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Minimal connected restrained monophonic sets in graphs

A. P. Santhakumaran,
Hindustan Institute of Technology and Science, India
P. Titus
University College of Engineering Nagercoil, India
and
K. Ganesamoorthy *
Coimbatore Institute of Technology, India
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Abstract

For a connected graph $G = (V, E)$ of order at least two, a connected restrained monophonic set S of G is a restrained monophonic set such that the subgraph $G[S]$ induced by S is connected. The minimum cardinality of a connected restrained monophonic set of G is the connected restrained monophonic number of G and is denoted by $m_{cr}(G)$. A connected restrained monophonic set S of G is called a minimal connected restrained monophonic set if no proper subset of S is a connected restrained monophonic set of G. The upper connected restrained monophonic number of G, denoted by $m^+_{cr}(G)$, is defined as the maximum cardinality of a minimal connected restrained monophonic set of G. We determine bounds for it and certain general properties satisfied by this parameter are studied. It is shown that, for positive integers a, b such that $4 \leq a \leq b$, there exists a connected graph G such that $m_{cr}(G) = a$ and $m^+_{cr}(G) = b$.

Key Words: *restrained monophonic set, restrained monophonic number, connected restrained monophonic set, connected restrained monophonic number, minimal connected restrained monophonic set.*

AMS Subject Classification: *05C12.*

1. Introduction

By a graph $G = (V, E)$ we mean a finite undirected connected graph without loops or multiple edges. The order and size of G are denoted by p and q, respectively. For basic graph theoretic terminology we refer to Harary [9]. The *distance* $d(x, y)$ between two vertices x and y in a connected graph G is the length of a shortest $x-y$ path in G. An $x-y$ path of length $d(x, y)$ is called an *$x - y$ geodesic* [1]. The *neighborhood* of a vertex v is the set $N(v)$ consisting of all vertices u which are adjacent with v. A vertex v is an *extreme vertex* if the subgraph induced by its neighbors is complete.

A *chord* of a path P is an edge joining two non-adjacent vertices of P. A path P is called a *monophonic path* if it is a chordless path. A set S of vertices of G is a *monophonic set* of G if each vertex v of G lies on an $x - y$ monophonic path for some x and y in S. The minimum cardinality of a monophonic set of G is the *monophonic number* of G and is denoted by $m(G)$, the monophonic number of a graph G and its related concepts have been studied by several authors [2, 3, 4, 5, 6, 7, 8, 10, 13, 16, 17]. A *restrained monophonic set* S of a graph G is a *monophonic set* such that either $S = V$ or the subgraph induced by $V - S$ has no isolated vertices. The minimum cardinality of a restrained monophonic set of G is the *restrained monophonic number* of G and is denoted by $m_r(G)$. The restrained monophonic number of a graph was introduced and studied in [14]. A *connected restrained monophonic set* S of G is a restrained monophonic set such that the subgraph $G[S]$ induced by S is connected. The minimum cardinality of a connected restrained monophonic set of G is the *connected restrained monophonic number* of G and is denoted by $m_{cr}(G)$. The connected restrained monophonic number of a graph was introduced and studied in [15].

For any two vertices u and v in a connected graph G, the *monophonic distance* $d_m(u, v)$ from u to v is defined as the length of a longest $u - v$ monophonic path in G. The *monophonic eccentricity* $e_m(v)$ of a vertex v in G is $e_m(v) = \max\ \{d_m(v, u) : u \in V(G)\}$. The *monophonic radius*, $rad_m(G)$ of G is $rad_m(G) = \min\ \{e_m(v) : v \in V(G)\}$ and the *monophonic diameter*, $diam_m(G)$ of G is $diam_m(G) = \max\ \{e_m(v) : v \in V(G)\}$. The monophonic distance was introduced and studied in [11, 12].

The following theorems will be used in the sequel.

Theorem 1.1. [15] Each extreme vertex of a connected graph G belongs to every connected restrained monophonic set of G.

Theorem 1.2. [15] Every cut-vertex of a connected graph G belongs to every connected restrained monophonic set of G.

Theorem 1.3. [15] For any non-trivial tree T of order p, $m_{cr}(T) = p$.

Throughout this paper G denotes a connected graph with at least two vertices.

