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Characterizations of a commutative semisimple modular annihilator Banach algebra through its socle

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Abstract:

Let A be a commutative complex semisimple Banach algebra. In this paper we continue the study of kh(soc(A)). Thus we will give, among other things, some new characterizations of this ideal in terms of the closure, in the spectral radius norm, of the socle of A.

Keywords: Commutative algebras; Socle; Inessential element; Completely continuous algebra; Modular annihilator algebra.

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1. Introduction

For a semisimple complex Banach algebra $\mathcal{A}$ we let $soc(\mathcal{A})$ be its socle. Now, let Φ be a set of primitive ideals of $\mathcal{A}$ and S a subset of $\mathcal{A}$. The kernel of Φ, in $\mathcal{A}$, is denoted by $k_{\mathcal{A}}(\Phi)$ and the hull of S, in the set of all primitive ideals of $\mathcal{A}$, is denoted by $h_{\mathcal{A}}(S)$ [20]. Then the kernel of the hull of S in $\mathcal{A}$, which is the intersection of all primitive ideals of A containing S, is simply denoted by $kh(S)$. By [20, Theorem 2.2.6], we can see that $kh(soc(\mathcal{A}))$ is the set of all elements $u \in \mathcal{A}$ such that $u + \overline{soc(\mathcal{A})} \in rad(\mathcal{A}/\overline{soc(\mathcal{A})})$. We recall that an element of $\mathcal{A}$ is called inessential if its spectrum is either finite or a sequence converging to zero, and an ideal is inessential if all its elements are inessential [4, page 106]. For example, it is known that $kh(soc(\mathcal{A}))$ is an inessential closed ideal [7, Theorem 3.4]. For instance, if we consider the special case where $\mathcal{A} = \mathcal{B}(\mathcal{X})$ is the algebra of all bounded operators on a Banach space $\mathcal{X}$, then one can see that $kh(soc(\mathcal{A}))$ is the ideal of all inessential operators on $\mathcal{X}$. In particular, if $\mathcal{X} = \mathcal{H}$ is a Hilbert space, then $kh(soc(\mathcal{A})) = \mathcal{K}(\mathcal{H}) = \overline{\mathcal{F}(\mathcal{H})}$, where $\mathcal{F}(\mathcal{H})$ (respectively, $\mathcal{K}(\mathcal{H})$) is the ideal of finite rank (respectively, compact) operators on $\mathcal{H}$. For the fundamental properties of $kh(soc(\mathcal{A}))$, the reader is referred to [5, 7, 21]. Now, note that an element a in $\mathcal{A}$ is compact if the operator ${}_aT_a$ defined by ${}_aT_a(x) = axa$ for all $x \in \mathcal{A}$ is compact on $\mathcal{A}$ [2]. The set of all compact elements of $\mathcal{A}$ is denoted by $\mathcal{K}(\mathcal{A})$. For example, if $\mathcal{A} = \mathcal{B}(\mathcal{X})$ then, by [22], $\mathcal{K}(\mathcal{A}) = \mathcal{K}(\mathcal{X})$ which is the ideal of all compact operators on $\mathcal{X}$. Using now [5, Theorem 2.1], we then obtain

$$\overline{soc(\mathcal{A})} \subseteq \mathcal{K}(\mathcal{A}) \subseteq kh(soc(\mathcal{A})) \tag{1.1}$$

where $\overline{soc(\mathcal{A})}$ is the closure, in the Banach norm of $\mathcal{A}$, of $soc(\mathcal{A})$.

Several works have dealt with relationship 1.1. Thus, it is known that both of these inclusions can generally be strict in the Banach space setting (see e.g. [14, 12]), and in the Banach commutative context (see e.g. [8, 15]). However, these inclusions become equalities if $\mathcal{A}$ is, for example, a C*-algebra [11, Lemma C.2.4]. Further, this fact also remains true in more special cases (see e.g. [17, Theorem 19.G], [21, Corollary 7.4] and [15, Proposition 4.1]). Thus, in this direction, a natural question is whether the three sets $\overline{soc(\mathcal{A})}$, $\mathcal{K}(\mathcal{A})$ and $kh(soc(\mathcal{A}))$ coincide for a given class of semisimple commutative Banach algebras. The current study was motivated by this question and, so too in the original, by the [10, Remark 2.5]. Next, in order to deal with this question we will also need to add some preliminaries.

