Proyecciones Journal of Mathematics
Vol. 41, N^o 1, pp. 111-135, February 2022.
Universidad Católica del Norte
Antofagasta - Chile

10.22199/issn.0717-6279-4457

An existence result for a strongly nonlinear parabolic equations with variable nonlinearity

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Received : September 2020. Accepted : July 2021

Abstract

We prove the existence of a solution for the strongly nonlinear parabolic initial boundary value problem associated to the equation

$$u_t - \ div\ a(x,t,\nabla u) + g(x,t,u,\nabla u) = f,$$

where the vector field $a(x,t,\xi)$ exhibits non-standard growth conditions.

Key words: *Strongly nonlinear parabolic equations, Variable exponents, Weak solution, Existence.*

MSC (2010): *35K55, 46E35, 35D30.*

1. Introduction

Let Ω be a bounded domain in $\mathbf{R}^N$, $N \geq 2$, with a Lipschitz boundary denoted by $\partial\Omega$. Fixing a final time $T > 0$, we denote by Q the cylinder $\Omega\times]0,T[$ and $\Gamma = \partial\Omega\times]0,T[$ its lateral surface. We consider the following strongly nonlinear parabolic initial-boundary problem :

$$(1.1)\qquad \begin{cases} \dfrac{\partial u}{\partial t} - \text{ div } a(x,t,\nabla u) + g(x,t,u,\nabla u) = f & \text{in } Q, \\ u(x,t) = 0 & \text{in } \Gamma, \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases}$$

The function a satisfying the Leray-Lions-like conditions with respect to the variable exponent $p(\cdot) : \overline{\Omega} \to [1, +\infty[$ which is a Log-Hölder continuous function only dependent on the space variable x (see definitions below). The nonlinear term $g(x,t,s,\xi)$ satisfies the more general natural growth condition with respect to ∇u

$$|g(x,t,u,\nabla u)| \leq b(|u|)(\theta(x,t) + |\nabla u|^{p(x)}),$$

for some continuous function $b : \mathbf{R}^+ \to \mathbf{R}^+$ and the sign condition

$$g(x,t,u,\nabla u)u \geq 0.$$

u_0 lies in $L^2(\Omega)$ and the right-hand side f is assumed to belong to X' where the space X, as introduced and discussed in [3] (see also [19]), is given by

$$X := \left\{ u \in L^{p^-}(0,T; W_0^{1,p(\cdot)}(\Omega)) : \ |\nabla u| \in L^{p(\cdot)}(Q) \right\},$$

Given the assumptions we have made, we think that this space is a reasonable framework to discuss our problem.

The specific attention accorded to problems with variable exponent is due to their applications in mathematical physics. Precisely, such equations are used to model phenomenon which arise in electrorheological fluids (see [15]) as well as in some model of image processing (see [5]) and elasticity ([21]).

For the problem (1.1) with $g \equiv 0$ having $p(x,t)$- structure, the authors proved in [6] and independently in [18] the existence of a least a weak solution. Besides, Fu and Pan in [9] have proved an existence result of weak solutions for the problem

$$\begin{cases} \dfrac{\partial u}{\partial t} - \text{ div } a(x,t,u,\nabla u) + a_0(x,t,u,\nabla u) = f & \text{in } Q = \Omega\times]0,T[, \\ u(x,t) = 0 & \text{in } \Gamma = \partial\Omega\times]0,T[, \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases}$$

where $u_0 \in L^2(\Omega)$ and $f \in W^{-1,x}L^{p'(x)}(Q)$ under some $p(x)$-growth conditions.

In this work, we prove the existence of solutions for nonlinear parabolic initial boundary value problems associated to equations whose prototype is:

$$\frac{\partial u}{\partial t} - \text{ div } (|\nabla u|^{p(x)-2}\nabla u) + u|\nabla u|^{p(x)} = f \text{ in } Q := \Omega\times]0,T[.$$

We are mainly concerned with the existence of weak solutions for the strongly nonlinear problem (1.1) in the variational framework where $p(\cdot)$ is only dependent on the space variable, $-$ div $a(x,t,\nabla u)$ is a Leray-Lions type operator which growths like $|\nabla u|^{p(x)-1}$ not depending on u and where the perturbation g has a critical growth with respect to ∇u. For this, we will use a Galerkin approximation to construct solutions. Then we want to conclude from a energy estimate and by using the sign condition on g that the approximated solution is uniformly bounded in X. We apply the time-regularizing convolution operator to prove the all everywhere convergence of the gradient of approximate solution to the gradient of the limit, which is important in the limiting process. Finally, we will see that the solutions of the approximate solution converge to the solution of the model problem in $C([0,T],L^2(\Omega))$ which gives meaning to the initial condition.