2. Upper Connected Restrained Monophonic Number

Definition 2.1. *A connected restrained monophonic set S of G is called a minimal connected restrained monophonic set if no proper subset of S is a connected restrained monophonic set of G. The upper connected restrained monophonic number of G, denoted by $m^+_{cr}(G)$, is defined as the maximum cardinality of a minimal connected restrained monophonic set of G.*

Example 2.2. *For the graph G given in Figure 2.1, the minimal connected restrained monophonic sets are $S_1 = \{v_1, v_2, v_3\}$, $S_2 = \{v_2, v_3, v_4\}$ and $S_3 = \{v_1, v_4, v_5, v_6\}$. Hence the connected restrained monophonic number of G is 3 and the upper connected restrained monophonic number of G is 4. Thus the connected restrained monophonic number and the upper connected restrained monophonic number of a graph G are different.*

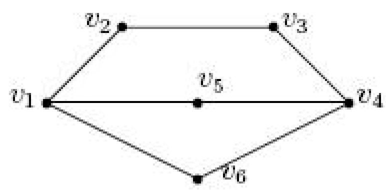

Figure 2.1: G

Every minimum connected restrained monophonic set of G is a minimal connected restrained monophonic set of G, but the converse need not be true. For the graph G given in Figure 2.1, S_3 is a minimal connected restrained monophonic set but it is not a minimum connected restrained monophonic set of G.

Theorem 2.3. *Each extreme vertex of a connected graph G belongs to every minimal connected restrained monophonic set of G.*

Proof. This follows from Theorem 1.1. $\square$

Corollary 2.4. *For the complete graph K_p, $m^+_{cr}(K_p) = p$.*

Theorem 2.5. *Let G be a connected graph with cut-vertices and let S be a minimal connected restrained monophonic set of G. If v is a cut-vertex of G, then every component of $G - v$ contains an element of S.*

Proof. Suppose that there is a component B of $G - v$ such that B contains no vertex of S. Let u be any vertex in B. Since S is a minimal connected restrained monophonic set, there exists a pair of vertices x and y in S such that u lies in some $x - y$ monophonic path $P : x = u_0, u_1, u_2, \cdots, u, \cdots, u_n = y$ in G with $u \neq x, y$. Since v is a cut-vertex of G, the $x - u$ subpath P_1 of P and the $u - y$ subpath P_2 of P both contain v, it follows that P is not a path, which is a contradiction. $\square$

Theorem 2.6. *Every cut-vertex of a connected graph G belongs to every minimal connected restrained monophonic set of G.*

Proof. This follows from Theorem 1.2. $\square$

Corollary 2.7. *For any non-trivial tree T of order p, $m^+_{cr}(T) = p$.*

Theorem 2.8. *For any connected graph G of order $p \geq 2$, $2 \leq m_{cr}(G) \leq m^+_{cr}(G) \leq p, m_{cr}(G) \neq p - 1, m^+_{cr}(G) \neq p - 1$.*

Proof. Any connected restrained monophonic set needs at least two vertices and so $m_{cr}(G) \geq 2$. Since every minimal connected restrained monophonic set of G is also a connected restrained monophonic set of G, it follows that $m_{cr}(G) \leq m^+_{cr}(G)$. It is clear that $V(G)$ induces a connected restrained monophonic set of G and $V(G)-\{z\}$ is not a connected restrained monophonic set of G for any vertex z in G. Hence $m^+_{cr}(G) \leq p$, $m_{cr}(G) \neq p - 1$ and $m^+_{cr}(G) \neq p - 1$. $\square$

The bounds in Theorem 2.8 are sharp. For the complete graph K_2, $m_{cr}(K_2) = m^+_{cr}(K_2) = 2$ and if G is a non-trivial tree of order p, then $m_{cr}(G) = m^+_{cr}(G) = p$. All the inequalities in Theorem 2.8 are strict. For graph G given in Figure 2.1, $m_{cr}(G) = 3$, $m^+_{cr}(G) = 4$ and $p = 6$. Thus we have $2 < m_{cr}(G) < m^+_{cr}(G) < p$.

Now we proceed to characterize graphs G for which the lower bound in Theorem 2.8 is attained.