Throughout the remaining of this paper, $\mathcal{A}$ will be a semisimple commutative Banach algebra with Gelfand spectrum $\Phi_{\mathcal{A}}$. Then $kh(\mathcal{I})$, where $\mathcal{I}$ is an ideal in $\mathcal{A}$, is $\bigcap\{N(\varphi) : \varphi \in \Phi_{\mathcal{A}}, \mathcal{I} \subseteq N(\varphi)\}$, where $N(\varphi)$ is the kernel of φ. Thus, by [20, Theorem 2.2.6], $kh(\mathcal{I})$ is constituted exactly by the elements $u \in \mathcal{A}$ such that $u + \overline{\mathcal{I}}$ is quasi-nilpotent in $\mathcal{A}/\overline{\mathcal{I}}$. Recall that $\mathcal{A}$ is a modular annihilator algebra if $\mathcal{A}/soc(\mathcal{A})$ is a radical algebra [18, Theorem 8.4.5] (or equivalently, in this context, $\mathcal{A}$ is a Riesz algebra [11, page 60]). It is easy to see, from the relation 1.1, that if $\mathcal{A}$ has a 'large' socle in the sense that it has dense, in the Banach norm of $\mathcal{A}$, socle then it is a modular annihilator algebra. However, as we will see below, the converse is not true. Yet, and in this regard, we will show, in section 2, that if $\mathcal{A}$ is a modular annihilator algebra then it always has dense, in the spectral radius norm of $\mathcal{A}$, socle.
Now, we recall that $\mathcal{A}$ is compact (respectively, completely continuous) if $\mathcal{A} = \mathcal{K}(\mathcal{A})$ (respectively, for every $a \in \mathcal{A}$, the multiplication operator $T_a : \mathcal{A} \to \mathcal{A}$, defined by $T_a(x) = ax$, is compact). For some studies of these algebras we can see (e.g. [2, 3, 9] and the references therein). It is easy to see that every completely continuous algebra is a compact algebra; the converse of this implication is true if $\mathcal{A}$ has a bounded approximate identity (see e.g. [1, Theorem 5.61]). In that direction, we will show, in Theorem 3.3, that this converse is also true if $\mathcal{A}$ is Tauberian.

2. on the closure, in the spectral radius norm, of the socle of $\mathcal{A}$.

Since $\mathcal{A}$ is semisimple then the spectral radius is a norm on $\mathcal{A}$. Thus, $\|.\|$ and $\|.\|_r$ denote the Banach norm of $\mathcal{A}$ and the spectral radius norm of $\mathcal{A}$, respectively. Also, if $S \subseteq \mathcal{A}$, $\overline{S}$ and S^- denote the closure in the norm $\|.\|$ of S and the closure in the norm $\|.\|_r$ of S, respectively. Throughout the remaining of this paper, we set

$$\mathcal{B} := \{\vec{x} = (x_n)_{n\geq 1} : x_n \in \mathbf{C}, \forall n \in \mathbf{N}^*, sup\{n|x_n|, n \in \mathbf{N}^*\} < \infty\}.$$

Then, $\mathcal{B}$ is a semisimple commutative Banach algebra, with the pointwise product and the following Banach norm $\|\vec{x}\| := sup\{n|x_n|, n \in \mathbf{N}^*\}$ for every $\vec{x}$ in $\mathcal{B}$. In [21], the author noticed that $\overline{soc(\mathcal{B})} = kh(soc(\mathcal{B}))$. However, in the following result, we show that the $kh(soc(\mathcal{A}))$ is always equal to $soc(\mathcal{A})^-$.