2. Preliminaries

Let

$$p^- := \operatorname{ess}\inf_{x\in\Omega} p(x) \text{ and } p^+ := \operatorname{ess}\sup_{x\in\Omega} p(x).$$

We will make use of the following assumption

$$1 < p^- \leq p(x) \leq p^+ < +\infty. \tag{2.1}$$

An interesting feature of generalized variable exponent Sobolev space is that smooth functions are not dense in it without additional assumptions on the exponent $p(\cdot)$. However, when the exponent satisfies the following so-called log-Hölder condition

$$\exists C > 0 \ : \quad |p(x)-p(y)|\log\left(e + \frac{1}{|x-y|}\right) \leq C, \qquad \text{for all } x,y \in \overline{\Omega}, \tag{2.2}$$

then $C_0^\infty(\overline{\Omega})$ is dense in $L^{p(\cdot)}(\Omega)$ (see [8, 16, 20]) and we have the Poincaré inequality (see [7, Theorem 8.2.4], [8, Theorem 2.7] and [10, Theorem 4.3]):

Lemma 2.1. *Let Ω be a bounded domain in $\mathbf{R}^N$. If $p \in C(\overline{\Omega})$ satisfy (2.1) and (2.2), then there exists a constant $C > 0$ depending only on Ω and the function p such that*

$$\|u\|_{L^{p(\cdot)}(\Omega)} \leq C\|\nabla u\|_{L^{p(\cdot)}(\Omega)} \quad \text{for all } u \in W_0^{1,p(\cdot)}(\Omega).$$

In particular, the space $W_0^{1,p(\cdot)}(\Omega)$ has a norm $\|\cdot\|_{W_0^{1,p(\cdot)}(\Omega)}$ given by

$$\|u\|_{W_0^{1,p(\cdot)}(\Omega)} = \|\nabla u\|_{L^{p(\cdot)}(\Omega)} \quad \text{for all } u \in W_0^{1,p(\cdot)}(\Omega),$$

which equivalent to $\|\cdot\|_{W^{1,p(\cdot)}(\Omega)}$. Moreover, the embedding $W_0^{1,p(\cdot)}(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega)$ is compact (see [11]).

As in [13] (lemma 1.3. p.12) we can prove the following lemma

Lemma 2.2. *Suppose that $1 \leq p(x) < \infty$. Let $\{v_n\}_n$ be bounded in $L^{p(\cdot)}(\Omega)$. If $v_n \to v$ a.e. in Ω, then $v_n \rightharpoonup v$ weakly in $L^{p(\cdot)}(\Omega)$.*

We extend a variable exponent function $p : \overline{\Omega} \to [1, +\infty[$ to $\overline{Q} \to [1, +\infty[$ by setting $p(x,t) = p(x)$ for all $(x,t) \in \overline{Q}$.

3. The space X

As in [3], we consider the following functional space

$$X := \Big\{ u \in L^{p^-}(0,T; W_0^{1,p(\cdot)}(\Omega)) : \ |\nabla u| \in L^{p(\cdot)}(Q) \Big\},$$

which is a separable and reflexive Banach space endowed with the norm

$$|u|_X := \|u\|_{L^{p^-}(0,T;W_0^{1,p(\cdot)}(\Omega))} + \|\nabla u\|_{L^{p(\cdot)}(Q)}$$

or the equivalent norm

$$\|u\|_X := \|\nabla u\|_{L^{p(\cdot)}(Q)}.$$

Lemma 3.1. *Assume that (2.1) and (2.2) are fulfilled. If $0 < |\Omega| < +\infty$, the Banach space X is continuously embedded in $L^1(Q)$. Moreover, there is a constant $c_0 > 0$ such that for every $u \in X$, one has $\|u\|_{L^1(Q)} \leq c_0|u|_X$.*

Proof. Let $u \in X$. Using the Hölder inequality one has

$$\int_\Omega |u(t)|dx \leq \left(1 + \frac{1}{p^-} - \frac{1}{p^+}\right)\max(|\Omega|^{\frac{1}{p'^-}}, |\Omega|^{\frac{1}{p'^+}})\|u(t)\|_{L^{p(\cdot)}(\Omega)}.$$

By Lemma 2.1, there is a constant $C > 0$ such that

$$\int_\Omega |u(t)|dx \leq C\left(1 + \frac{1}{p^-} - \frac{1}{p^+}\right)\max(|\Omega|^{\frac{1}{p'^-}}, |\Omega|^{\frac{1}{p'^+}})\|\nabla u(t)\|_{L^{p(\cdot)}(\Omega)}.$$