Theorem 2.9. *Let G be a connected graph of order $p \geq 2$. Then $G = K_2$ if and only if $m^+_{cr}(G) = 2$.*

Proof. If $G = K_2$, then by Corollary 2.4, we have $m^+_{cr}(G) = 2$. Conversely, let $m^+_{cr}(G) = 2$. Let $S = \{u, v\}$ be a minimal connected restrained monophonic set of G. Then uv is an edge. If $G \neq K_2$, there exists a vertex w different from u and v. Since uv is an edge, w can not lie on any $u - v$ monophonic path and so S is not a connected restrained monophonic set, which is a contradiction. Thus $G = K_2$. $\square$

Theorem 2.10. *Let G be a connected graph with every vertex of G is either a cut-vertex or an extreme vertex. Then $m^+_{cr}(G) = p$.*

Proof. Let G be a connected graph with every vertex of G is either a cut-vertex or an extreme vertex. Then by Theorems 2.3 and 2.6, we have $m^+_{cr}(G) = p$. $\square$

The converse of Theorem 2.10 need not be true. For the graph G given in Figure 2.2, $m^+_{cr}(G) = 6 = p$, but the vertices v_3 and v_4 are neither cut-vertices nor extreme vertices of G.

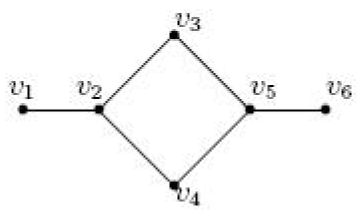

Figure 2.2: G

Theorem 2.11. *For a connected graph G, $m^+_{cr}(G) = p$ if and only if $m_{cr}(G) = p$.*

Proof. Let $m^+_{cr}(G) = p$. Then $S = V(G)$ is the unique minimal connected restrained monophonic set of G. Since no proper subset of S is a connected restrained monophonic set of G, it is clear that S is the unique minimum connected restrained monophonic set of G and so $m_{cr}(G) = p$. The converse follows from Theorem 2.8. $\square$

Theorem 2.12. *Let G be a connected graph of order $p \geq 2$. If $m_{cr}(G) = p - 2$ then $m^+_{cr}(G) = p - 2$.*

Proof. If $m_{cr}(G) = p - 2$, it follows from Theorem 2.8 that $m^+_{cr}(G) = p - 2$ or $m^+_{cr}(G) = p$. If $m^+_{cr}(G) = p$, then by Theorem 2.11, $m_{cr}(G) = p$, which is a contradiction. Hence $m^+_{cr}(G) = p - 2$. □

The converse of Theorem 2.12 need not be true. For the graph G given in Figure 2.1, the upper connected restrained monophonic number of G is $m^+_{cr}(G) = 4 = p - 2$ and the connected restrained monophonic number of G is $m_{cr}(G) = 3 \neq p - 2$.

We leave the following problem as an open question.

Problem 2.13. *Characterize graphs G for which $m_{cr}(G) = m^+_{cr}(G)$.*

3. Realization results for $m^+_{cr}(G)$

In view of Theorem 2.8, we have the following realization theorem.

Theorem 3.1. *For every pair a, b of positive integers with $4 \leq a \leq b$, there is a connected graph G with $m_{cr}(G) = a$ and $m^+_{cr}(G) = b$.*

Proof. We prove this theorem by considering two cases.

Case 1. $4 \leq a = b$. Let G be any tree with a vertices. Then by Theorem 1.3 and Corollary 2.7, G has the desired property.

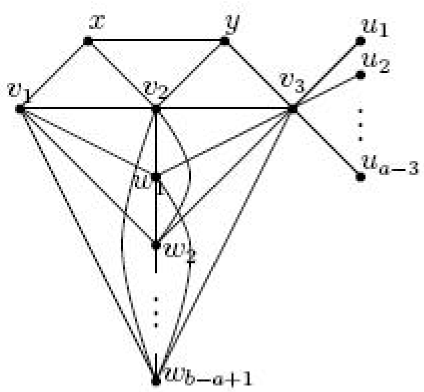

Figure 3.1: G

Case 2. $4 \leq a < b$. Let H be the graph obtained from the path P_3 : v_1, v_2, v_3 of order 3 by adding $b-2$ new vertices $w_1, w_2, ..., w_{b-a+1}, u_1, u_2, \cdots, u_{a-3}$