Theorem 2.1. *Let $\mathcal{A}$ be a semisimple commutative Banach algebra. Then $kh(soc(\mathcal{A})) = soc(A)^-$.*

Proof. It is known that $soc(\mathcal{A}) = soc(\mathcal{A}^\sharp)$, $k_\mathcal{A} h_\mathcal{A}(soc(\mathcal{A})) = k_{\mathcal{A}^\sharp} h_{\mathcal{A}^\sharp}(soc(\mathcal{A}^\sharp))$ and $r_\mathcal{A}(x) = r_{\mathcal{A}^\sharp}(x)$ for any $x \in \mathcal{A}$, where $\mathcal{A}^\sharp$ is the unitization of $\mathcal{A}$. Then we can suppose that $\mathcal{A}$ has identity element. Let $x \in kh(soc(\mathcal{A}))$ and fix $\epsilon > 0$. Since x is inessential then there exist a finite set $\{\alpha_1, ..., \alpha_n\} \subset sp(x) \setminus \{0\}$ such that $\{\lambda \in sp(x) : |\lambda| \geq \epsilon\} = \{\alpha_1, ..., \alpha_n\}$. Set $e_k := p(\alpha_k, x)$ the projection associated to x and α_k (see e.g. page 106 in [4]). Then, by the analytic functional calculus, $\|x - \sum_{k=1}^n e_k x\|_r < \epsilon$. Therefore, since $\mathcal{A}$ is semisimple, $x \in soc(\mathcal{A})^-$, by [4, Lemma 5.7.1] .
Conversely, let $x \in soc(\mathcal{A})^-$. Given an $\epsilon > 0$, then there exist $s \in soc(\mathcal{A})$ such that $r(x-s) < \epsilon$. Thus $r(x+\overline{soc(\mathcal{A})}) < \epsilon$. Therefore $r(x+\overline{soc(\mathcal{A})}) = 0$. Consequently, $x \in kh(soc(\mathcal{A}))$.
□

The proof given here of $kh(soc(\mathcal{A})) \subseteq soc(A)^-$ is adaptation of the proof of [3, Theorem 2.4 page 17]. However, another proof of this inclusion my be deduced directly from [15, Proposition 3.1] (without use of the machinery of analytic functional calculus).

One of our original motivations for introducing Theorem 2.3, is [15, Question 5.2]. After trying to answer that question, we found another characterization of a modular annihilator algebra. To give this we need the following preparation. Here $\mathcal{A}_r$ and $\mathcal{B}(\mathcal{A}_r)$ denote the spectral radius normed algebra $\mathcal{A}$ and the normed algebra of all bounded linear operators on $\mathcal{A}_r$, respectively. Thus, $\mathcal{K}(\mathcal{A}_r)$ is the set of all elements $a \in \mathcal{A}$ such that T_{a^2} is a compact on $\mathcal{A}_r$. The following lemma tells us that $sp_\mathcal{A}(u)$ is either finite or a sequence converging to zero for any $u \in \mathcal{K}(\mathcal{A}_r)$.

Lemma 2.2. *Let $\mathcal{A}$ be a semisimple commutative Banach algebra. Then*

(i) $\mathcal{A}\mathcal{K}(\mathcal{A}_r) \subseteq \mathcal{K}(\mathcal{A}_r)$.

(ii) *For every $u \in \mathcal{K}(\mathcal{A}_r)$, $sp_{\mathcal{B}(\mathcal{A}_r)}(T_{u^2})$ is either finite or a sequence converging to zero, where $sp(.)$ is the spectrum.*

(iii) *For every $u \in \mathcal{K}(\mathcal{A}_r)$, $sp_{\mathcal{B}(\mathcal{A}_r)}(T_u)$ is either finite or a sequence converging to zero.*

(iv) *For every $u \in \mathcal{A}_r$, $sp_{\mathcal{B}(\mathcal{A}_r)}(u) \setminus \{0\} \subseteq sp_{\mathcal{B}(\mathcal{A}_r)}(T_u) \setminus \{0\}$.*

(v) *For every* $u \in \mathcal{K}(\mathcal{A}_r)$, $sp_{\mathcal{A}}(u)$ *is either finite or a sequence converging to zero.*

