2}n_1n_2r_2}.]$

From the above result the following example follows:

Example 4.

$$\begin{aligned} (i)\ NK_E(C_n\circ K_2) &= 3^n2^{6n} \\ (ii)\ NK_E(K_2\circ C_n) &= 2^{4n+1}n(n+2)^{2n} \end{aligned}$$

Conclusion

In this paper, we compute the Narumi-Katayama index of some graph operations such as join of graphs, Cartesian product of graphs and corona product of graphs and consider some special case. From our derived results, we compute this index for some special graphs known as path graph, complete graphs and cycle. For further study, we consider some other graph operations to compute this Narumi-Katayama index.

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