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λ-quasi Cauchy sequence of fuzzy numbers

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Abstract:

In this paper we introduce the λ-quasi Cauchy sequence of fuzzy numbers. We obtain the relation between strongly λ-quasi Cauchy convergence and statistically λ-quasi Cauchy convergence for fuzzy numbers.

Keywords: Fuzzy number; Quasi Cauchy sequence; Statistical convergence; Solidness; Symmetricity and convergence free.

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1. Introduction

Fuzzy set was first suggested by Zadeh [24]. Bounded and convergent sequences of fuzzy numbers were studied by Matloka [17], where it is shown that every convergent sequence is bounded. Later on different classes of sequences of fuzzy numbers have been studied by Ganie and Sheikh [2] and many others. The idea of statistical convergence for single sequences was introduced by Fast [10] in 1951. Schoenberg [15], studied the statistical convergence as a summability method and listed some of elementary properties of statistical convergence. Both of these authors noted that if bounded sequence is statistically convergent, then it is Cesàro summable. Mursaleen [16] defined and studied λ-statistically convergent sequences (also see [1, 14]). The notion was further investigated and different properties in the field of summability theory has been investigated by Tripathy [18], Tripathy and Sen [19], Tripathy and Esi [20], Tripathy and Das [23], and many others.

2. Preliminaries

A fuzzy real number X is fuzzy set on $\mathbf{R}$ and is a mapping $X : \mathbf{R} \rightarrow I(= [0, 1])$ associating each real number t with its grade membership $X(t)$. For the definition of normal, upper-semi continuous and convex fuzzy real numbers denoted by $\mathbf{R}(I)$ (one may refer Tripathy and Baruah [21]).

Throughout the article $w^F, c^F, c_0^F, \ell_\infty^F$ denote the classes of all, convergent, null, bounded sequences of fuzzy real numbers, respectively.

We denote D, the set of all the closed and bounded intervals on the real line $\mathbf{R}$. For any $X, Y \in D$ we define

$$d(X, Y) = \max(|a_1 - a_2|, |b_1 - b_2|) \text{ where } X = [a_1, b_1], Y = [a_2, b_2].$$

Clearly (D, d) is a complete metric space.

Let $\bar{d} : \mathbf{R}(I) \times \mathbf{R}(I) \rightarrow \mathbf{R}$ be defined by

$$\bar{d}(X, Y) = \sup_{0 \leq \alpha \leq 1} d(X^\alpha, Y^\alpha).$$

Then $\bar{d}$ is a metric on $\mathbf{R}(I)$ (One may refer to Matloka [17]). The additive identity and the multiplicative identity in $\mathbf{R}(I)$ are denoted by $\bar{0}$ and $\bar{1}$, respectively.

A fuzzy real valued sequence (X_k) is said to be convergent to a fuzzy real number X_0, if for every given $\varepsilon > 0$, there exists $n_0 \in \mathbf{N}$ such that $\bar{d}(X_k, X_0) < \varepsilon$ for all $k \geq n_0$.

A fuzzy real valued sequence (X_k) is said to be Cauchy sequence if for a given $\varepsilon > 0$, there exists $n_0 \in \mathbf{N}$ such that $\bar{d}(X_k, X_n) < \varepsilon$ for all $k, n \geq n_0$.

For $X = (X_k)$, a given sequence $S(X)$ denotes the set of all permutation of the elements of (X_k) that is $S(X) = (X_{\pi(k)})$.

A sequence space E^F is said to be solid if $(Y_k) \in E^F$ whenever $|Y_k| \leq |X_k|$ for all $k \in \mathbf{N}$ and $(X_k) \in E^F$.

A sequence space E^F is said to be sequence algebra if $(X_k Y_k) \in E^F$ whenever (X_k) and (Y_k) in E^F.

A sequence space E^F is said to be convergence free if $(Y_k) \in E^F$ whenever $(X_k) \in E^F$ and $X_k = 0$ implies $Y_k = 0$.

Definition 2.1. A sequence $X = (X_k)$ of fuzzy numbers is said to be convergent to the fuzzy number X_0 if for each $\varepsilon > 0$ there exists a positive integer k_0 such that $\bar{d}(X_k, X_0) < \varepsilon$ for all $k \geq k_0$ and we denote it by $\lim_k X_k = X_0$. Thus, $\lim_k X_k = X_0 \iff \lim_l X_{k_l} = X_{0_l}$ and $\lim_r X_{k_r} = X_{0_r}$.

In the following way Burton and Coleman [3] defined quasi Cauchy sequence.

A sequence $X = (X_k)$ of points in $\mathbf{R}$ is called quasi Cauchy if Δx_k is a null sequence, where $\Delta x_k = x_k - x_{k+1}$.

For quasi-Cauchy sequences one can see [4, 5, 6, 8, 7, 9]. We refer ([11, 12, 13, 22] and references therein) for difference sequence of fuzzy numbers.

In section 3, we introduce and discuss the concept of strongly λ-quasi Cauchy convergence and statistically λ-quasi Cauchy convergence of fuzzy numbers.