Proof. (i) By [19, Theorem 7.2].
(ii) By [19, Theorem 7.10].
(iii) It is enough to see that, for every $\lambda \in \mathbf{C}$ such that $\lambda^2 \in \rho_{\mathcal{B}(\mathcal{A}_r)}(T_{u^2})$ we have $\lambda \in \rho_{\mathcal{B}(\mathcal{A}_r)}(T_u)$, with $\rho_{\mathcal{B}(\mathcal{A}_r)}(.)$ is the resolvent.
(iv) Let $u \in \mathcal{A}$ and $\lambda \in \rho_{\mathcal{B}(\mathcal{A}_r)}(T_u) \setminus \{0\}$. Then $\lambda \in \rho_{\mathcal{B}(\mathcal{A}_r^\sharp)}(T_u^\sharp)$, where $T_u^\sharp : \alpha + a \to u(\alpha + a)$ is the multiplication operator by u on $\mathcal{A}^\sharp$. Thus we can suppose that $\mathcal{A}$ has identity element which is denoted by 1. Since $\lambda \in \rho_{\mathcal{B}(\mathcal{A}_r)}(T_u) \setminus \{0\}$ there exists a sequence (x_n) of elements of $\mathcal{A}$ such that $(\lambda - T_u)x_n \to 1$ in the norm $\|.\|_r$. Therefore $\lambda \in \rho(u)$.
(v) It follows from (iii) and (iv). □

Theorem 2.3. *Let $\mathcal{A}$ be a semisimple commutative Banach algebra. Then*

$$kh(soc(\mathcal{A})) = \mathcal{K}(\mathcal{A}_r) = soc(\mathcal{A})^-$$

Proof. We can assume that $\mathcal{A}$ has an identity element since $T_{u^2}^\sharp$ is a compact, in the spectral radius norm, operator on $\mathcal{A}^\sharp$ when T_{u^2} is a compact, in the spectral radius norm, operator on $\mathcal{A}$ for every $u \in \mathcal{A}$. Then by using Lemma 2.2 and [5, Theorem 2.1], we get firstly $\mathcal{K}(\mathcal{A}_r) \subseteq kh(soc(\mathcal{A}))$. Now fix $u \in kh(soc(\mathcal{A}))$. By Theorem 2.1 there exists a sequence (s_n) of elements of $soc(\mathcal{A})$ such that $T_{s_n^2} \to T_{x^2}$ in the norm of $\mathcal{B}(\mathcal{A}_r)$. Since $T_{s_n^2}$ is a finite rank operator for every $n \in \mathbf{N}$, L_{x^2} is a compact, in the spectral radius norm, operator on $\mathcal{A}$. That is $u \in \mathcal{K}(\mathcal{A}_r)$. □

The following result is a new characterization of a modular annihilator algebra. It says that any semisimple commutative modular annihilator Banach algebra is compact and have a 'large' socle in some sense (see e.g [18, Page 708]).

Corollary 2.4. *Let $\mathcal{A}$ be a semisimple commutative Banach algebra. Then the assertions are equivalent:*

(i) *$\mathcal{A}$ is a modular annihilator;*

(ii) *$\mathcal{A}$ is compact, in the spectral radius norm (that is $\mathcal{A} = \mathcal{K}(\mathcal{A}_r)$);*

(ii) *$\mathcal{A}$ has dense, in the spectral radius norm, socle.*

Remarks 2.5. *However, the results 2.3 and 2.4 are not true if the spectral radius norm is replaced by the Banach norm of $\mathcal{A}$. Indeed:*

(i) *In [15, Example 5.1], $\mathcal{K}(\mathcal{B}) \neq \overline{soc(\mathcal{B})}$ has been demonstrated.*

(ii) *The algebra above $\mathcal{B}$ is modular annihilator. Indeed, it is clear that the minimal idempotents in $\mathcal{B}$ are precisely the vectors $\vec{e_n}$ with $\vec{e_n} = (\delta_{nk})_k$. But for every n there is, by [15, Lemma 3.2] only one element $\chi_n \in iso(\Phi_{\mathcal{B}})$ such that $\widehat{\vec{e_n}} = \delta_{\chi_n}$, where $iso(\Phi_{\mathcal{B}})$ is the set of all isolated points of $\Phi_{\mathcal{B}}$ and $\widehat{\vec{e_n}}$ is the Gelfand transform of $\vec{e_n}$. We note here that $\Phi_{\mathcal{B}} = \{\chi_n\}$, then $\Phi_{\mathcal{B}}$ is discrete. Therefore $\mathcal{B}$ is modular annihilator. However, since $\mathcal{K}(\mathcal{B}) \neq \overline{soc(\mathcal{B})}$, $\mathcal{B}$ has not dense, in the Banach norm, socle .*

(iii) *In section 5 of [8] the authors have built a (semisimple commutative Banach algebra) modular annihilator, not compact, algebra $\mathcal{A}$; (this example will be studied in more detail in (1-i) of the remark 3.9).*

