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Existence and uniqueness of the generalized solution of a non-homogeneous hyperbolic differential equation modeling the vibrations of a dissipating elastic rod

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Abstract

The purpose of this mathematical paper is to establish a qualitative research of the existence and uniqueness of the generalized solution to a non-homogeneous hyperbolic partial differential equation problem

$$\frac{\partial^2 u}{\partial t^2} - \Delta u - \Delta \frac{\partial u}{\partial t} = f$$

subject to the contour condition $u = 0$ over $\sum$, and with initial conditions $u(x,0) = u_0(x)$ in Ω, $\partial u_t(x,0) = u_1(x)$ in Ω. In the development of the research, the deductive method of Faedo-Garleskin and Medeiro is used to demonstrate the existence of the generalized solution that consists in the construction of approximate solutions in a finite dimensional space, obtaining a succession of approximate solutions to the non-homogeneous hyperbolic problem, that is, by means of a priori estimations, these successions of approximate solutions are passed to limit in a suitable topology. Then the initial conditions are verified and the uniqueness of the generalized solution is proved.

Keywords: Dissipative evolution equation, Existence and Uniqueness of the generalized solution.

MSC (2020): 35A01; 35D30; 35L05.

1. Introduction

The evolution equations are equations in partial derivatives, which describe processes that develop over time. Likewise, let us consider an non-homogeneous evolution equation that describes the vibrations of an elastic rod subject to dissipative constraints caused by the medium where the motion occurs in a cylindrical domain; with these physical properties we need to add to the evolution equation certain conditions such as: Dirichlet condition that expresses that the elastic rod is fixed on the lateral boundary of a cylindrical and the Cauchy conditions that fix the state of the elastic rod at the initial instant, also called initial displacement and initial velocity.

The problem encountered when investigating the partial differential equations of non-homogeneous evolution with dissipation, coupled with certain initial and contour values, is reduced to analyzing at first the existence and uniqueness of the solution for the model describing the vibrations of an elastic rod subject to dissipative constraints. But a difficulty presents this non-homogeneous evolution equation with dissipative terms, for example, the data may have initial conditions of functions (solution of this PDE) that are not regular or sufficient to have non-differentiable functions in the classical sense and even be discontinuous, here is the importance of the weak or generalized solution in the study of the wave equations. This new solution concept was introduced by Sobolev and Schwartz (1969). These function spaces are based on the concept of weak solution consisting of $L^p(\Omega)$ functions whose derivatives in the generalized sense also belong to such a space. Those spaces become the natural environment for the theorems of existence and uniqueness of generalized solutions of partial derivative equations, used by prestigious mathematicians in their research [1,2,3,4,6,7,8,9,10,13,14].

2. Materials and Methods

In the present article a mathematical model has been considered concerning the vibrations of a bar (acoustic, electromagnetic, etc.) in an elastic medium with dissipation caused by the medium where the motion occurs, subjected to an external force:

$$
(2.1) \qquad \begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \Delta u - \Delta \dfrac{\partial u}{\partial t} = f \text{ , } in \;\; Q = \Omega x(0,T), \\[2ex] u = 0 \text{ , } over \; \Sigma = \Gamma x(0,T), \\[2ex] u(x,0) = u_0(x) \text{ , } in \;\; \Omega, \\[2ex] \dfrac{\partial u(x,0)}{\partial t} = u_1(x) \text{ , } \;\; in \;\; \Omega, \end{cases}
$$

where $u\,(x,t)$ are the configurations of the vibrations at instant t and at point x of Ω, for Ω an open, bounded and well-regular set of $\mathbf{R}^n$ with boundary $\Gamma = \partial\Omega$, u_0 and u_1 are given functions.

The method used is from Faedo-Galerkin and Medeiro and consists of approximating the initial evolution problem by equivalent approximate systems, but in finite dimension. Then it is divided into stages: Approximate solution bounding, convergence of the approximate solutions, verification of the initial conditions and finally, uniqueness of the solution.