and joining $w_i(1 \leq i \leq b-a+1)$ to the vertices v_1, v_2 and v_3; joining $u_j(1 \leq j \leq a-3)$ to the vertex v_3; and also joining each $w_i(1 \leq i \leq b-a)$ with $w_j(i+1 \leq j \leq b-a+1)$. The graph G is obtained from H and the path $P_2 : x, y$ of order 2 by joining the vertex x to the vertices v_1 and v_2; also joining the vertex y to the vertices v_2 and v_3, which is shown in Figure 3.1. Let $S = \{u_1, u_2, \cdots, u_{a-3}, v_3\}$ be the set of all extreme vertices and cut-vertex of G. By Theorems 1.1, 1.2, 2.3 and 2.6, every connected restrained monophonic set and every minimal connected restrained monophonic set of G contain S. Clearly, S is not a connected restrained monophonic set of G. Also, for any vertex $v \in V(G) - S$, $S_1 = S \cup \{v\}$ is not a connected restrained monophonic set of G. Let $S_2 = S \cup \{v_1, v_2\}$. It is easily verified that S_2 is a connected restrained monophonic set of G and so $m_{cr}(G) = a$.

Next we show that $m^+_{cr}(G) = b$. Clearly $T = S \cup \{y, w_1, w_2, \ldots, w_{b-a+1}\}$ is a connected restrained monophonic set of G. We claim that T is a minimal connected restrained monophonic set of G. Let W be any proper subset of T. Then there exists a vertex, say v, such that $v \in T$ and $v \notin W$. By Theorems 2.3 and 2.6, $v \in \{y, w_1, w_2, \ldots, w_{b-a+1}\}$. It is easily verified that v is not an internal vertex of any $x - y$ monophonic path for some $x, y \in W$, it follows that W is not a connected restrained monophonic set of G. Hence T is a minimal connected restrained monophonic set of G and so $m^+_{cr}(G) \geq b$. Suppose that $m^+_{cr}(G) > b$. Let M be a minimal connected restrained monophonic set of G with $|M| > b$. Then there exists at least one vertex, say, $v \in M$ such that $v \notin T$. Thus $v \in \{v_1, v_2, x\}$. If $v = v_1$, then $M_1 = S \cup \{v_1, w_1\}$ is a connected restrained monophonic set of G and also it is a proper subset of M, which is a contradiction to M a minimal connected restrained monophonic set of G. If $v = v_2$, then $M_2 = S \cup \{v_2, w_1, y\}$ is a connected restrained monophonic set of G and also it is a proper subset of M, which is a contradiction to M a minimal connected restrained monophonic set of G. If $v = x$, then $M_3 = S \cup \{x, y\}$ is a connected restrained monophonic set of G and also it is a proper subset of M, which is a contradiction to M a minimal connected restrained monophonic set of G. Hence $m^+_{cr}(G) = b$. □

Theorem 3.2. *If p, d and k are positive integers such that $2 \leq d \leq p-2$, $k \geq 4$, $k \neq p-1$ and $p-d-k \geq 0$, then there exists a connected graph G of order p, monophonic diameter d and $m^+_{cr}(G) = k$.*

Proof. We prove this theorem by considering three cases.

Case 1. $d = 2$ and $k \geq 4$. Let $P_3 : x, y, z$ be a path of order 3. Let G be the graph obtained by adding $p-3$ new vertices $v_1, v_2, \ldots, v_{p-k}, w_1, w_2, \ldots, w_{k-3}$ to P_3 and joining each $w_i (1 \leq i \leq k-3)$ to y; and joining each $v_i(1 \leq i \leq p-k)$ with x, y and z; and joining each $v_i(1 \leq i \leq p-k-1)$ with $v_j(i+1 \leq j \leq p-k)$. The graph G is shown in Figure 3.2. Then G has order p and monophonic diameter $d = 2$. Let $S = \{w_1, w_2, w_3, \ldots, w_{k-3}, x, z, y\}$ be the set of all extreme vertices and cut-vertex of G. By Theorems 2.3 and 2.6, every minimal connected restrained monophonic set of G contains S. It is easily verified that S is the unique minimal connected restrained monophonic set of G and so $m^+_{cr}(G) = k$.