Remark 2.6. *It is important to emphasize here that [8] contains an algebra which would have been answered, in the negative, of the [15, Question 5.2]. Indeed, let $\mathcal{A}$ be the semisimple commutative Banach algebra constructed in section 5 of [8], it is also modular annihilator but not compact. Then $kh(soc(\mathcal{A}^\sharp)) \neq \mathcal{K}(\mathcal{A}^\sharp)$, despite the fact that $\mathcal{A}^\sharp$ is a semisimple commutative Banach algebra with identity element. Thus, $\mathcal{A}^\sharp$ is a negative answer to [15, Question 5.2].*

3. on the compactness of the Algebra $\mathcal{A}$.

It is necessary here to point out that the results of this section are largely inspired by [1, Theorem 5.61] containing the following result which says that the definition of compact algebra coincides with the given definition for a completely continuous algebra when $\mathcal{A}$ has bounded approximate identity (without the assumption that $\mathcal{A}$ is semsimple). The proof we present here is more elementary, since we do not use the Cohen-Hewitt Factorization Theorem as in the proof of [1, Theorem 5.61].

Theorem 3.1. *[1, Theorem 5.61] Suppose that $\mathcal{A}$ is a commutative Banach algebra with a bounded approximate identity $(e_\lambda)_\lambda$. Then the following statements are equivalent:*

(i) *$\mathcal{A}$ is a completely continuous algebra;*

(ii) *$\mathcal{A}$ is a compact algebra;*

(iii) *$T_{e_\lambda^2}$ is a compact operator on $\mathcal{A}$ for any λ.*

Proof. It is enough to notice that for every $u \in \mathcal{A}$ we have $\lim_\lambda T_{e_\lambda^2} T_u = T_u$, since $(e_\lambda)_\lambda$ is a bounded approximate identity of $\mathcal{A}$.
(i) $\Rightarrow$ (ii) and (ii) $\Leftrightarrow$ (iii) are trivial.
(ii) $\Rightarrow$ (i) Since $T_{e_\lambda^2}$ is a compact operator on $\mathcal{A}$ for any λ, T_u is also a compact operator on $\mathcal{A}$. This being true for any u in $\mathcal{A}$, we conclude that $\mathcal{A}$ is a completely continuous algebra. □

Remark 3.2. *This result can not directly apply, for example, on the whole class of Herz algebra. Indeed, firstly we recall that the Herz algebra of G is denoted by $A_p(G)$, where G is a locally compact group and $1 < p < \infty$. By [16, Theorem 6], there exist a G and $1 < p < \infty$ such that $A_p(G)$ has not bounded approximate identities. Nevertheless, this case will be covered, thanks to Theorem 3.3, by Example 3.5.*

Before stating the main results of this section, we shall need some preliminaries. Recall that $\mathcal{A}$ is said to be Tauberian if the elements of compact support are dense in $\mathcal{A}$, where the support of any $x \in \mathcal{A}$ is defined by $supp(x) := \overline{\{\varphi : \varphi \in \Phi_A; \varphi(x) = 0\}}$. This class contains many special important algebras. For example, obviously, if $\mathcal{A}$ has dense socle then it is Tauberian (whilst the converse need not be true, since for example $\mathcal{B}^\sharp$ is Tauberian with socle not dense). Thus, the algebra l^p $(1 \le p < \infty)$ with the coordinatewise multiplication; $\mathcal{C}_0(\Omega)$, the algebra of all complex continuous functions which vanish at infinity on a locally compact Hausdorff space Ω; the group algebras $L^1(G)$ for any locally abelian compact group G; the algebra $L^p(G)$, $1 < p < \infty$, for a compact abelian group G, are Tauberian. Also, the Herz algebra $A_p(G)$, for any locally compact group G and $1 < p < \infty$, is Tauberian [16, Proposition 3]. On the other hand, If $\mathcal{A}$ has identity then it is Tauberian; but, as we shall see in (i) of Remark 3.9, this implication is false if the algebra $\mathcal{A}$ has just a bounded approximate identity.