3. Results and Discussion

3.1. Existence of the generalized solution of the non-homogeneous hyperbolic differential equation

Definition 3.1.1. (Generalized solution)

The function $u : [0,T] \to H_0^1\,(\Omega)$ is a generalized solution to the non-homogeneous hyperbolic evolution problem (2.1), if and only if u satisfies the following condition:

$$
(u''(t), v) + ((u(t), v)) + ((u'(t), v)) = (f(t), v), \text{ for all } v \in H_0^1(\Omega)
$$

being equality in the sense of $D\,(0,T)\,.$

Theorem 3.1.1. (Existence of the generalized solution)

Let $f \in L^2(Q)$, $u_0 \in H_0^1(\Omega)$ and $u_1 \in L^2(\Omega)$. Then there exists the function $u : [0,T] \to H_0^1\,(\Omega)$ fulfilling the conditions:

(3.1) $$u \in L^{\infty}\left(0,T;H_0^1(\Omega)\right),$$

(3.2) $$u' \in L^{\infty}\left(0,T;L^2(\Omega)\right),$$

(3.3) $$u'' \in L^{2}\left(0,T;H^{-1}(\Omega)\right),$$

$(u''(t),v)+((u(t),v))+((u'(t),v)) = (f(t),v)$ for all $v \in H_0^1(\Omega)$,
(3.4)
in the sense of $D'(0,T)$.

(3.5) $$u(0) = u_0,$$

(3.6) $$u'(0) = u_1.$$

Proof . The demonstration is divided into four stages:

Stage 1 - Approximate problem

The demonstration begins by analyzing $H_0^1(\Omega)$ that the space is a separable Hilbert space, then there exists a succession of vectors (w_i), $w_i \in H_0^1(\Omega)$ for all i ,satisfying the conditions: for each m the vectors $w_1, w_2, ..., w_m$ are linearly independent and the finite linear combinations of (w_i) are dense in $H_0^1(\Omega)$.

Let $W_m = [w_1, w_2, ..., w_m]$ a subspace of $H_0^1(\Omega)$, of dimension m, generated by the first m vectors. Consider the approximate solution $u_m : [0,t_m] \to H_0^1(\Omega)$ such that $u_m(t) = \sum_{i=1}^{m} g_{im}(t)w_i$, where $u_m(t)$ is the solution of the approximate problem :

(3.7) $$(u''_m(\mathrm{t}),v) + ((u_m(t),v)) + ((u'_m(t),v)) = (\mathrm{f}(\mathrm{t}),v)$$

for all v in V_m, in the sense of $D'(0,\mathrm{T})$.

(3.8) $$u_m(0) = u_{0m}, u_{0m} = \sum_{i=1}^{m} \alpha_{im} w_i \to u_0, \mathrm{u}_0 in \mathrm{H}_0^1(\Omega);$$

$$u'_m(0) = u_{1m}, u_{1m} = \sum_{i=1}^{m} \beta_{im} w_i \to u_1, \mathrm{u}_1 in \mathrm{L}^2(\Omega). \tag{3.9}$$

Stage 2 - Approximate solution bounding

Two a priori estimates are presented:

A priori estimate I

Let $v = u'_m(t) \in V_m$, be the first a priori estimate and then replacing it in the approximate equation (3.7) we have:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\left[\left|\mathrm{u}'_{\mathrm{m}}(\mathrm{t})\right|^2 + \|u_m(t)\|^2\right] + \|u'{}_m(t)\|^2 = \left(\mathrm{f}(\mathrm{t}), u'_m(\mathrm{t})\right). \tag{3.10}$$

The approximate solution $u_m(t)$ exists in $[0, t_m)$. Also, integrating equation (3.10) we obtain:

$$|u'{}_m(t)|^2 + \|u_m(t)\|^2 + 2\int_0^t \|u'_m(s)\|^2 ds \le c_1 + \int_0^t |u'_m(s)|^2\, ds. \tag{3.11}$$

Now, applying Gronwall's inequality [7] in (3.11), it is concluded that:

$$|u'{}_m(t)| \le c, \text{independent of } m \text{ and } t. \tag{3.12}$$

Furthermore, by the Poincaré-Friedricks inequality [8], it follows that:

$$\|u_m(t)\| \le c,$$

(3.13)

independent of m and t.