3. λ-quasi Cauchy convergence of fuzzy numbers

Definition 3.1. Let $\lambda = \{\lambda_n\}$ be a non-decreasing sequence of positive numbers such that $\lambda_{n+1} \leq \lambda_n + 1, \lambda_1 = 1, \lambda_n \to \infty$ as $n \to \infty$ and $I_n = [n - \lambda_n + 1, n]$. A sequence $X = (X_k)$ of fuzzy numbers is said to be strongly λ-quasi Cauchy summable if there is an fuzzy number X_0 such that

$$\lim_n \frac{1}{\lambda_n} \sum_{k \in I_n} \bar{d}(\Delta X_k, X_0) = 0.$$

In which case we say that the sequence $X = (X_k)$ of fuzzy numbers is said to be strongly λ-quasi Cauchy summable to fuzzy number X_0 and it is denoted by $w^F(\lambda)$.

Definition 3.2. A sequence $X = (X_k)$ of fuzzy numbers is said to be statistically λ-quasi Cauchy convergent to fuzzy number X_0 if for every $\varepsilon > 0$

$$\lim_k \frac{1}{\lambda_n}|\{k \in I_n : \bar{d}(\Delta X_k, X_0) \geq \varepsilon\}| = 0.$$

In this case we write $S_\lambda - \lim \Delta X_k = X_0$.

Theorem 3.3. Let $X = (X_k)$ and $Y = \{Y_k\}$ be two sequences of fuzzy numbers.

(i) If $S_\lambda - \lim \Delta X_k = X_0$ and $\alpha \in \mathbf{R}$, then $S_\lambda - \lim \alpha \Delta X_k = \alpha X_0$.

(ii) If $S_\lambda - \lim \Delta X_k = X_0$ and $S_\lambda - \lim \Delta Y_k = Y_0$. Then $S_\lambda - \lim(\Delta X_k + \Delta Y_k) = X_0 + Y_0$.

Proof. (i) Let $\alpha \in \mathbf{R}$. We have $\bar{d}(\alpha \Delta X_k, \alpha X_0) = |\alpha| \bar{d}(\Delta X_k, X_0)$. For a given $\varepsilon > 0$ we have

$$\frac{1}{\lambda_n}|\{k \in I_n : \bar{d}(\alpha \Delta X_k, \alpha X_0) \geq \varepsilon\}| \leq \frac{1}{\lambda_n}\left|\left\{k \in I_n : \bar{d}(\Delta X_k, X_0) \geq \frac{\varepsilon}{|\alpha|}\right\}\right|.$$

Hence $S_\lambda - \lim \alpha \Delta X_k = \alpha X_0$.
(ii) Suppose $S_\lambda - \lim \Delta X_k = X_0$ and $S_\lambda - \lim \Delta Y_k = Y_0$. We have

$$\bar{d}(\Delta X_k + \Delta Y_k, X_0 + Y_0) \leq \bar{d}(\Delta X_k, X_0) + \bar{d}(\Delta Y_k, Y_0).$$

Therefore, for given $\varepsilon > 0$ we have

$$\begin{aligned} &\tfrac{1}{\lambda_n}\left|\{k \in I_n : \bar{d}(\Delta X_k + \Delta Y_k, X_0 + Y_0) \geq \varepsilon\}\right| \\ &\leq \tfrac{1}{\lambda_n}\left|\{k \in I_n : \bar{d}(\Delta X_k, X_0) + \bar{d}(\Delta Y_k, Y_0) \geq \varepsilon\}\right| \\ &\leq \tfrac{1}{\lambda_n}\left|\{k \in I_n : \bar{d}(\Delta X_k, X_0) \geq \tfrac{\varepsilon}{2}\}\right| + \tfrac{1}{\lambda_n}\left|\{k \in I_n : \bar{d}(\Delta Y_k, Y_0) \geq \tfrac{\varepsilon}{2}\}\right|. \end{aligned}$$

Hence $S_\lambda - \lim(\Delta X_k + \Delta Y_k) = X_0 + Y_0$. □

In the following theorems, we exhibit some connections between strongly λ-quasi Cauchy summable and statistically λ-quasi Cauchy convergence of sequences of fuzzy numbers.

Theorem 3.4. If a fuzzy sequence $X = (X_k)$ is strongly λ-quasi Cauchy summable to fuzzy number X_0 then it is statistically λ-quasi Cauchy convergent to fuzzy number X_0.

Proof. Let $\varepsilon > 0$, be given. We have the result from the following inequality.
$$\begin{aligned}\frac{1}{\lambda_n}\sum_{k\in I_n}\bar{d}(\Delta X_k, X_0) &\geq \frac{1}{\lambda_n}\sum_{k\in I_n, \bar{d}(\Delta X_k, X_0)\geq\varepsilon}\bar{d}(\Delta X_k, X_0)\\ &\geq \frac{1}{\lambda_n}\left|\{k \in I_n : \bar{d}(\Delta Y_k, Y_0) \geq \varepsilon\}\right|.\end{aligned}$$

Hence the result follows. to fuzzy number X_0 then it is statistically λ-quasi Cauchy convergent to fuzzy number X_0. □

Theorem 3.5. If a fuzzy sequence $X = (X_k)$ is statistically quasi Cauchy convergent to fuzzy number X_0 and $\liminf_n \frac{\lambda_n}{n} > 0$, then it is statistically λ-quasi Cauchy convergent to X_0.