Now we set here $\mathcal{A}_0$ the set of all elements of $\mathcal{A}$ with compact support, and let $\mathcal{C}_c(\mathcal{A})$ denote the set of those $a \in \mathcal{A}$ such that T_a is a compact operator on $\mathcal{A}$. It is easy to see that $\mathcal{C}_c(\mathcal{A})$ is a closed ideal of $\mathcal{A}$. The main result of this section is the following. It is similar to the last proposition, replacing the assumption that $\mathcal{A}$ possess a bounded approximate identity with that of Tauberianess.

Theorem 3.3. *Suppose that $\mathcal{A}$ is a semisimple commutative Tauberian Banach algebra. Then the following statements are equivalent:*

(i) *$\mathcal{A}$ is a compact algebra;*

(ii) *T_a is compact for any $a \in \mathcal{A}$ with compact support;*

(iii) *$\mathcal{A}$ is a completely continuous algebra.*

Proof. (i) $\Rightarrow$ (ii) Assume that $\mathcal{A}$ is compact. Then $\mathcal{A}_0 \subseteq \mathcal{C}_c(\mathcal{A})$. Indeed, Let $a \in \mathcal{A}$ with compact support. If the spectrum of $\mathcal{A}$ is compact then, by [20, Theorem 3.6.6], $\mathcal{A}$ has an identity element. Hence $\mathcal{A}$ would be finite dimensional since it is a compact algebra by [6, Proposition 6.3]. It follows that T_a is compact. Suppose now that the spectrum of $\mathcal{A}$ is not compact. There exists $\varphi \in \Phi_{\mathcal{A}}$ such that φ is not in *supp*(a). Since $\mathcal{A}$ is compact then, by Relation 1.1, it is modular annihilator. Hence $\mathcal{A}$ is regular, so by [17, Lemma 25.C], there exist $e \in \mathcal{A}$ such that $\hat{e} = 1$ on *supp*(a) and $\hat{e}(\varphi) = 0$. Therefore $a = ae$ since $\mathcal{A}$ is semisimple. Hence

$$T_a = \frac{1}{2}(T_{(a+e)^2} - T_{a^2} - T_{e^2}).$$

Since $\mathcal{A}$ is a compact algebra then $T_{(a+e)^2}$, T_{a^2} and T_{e^2} are compact operators, and therefore T_a is compact.
(ii) $\Rightarrow$ (iii) Suppose now that $\mathcal{A}_0 \subseteq \mathcal{C}_c(\mathcal{A})$. Since $\mathcal{C}_c(\mathcal{A})$ is closed, and by using the fact that $\mathcal{A}$ is Tauberian we then conclude that $\mathcal{A}$ is completely continuous.
(iii) $\Rightarrow$ (i) is trivial. □

Remark 3.4. *We do not know if Theorem 3.3 remains true if we remove the assumption that $\mathcal{A}$ is Tauberian.*

Knowing that $A_p(G)$ is a semisimple commutative Tauberian Banach algebra [16], for any locally compact group G and $1 < p < \infty$, the following example is a direct application of Theorem 3.3.

Example 3.5. *Let G be any locally compact group and $1 < p < \infty$. Then the following are equivalent:*

(i) *$A_p(G)$ is a compact algebra;*

(ii) *$A_p(G)$ is a completely continuous algebra.*

Not surprisingly, the following result is, more or less, similar, in formulation and proof, to the second part of [1, Theorem 5.61].

Corollary 3.6. *Suppose that $\mathcal{A}$ is a semisimple commutative Tauberian Banach algebra. Then the following statements are equivalent:*