Likewise, using (3.12) and (3.13) we obtain:

$$(u'_m) \tag{3.14}$$

is a succesion bounded in $L^\infty\left(0, T; L^2(\Omega)\right)$,

$$(u_m) \tag{3.15}$$

is a succesion bounded in $L^\infty\left(0, T; H_0^1(\Omega)\right)$.

If $t = T$ in equality (3.11), then:

$$(u'_m(T)) \tag{3.16}$$

is a succesion bounded in $L^2(\Omega)$,

$$(u_m(T)) \tag{3.17}$$

is a succesion bounded in $H_0^1(\Omega)$.

A priori estimate II

Deriving the approximate equation (3.7) in relation at t, we obtain:

$$(\mathrm{u}'''_{\mathrm{m}}(\mathrm{t}), v) + ((\mathrm{u}'_{\mathrm{m}}(\mathrm{t}), v)) + ((\mathrm{u}''_{\mathrm{m}}(\mathrm{t}), v)) = (f'(\mathrm{t}), v). \tag{3.18}$$

Making $v = u''_m(t) \in V_m$, second estimate and replacing in equation (3.18) we obtain:

$$\frac{1}{2}\frac{d}{\mathrm{dt}}\left[\left|u''_m(t)\right|^2 + \left\|u'_m(t)\right\|^2\right] + \left\|u''_m(t)\right\|^2 = (f'(t), u''_m(t)). \tag{3.19}$$

Integrating from 0 to t in (3.19), we have:

$$\left|u''_m(t)\right|^2 \leq c_2 + \int_0^t \left|u''_{\mathrm{m}}(s)\right|^2 ds. \tag{3.20}$$

Again, applying Gronwall's inequality in (3.20) we conclude: $|u''_m(t)|$ is bounded by a constant in $[0, T]$ and independent of t and m. Likewise, by the continuous immersion of $L^2(\Omega) \hookrightarrow H^{-1}(\Omega)$ we obtain a bounded sequence (u''_m) in the space of vector distributions

$$L^2\left(0, T; H^{-1}(\Omega)\right). \tag{3.21}$$

Stage 3 - Convergence of the approximate solutions

In the bounds obtained in (3.14), (3.15), (3.16), (3.17) and (3.21), it follows that there are subsequences such that:

$$u_m \rightarrow u \tag{3.22}$$

weak star in $L^\infty\left(0, T; H_0^1(\Omega)\right)$,

$$u'_m \to u' \tag{3.23}$$

weak star in $L^\infty(0,T;L^2(\Omega))$,

$$u''_m \to u'' \tag{3.24}$$

weak in $L^2(0,T;H^{-1}(\Omega))$,

$$u_m(T) \to u(T) \tag{3.25}$$

weak in $H_0^1(\Omega)$,

$$u'_m(T) \to u'(T) \tag{3.26}$$

weak in $L^2(\Omega)$.

Also, since the immersion of $H_0^1(\Omega)$ in $L^2(\Omega)$ is compact and by Rellich's Theorem [8], the relation (3.26) implies that: $u_m(T) \to u(T)$ weak in $L^2(\Omega)$.

Now multiplying the approximate equation (3.7) by $\theta \in D(0,T)$ and integrating from 0 to T, we obtain:

$$\int_0^T (u''_m(t), v)\,\theta(t)\,dt + \int_0^T ((u_m(\mathrm{t}), v))\theta(t)\,dt + \int_0^T ((u'_m(t), v))\theta(t)\,dt = \int_0^T (\mathrm{f}(t), v)\,\theta(t)\,dt \tag{3.27}$$

for all $v \in V_m$ and all $\theta \in D(0,T)$.

Deriving in the sense of the distributions, the first and third integral of equation (3.27) and then, passing to the limit when $m \to +\infty$ are obtained:

$$\int_0^T -(u'(t), v)\,\theta'(t)\,dt + \int_0^T ((u(t), v))\theta(t)\,dt - \int_0^T ((u(t), v))\theta'(t)\,dt = \int_0^T (f(t), v)\,\theta(t)\,dt, \tag{3.28}$$

for all $v \in V_m$ and all $\theta \in D(0,T)$.