Proof. For given $\varepsilon > 0$ we have

$$\{k \leq n : \bar{d}(\Delta X_k, X_0) \geq \varepsilon\} \supset \{k \in I_n : \bar{d}(\Delta X_k, X_0) \geq \varepsilon\}.$$

Therefore
$$\begin{aligned}&\frac{1}{n}\left|\{k \leq n : \bar{d}(\Delta X_k, X_0) \geq \varepsilon\}\right|\\ &\geq \frac{1}{n}\left|\{k \in I_n : \bar{d}(\Delta X_k, X_0) \geq \varepsilon\}\right|\\ &\geq \frac{\lambda_n}{n}\frac{1}{\lambda_n}\left|\{k \in I_n : \bar{d}(\Delta X_k, X_0) \geq \varepsilon\}\right|.\end{aligned}$$
Taking limits as $n \to \infty$ and using the condition $\liminf_n \frac{\lambda_n}{n} > 0$, we get (X_k) is statistically λ-quasi Cauchy convergent to X_0. □

Theorem 3.6. $w^F(\lambda)$ is a solid space.

Proof. Let (X_k) and $\{Y_k\}$ be sequences in $w^F(\lambda)$ such that $\bar{d}(Y_k, 0) \leq \bar{d}(X_k, 0)$ for each $k \in \mathbf{N}$. Then we get required result from the following inequality
$$\frac{1}{\lambda_n}\sum_{k\in I_n}\bar{d}(Y_k, 0) \leq \frac{1}{\lambda_n}\sum_{k\in I_n}\bar{d}(X_k, 0).$$

□

Theorem 3.7. $w^F(\lambda)$ is not a symmetric space.

Proof. For each $k \in \mathbf{N}$ and let us consider the sequences

$$X = \{A, B, A, B, \ldots\}$$

where
$$A = \begin{cases} t+1, \text{ if } -1 \leq t \leq 0\\ 1-t, \text{ if } 0 \leq t \leq 1\\ 0, \text{ otherwise}\end{cases}$$

and

$$B = \begin{cases} \frac{t+2}{2}, \text{ if } -2 \leq t \leq 0 \\ \frac{2-t}{2}, \text{ if } 0 \leq t \leq 2 \\ 0, \text{ otherwise} \end{cases}$$

Now consider the re-arrangement Y of the sequence X

$$Y = \{A, A, B, B, A, A, B, B, \ldots\} \notin w^F(\lambda).$$

But $X \in w^F(\lambda)$. This completes the proof. □

Theorem 3.8. $w^F(\lambda)$ is not a convergence free space in general.

Proof. We consider the sequences

$$X_k(t) = \begin{cases} kt + 1, \text{ if } -\frac{1}{k} \leq t \leq 0 \\ 1 - kt, \text{ if } 0 \leq t \leq \frac{1}{k} \\ 0, \text{ otherwise} \end{cases}$$

$$X_{k+1}(t) = \begin{cases} (k+1)t + 1, \text{ if } -\frac{1}{k+1} \leq t \leq 0 \\ 1 - (k+1)t, \text{ if } 0 \leq t \leq \frac{1}{k+1} \\ 0, \text{ otherwise} \end{cases}$$

Then

$$\Delta X_k(t) = \begin{cases} \frac{t(k^2+k)+2k+1}{2k+1}, \text{ if } -\frac{1}{k} - \frac{1}{k+1} \leq t \leq 0 \\ \frac{(2k+1)-t(k^2+k)}{2k+1}, \text{ if } 0 \leq t \leq \frac{1}{k} + \frac{1}{k+1} \\ 0, \text{ otherwise} \end{cases}$$

Now $\lim\limits_{k\to\infty} \Delta X_k(t) = 0$. Thus $(X_k) \in w^F(\lambda)$.

Again let

$$Y_k(t) = \begin{cases} \frac{t+k}{k}, \text{ if } -k \leq t \leq 0 \\ \frac{k-t}{k}, \text{ if } 0 \leq t \leq k \\ 0, \text{ otherwise} \end{cases}$$

$$Y_{k+1}(t) = \begin{cases} \frac{t+(k+1)}{k+1}, \text{ if } -(k+1) \leq t \leq 0 \\ \frac{(k+1)-t}{k+1}, \text{ if } 0 \leq t \leq (k+1) \\ 0, \text{ otherwise} \end{cases}$$

Then

$$\Delta Y_k(t) = \begin{cases} \frac{t+(2k+1)}{2k+1}, \text{ if } -(2k+1) \leq t \leq 0 \\ \frac{(2k+1)-t}{2k+1}, \text{ if } 0 \leq t \leq (2k+1) \\ 0, \text{ otherwise} \end{cases}$$

Clearly $(Y_k) \notin w^F(\lambda)$. This completes the proof. □

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