(i) *$\mathcal{A}$ is a compact algebra;*

(ii) *$\mathcal{A}$ is a completely continuous algebra;*

(iii) *$\mathcal{A}$ is a modular annihilator algebra ;*

(iv) *$\Phi_{\mathcal{A}}$ is discrete;*

(v) *$\mathcal{A}$ is inessential;*

(vi) *$\mathcal{A}$ has dense, in the Banach norm, socle;*

(vii) *$\mathcal{A}$ has dense, in the spectral radius norm, socle.*

Proof. (i) $\Rightarrow$ (ii) by Theorem 3.3.
(ii) $\Rightarrow$ (i) is trivial.
(i) $\Rightarrow$ (iii) follows immediately from the relation 1.1.
(iii) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (v) It follows from [18, Theorem 8.6.4].
(v) $\Rightarrow$ (vi) Assume that $\mathcal{A}$ is inessential. This means that the spectrum of $\mathcal{A}$ is discrete. Thus to conclude that $\mathcal{A}$ has dense socle, it is enough to see, by the fact that $\mathcal{A}$ is Tauberian, that the socle of $\mathcal{A}$ equals the set of all elements of $\mathcal{A}$ with compact support.
(vi) $\Rightarrow$ (vii) It follows from Theorem 2.1.
(vii) $\Rightarrow$ (i) Suppose that $\mathcal{A} = soc(\mathcal{A})^{-}$. Then, by Corollary 2.4, $\mathcal{A}$ is modular annihilator. From (v) $\Rightarrow$ (vi) we may infer that $\mathcal{A} = \overline{soc(\mathcal{A})}$. Therefore, by the relation 1.1, $\mathcal{A}$ is compact.
□

A direct application of this result can be done to the following examples.

Example 3.7. (i) *For any locally compact abelian group G we have the following equivalence: $L^1(G)$ is completely continuous if and only if G is compact, since $\Phi_{L^1(G)} = \hat{G}$ and by the equivalence between compactness of G and discreteness of dual group $\hat{G}$ (see e.g. [1, page 220]).*

(ii) *For every locally compact group G and $1 < p < \infty$, G is the Gelfand spectrum of $A_p(G)$ (by [16, Theorem 3]). Then, we have the following equivalence: $A_p(G)$ is completely continuous if and only if G is discrete.*

Corollary 3.8. *Suppose that $\mathcal{A}$ is a semisimple commutative Tauberian Banach algebra. If $\mathcal{A}$ is a modular annihilator then $\mathcal{A}$ (also its unitization) satisfies:*

$$kh(soc(\mathcal{A})) = \mathcal{K}(\mathcal{A}) = \overline{soc(\mathcal{A})}.$$

Remarks 3.9. (1) *In this corollary the Tauberian condition cannot removed. Indeed:*

(i) *Consider the algebra $\mathcal{A}$ constructed in section 5 of [8]. It is a semisimple commutative, not Tauberian, modular annihilator Banach algebra (with a bounded approximate identity), by [8, Theorem 6.4]. However, $kh(soc(\mathcal{A})) \neq \mathcal{K}(\mathcal{A})$, by [8, Corollary 6.2].*

(ii) *We have seen in Remarks 2.5 that the algebra $\mathcal{B}$ is a semisimple commutative modular annihilator Banach algebra such that $\overline{soc(\mathcal{B})} \neq \mathcal{B}$. Then $\mathcal{B}$ is not Tauberian, by Corollary 3.6. However, it satisfies $\mathcal{K}(\mathcal{B}) \neq \overline{soc(\mathcal{B})}$, by [15, Example 5.1].*

(2) *Also, the example in (i) shows that the assumption of Tauberianess in the last corollary cannot replaced by the assumption that $\mathcal{A}$ has a bounded approximate identity.*

We can now give the following example.

Example 3.10. *Let G be any locally compact group and $1 < p < \infty$. If G is discrete then $A_p(G)$ (also its unitization) satisfies*

$$kh(soc(A_p(G))) = \mathcal{K}(A_p(G)) = \overline{soc(A_p(G))}. \tag{3.1}$$

In particular, using [13, Theorem 3.2], these equalities are also true if $A_p(G)$ is Arens regular (in the sense of [13, page 218]).

Remark 3.11. *As we have seen above, in the general context of semisimple commutative Banach algebras, the equivalence between being Tauberian and being having dense socle is not generally true. However, this equivalence becomes true in the case of modular annihilator semisimple* *commutative Banach algebras. More precisely, the following fact gives an characterization, in terms of Tauberianess, of the equality* $kh(soc(\mathcal{A})) = \overline{soc(\mathcal{A})}$. *Assume that* $\mathcal{A}$ *is a semisimple commutative Banach algebra. Since* $soc(kh(soc(\mathcal{A}))) = soc(\mathcal{A})$ *and* $kh(soc(\mathcal{A}))$ *is inessential, then we have the following equivalence:* $kh(soc(\mathcal{A}))$ *is Tauberian if and only if* $kh(soc(\mathcal{A})) = \mathcal{K}(\mathcal{A}) = \overline{soc(\mathcal{A})}$.

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