Again, applying the distributional derivative in (3.28), we have:

$$\int_0^T \frac{d}{dt}\left(u'(t), v\right)\theta(t)\,dt + \int_0^T ((u(t), v))\theta(t)\,\mathrm{dt} + \int_0^T \frac{d}{dt}((u(t), \mathrm{v}))\theta(\mathrm{t})\,\mathrm{dt}$$

$$= \int_0^T (f(t), v)\,\theta(t)\,dt \tag{3.29}$$

for all $v \in V_m$ and all $\theta \in D(0,T)$.

Then equation (3.29) can be put in the following form:

$\frac{d}{dt}(u'(t), v) + ((u(t), v)) + \frac{d}{dt}((u(t), v)) = (f(t), v)$, for all $v \in H_0^1(\Omega)$, in the sense of $D'(0,T)$.

Stage 4 - Initial conditions

Once again, we present two tests:

1. $u(0) = u_0$

Indeed, from (3.22) and (3.23) we have that:

$$u \in L^\infty\left(0,T; H_0^1(\Omega)\right), u' \in L^\infty\left(0,T; L^2(\Omega)\right).$$

Then $u \in C^0([0,T]; L^2(\Omega))$ and therefore it makes sense to calculate $u(0)$.

$$\int_0^T ((u_m(t), z(t)))dt \to \int_0^T ((u(t), z(t)))dt, \text{ for all } z \in L^1\left(0,T; H^{-1}(\Omega)\right).$$

Taking in particular $z \in L^1(0,T; H_0^1(\Omega))$, the convergence becomes:

$$\int_0^T ((u_m(t), z(t)))dt \to \int_0^T ((u(t), z(t)))dt, \tag{3.30}$$

for all $z \in L^1\left(0,T; H_0^1(\Omega)\right)$.

Let us now consider $\theta \in C^1([0,T]; \mathbf{R})$, with $\theta(0) = 1$ and $\theta(T) = 0$. The $\theta' \in C^0([0,T]; \mathbf{R})$.

Then, taking $z(t) = v\theta'(t)$, we have from (3.30) that:

$$\int_0^T (u_m(t), v)\,\theta'(t)\,dt \;\to\; \int_0^T (u(t), v)\,\theta'(t)\,dt, \tag{3.31}$$

for all $v \in H_0^1(\Omega)$, all $\theta \in C^1([0,T];\mathbf{R})$.

Using the method of integration by parts in the first member of (3.31) and applying the limit when $m \to +\infty$, it results:

$$\int_0^T (u_m(t), v)\,\theta'(t)\,dt = -(u_0, v) - \int_0^T \frac{d}{dt}(u(t), v)\,\theta(t)\,dt. \tag{3.32}$$

Again, integrating by parts the second member of (3.31) implies that:

$$\int_0^T (u(t), v)\,\theta'(t)\,dt = -\;(u(0), v) - \int_0^T \frac{d}{dt}(u(t), v)\,\theta(t)\,dt. \tag{3.33}$$

Then, from (3.32) and (3.33) in (3.31), we obtain:
$(u(0), \mathrm{v}) = (u_0, \mathrm{v})$ for all $v \in H_0^1(\Omega)$, proving that $u(0) = u_0$.

2. $u'(0) = u_1$

Indeed, from (3.24) and (3.25) we have that:

$u' \in L^\infty(0,T;L^2(\Omega))$, $u'' \in L^2(0,T;H^{-1}(\Omega))$. Therefore, $u' \in C^0([0,T];H^{-1}(\Omega))$ and therefore it makes sense to calculate $u'(0)$.

Let us consider $\theta \in C^1([0,T];\mathbf{R})$, with $\theta(0) = 1$ and $\theta(T) = 0$. Multiplied in (3.7) by θ and integrating by parts and applying the limit when $m \to +\infty$ is obtained:

$$-(u_1, v) - \int_0^T (u'(t), v)\,\theta'(t)\,dt + \int_0^T ((u(t), v))\theta(t)\,dt + \int_0^T \frac{d}{dt}((u(t), v))\theta(t)\,dt$$

$$= \int_0^T (f(t), v)\,\theta(t)\,dt, \tag{3.34}$$

for all $\theta \in C^1([0,T];\mathbf{R})$ with $\theta(0) = 1$ and $\theta(T) = 0$, and for all $v \in H_0^1(\Omega)$.