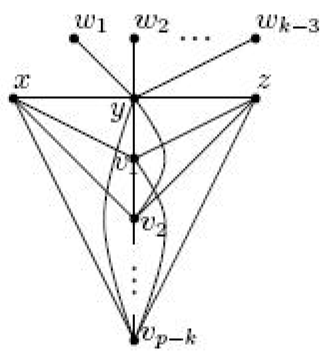

Figure 3.2: G

Case 2. $d = 3$ and $k \geq 4$. Let $P_3 : x, y, z$ be a path of order 3. Let G be the graph obtained by adding $p-3$ new vertices $v_1, v_2, \ldots, v_{p-k}, w_1, w_2, \ldots, w_{k-3}$ to P_3 and joining each $w_i(1 \leq i \leq k-3)$ to z; and joining each $v_i(1 \leq i \leq p-k)$ with x, y and z; and joining each $v_i(1 \leq i \leq p-k-1)$ with $v_j(i+1 \leq j \leq p-k)$. The graph G is shown in Figure 3.3. Then G has order p and monophonic diameter $d = 3$. Let $S = \{w_1, w_2, w_3, \ldots, w_{k-3}, x, z\}$ be the set of all extreme vertices and cut-vertex of G. By Theorems 2.3 and 2.6, every minimal connected restrained monophonic set of G contains S. Clearly, S is not a connected restrained monophonic set of G. It is easy to observe that, $S \cup \{y\}$ and $S \cup \{v_i\}$ $(1 \leq i \leq p-k)$ are the minimal connected restrained monophonic sets of G each of cardinality k and so $m^+_{cr}(G) = k$.

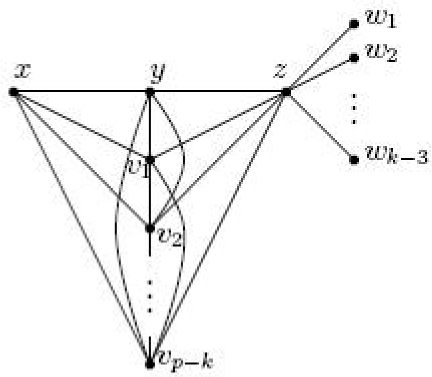

Figure 3.3: G

Case 3. $4 \leq d \leq p-2$ and $k \geq 4$. Let $C_{d+1} : v_1, v_2, ..., v_{d+1}, v_1$ be the cycle of order $d+1$. The required graph G is obtained from C_{d+1} by adding $p-d-1$ new vertices $w_1, w_2, \ldots, w_{k-2}, u_1, u_2, \ldots, u_{p-d-k+1}$ and joining each vertex $w_i (1 \leq i \leq k-2)$ to both v_1 and v_2; and also joining each vertex $u_j (1 \leq j \leq p-d-k+1)$ to both v_3 and v_5. The graph G is shown in Figure 3.4. Then G has order p and monophonic diameter d. Let $S = \{w_1, w_2, \ldots, w_{k-2}\}$ be the set of all extreme vertices of G. Then by Theorem 2.3, S is contained in every minimal connected restrained monophonic set of G. It is clear that $S_1 = S \cup \{v_2, v_3\}$ and $S_2 = S \cup \{v_1, v_{d+1}\}$ are the only two minimal connected restrained monophonic sets of G and so $m^+_{cr}(G) = k$. □

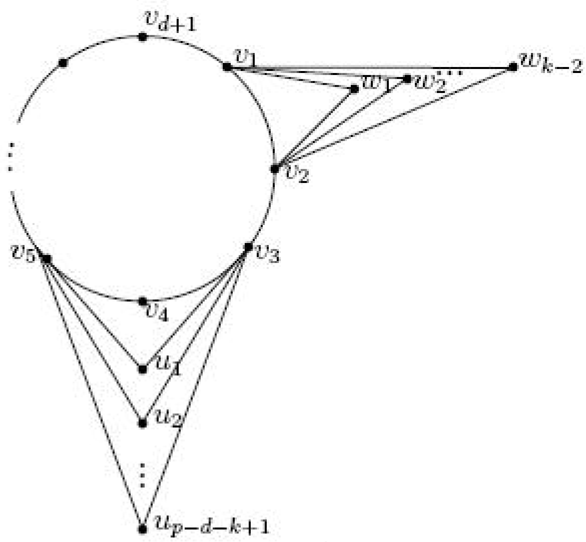

Figure 3.4: G

Acknowledgments

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A. P. Santhakumaran
Department of Mathematics
Hindustan Institute of Technology and Science
Chennai-603 103,
India
e-mail: apskumar1953@gmail.com

P. Titus
Department of Mathematics,
University College of Engineering Nagercoil
Anna University,
Tirunelveli Region
Nagercoil-629 004,
India
e-mail: titusvino@yahoo.com

and

K. Ganesamoorthy
Department of Mathematics,
Coimbatore Institute of Technology
Coimbatore-641 014,
India
e-mail: kvgm_2005@yahoo.co.in
Corresponding author