On the other hand, being u solution of (3.7) and multiplying by $\theta \in C^1([0,T];\mathbf{R})$, with $\theta(0) = 1$ and $\theta(T) = 0$, and integrated by parts, we obtain:

$$-(u'(0),v) - \int_0^T (u'(t),v)\,\theta'(t)\,dt + \int_0^T ((u(t),v))\theta(t)\,dt$$
$$+\int_0^T \frac{d}{dt}((u(t),v))\theta(t)\,dt = \int_0^T (f(t),v)\,\theta(t)\,dt. \tag{3.35}$$

It follows from (3.34) and equation (3.35) that:

$(u_1, \mathrm{v}) = (u'(0), \mathrm{v})$ for all $v \in H_0^1(\Omega)$, proving that $u_1 = u'(0)$. □

3.2. Uniqueness of the generalized solution of a non-homogeneous hyperbolic differential equation

Corollary 3.2.1. (Uniqueness of the generalized solution)

If u_2 and u_3 are solutions of the existence theorem that satisfy the condition:

$$-(u_1,v) - \int_0^T (u'(t),v)\,\theta'(t)\,dt + \int_0^T ((u(t),v))\theta(t)\,dt$$
$$+\int_0^T \frac{d}{dt}((u(t),v))\theta(t)\,dt = \int_0^T (f(t),v)\,\theta(t)\,dt, \tag{3.36}$$

for all $\theta \in C^1([0,T];\mathbf{R})$, $\theta(0) = 1$, $\theta(T) = 0$, for all $\theta \in H_0^1(\Omega)$. Then $u_2 = u_3$.

Proof. Assuming that u_2 and u_3 satisfy (3.36) and subtracting, we obtain:

$$-\int_0^{\mathrm{T}} (u_2'(t) - u_3'(t), v\theta'(t))\,dt + \int_0^T ((u_2(t) - u_3(t), v\theta(t)))dt$$
$$+\int_0^T \frac{d}{dt}((u_2(t) - u_3(t), v\theta(t)))dt = 0. \tag{3.37}$$

Then, if u_2 and u_3 are solutions of the wave existence theorem, it follows that $w(t) = u_2(t) - u_3(t)$ satisfies equation (3.37), i.e. :

$$-\int_0^T (w'(t), v\theta'(t))\, dt + \int_0^T ((w(t), v\theta(t)))\, dt + \int_0^T \frac{d}{dt}((w(t), v\theta(t)))\, dt = 0. \tag{3.38}$$

Let $z(t) = \theta(t)v$ be a function of $L^1(0, T; H_0^1(\Omega))$.

Replacing in (3.38) we obtain:

$$-\int_0^T (w'(\mathrm{t}), z'(t))\, dt + \int_0^T ((w(t), z(t)))\, dt + \int_0^T \frac{d}{dt}((\mathrm{w}(t), z(t)))dt = 0, \tag{3.39}$$

for all $z \in L^1(0, T; H_0^1(\Omega))$.

Let w be a solution under the conditions of the wave existence theorem with $w(0) = w'(0) = 0$. We have that $w \in L^1(0, T; H_0^1(\Omega))$, then $z(t)$, the integral in $H_0^1(\Omega)$ is a function of $C^1([0, T]; H_0^1(\Omega))$, then $z'(t) = w(t)$. Substituting z in (3.39) we obtain:

$$-\int_0^s (w'(t), w(t))\, dt + \int_0^s ((z'(t), z(t)))dt + \int_0^s \frac{d}{dt}((z'(t), z(t)))dt = 0,$$

where $z(s) = 0$, $w(0) = 0$, it turns out that:

$$\frac{1}{2}|w(s)|^2 + \|z(0)\|^2 = 0.$$

As $\|z(0)\| \geq 0$, we obtain $|w(s)| = 0$ for all s in $[0, T]$, proving: $u_2 = u_3$. $\square$

4. Conclusions

With the theory of distributions, Sobolev spaces and the Faedo-Garleskin - Medeiros method, the existence and uniqueness of weak or generalized solutions $u(x,t)$ defined on Q satisfying the hyperbolic problem (2.1) modeling the vibrations of an elastic bar with dissipation was proved.

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