Integrating between 0 and T and using the Hölder inequality, we get

$$\int_Q |u(t)|dx \leq C\left(1+\frac{1}{p^-}-\frac{1}{p^+}\right)\max(|\Omega|^{\frac{1}{p'^-}}, |\Omega|^{\frac{1}{p'^+}})T^{1-\frac{1}{p^-}}\left(\int_0^T \|\nabla u(t)\|^{p^-}_{L^{p(\cdot)}(\Omega)}\right)^{\frac{1}{p^-}},$$

which yields

$$\int_Q |u(t)|dx \leq C\left(1 + \frac{1}{p^-} - \frac{1}{p^+}\right)\max(|\Omega|^{\frac{1}{p'^-}}, |\Omega|^{\frac{1}{p'^+}})T^{1-\frac{1}{p^-}}|u|_X.$$

□

Remark 3.2. *[3, Lemma 3.1] $\mathcal{C}_0^\infty(Q)$ is dense in X. Moreover we have the following continuous dense embedding*

$$L^{p^+}(0,T;W_0^{1,p(\cdot)}(\Omega)) \hookrightarrow_d X \hookrightarrow_d L^{p^-}(0,T;W_0^{1,p(\cdot)}(\Omega)).$$

For the corresponding dual spaces, we have

$$L^{(p^-)'}(0,T;W^{-1,p'(\cdot)}(\Omega)) \hookrightarrow X' \hookrightarrow L^{(p^+)'}(0,T;W_0^{-1,p'(\cdot)}(\Omega)).$$

Lemma 3.3. *Let Y be a Banach space such that the embedding $L^1(\Omega) \hookrightarrow Y$ is continuous. If $\mathcal{F}$ is bounded in X and relatively compact in $L^1(0,T;Y)$ then $\mathcal{F}$ is relatively compact in $L^1(Q)$.*

Proof. Let $\epsilon > 0$. Since the embedding $W_0^{1,p(\cdot)}(\Omega) \hookrightarrow L^1(\Omega)$ is compact, by [17, Lemma 8] there exists a finite constant $N > 0$ such that for every $v \in W_0^{1,p(\cdot)}(\Omega)$ one has

$$\|v\|_{L^1(Q)} \leq \epsilon \int_0^T \|v\|_{W_0^{1,p(\cdot)}(\Omega)}dt + N\|v\|_{L^1(0,T;Y)}.$$

Being $\mathcal{F}$ relatively compact, for ϵ, there is a finite sequence $u_1, u_2, \ldots, u_m$ in $\mathcal{F}$ satisfying

$$\forall u \in \mathcal{F},\ \exists u_n, 1 \leq n \leq m, \ \text{ such that } \ \|u_n - u\|_{L^1(0,T;Y)} \leq \epsilon.$$

Hence, we get

$$\|u_n - u\|_{L^1(Q)} \leq \epsilon \int_0^T \|u_n - u\|_{W_0^{1,p(\cdot)}(\Omega)} dt + N\|u_n - u\|_{L^1(0,T;Y)}.$$

By the fact that $\int_0^T \|u_n - u\|_{W_0^{1,p(\cdot)}(\Omega)} dt$ is bounded, as a consequence of the continuous embedding $X \hookrightarrow L^1(0,T;W_0^{1,p(\cdot)}(\Omega))$, we conclude that $\mathcal{F}$ is relatively compact in $L^1(Q)$. $\square$

Lemma 3.4. *Let (u_n) be a sequence of elements of X such that*

$$u_n \rightharpoonup u \textit{weakly in } X$$

and

$$\frac{\partial u_n}{\partial t} = h_n + k_n \ \textit{in} \mathcal{D}'(Q),$$

where (h_n) is a bounded sequence in X' and (k_n) is a bounded sequence in $L^1(Q)$. Then

$$u \in \mathcal{C}(0,T;W^{-1,1}(\Omega)) \textit{ and } u_n \to u \textit{ strongly in } L^1(Q).$$

Proof. Observe that the sequence (u_n) is bounded in X and in $L^1(0,T;W^{-1,1}(\Omega))$ as well, since the embedding $L^1(\Omega) \hookrightarrow W^{-1,1}(\Omega)$ is continuous, by Remark 3.2 we conclude that the following embedding are continuous

$$X \hookrightarrow L^1(0,T;W_0^{1,p(\cdot)}(\Omega)) \hookrightarrow L^1(0,T;L^1(\Omega)) \hookrightarrow L^1(0,T;W^{-1,1}(\Omega)).$$

On the other hand, $\frac{\partial u_n}{\partial t}$ is bounded in $X' + L^1(Q)$ and in $L^1(0,T;W^{-1,1}(\Omega))$ too, since

$$X' + L^1(Q) \hookrightarrow L^1(0,T;W^{-1,1}(\Omega)) + L^1(0,T;L^1(\Omega)) \hookrightarrow L^1(0,T;W^{-1,1}(\Omega))$$

with continuous imbedding. Thus, by [17, Lemma 4] one has

$$u_n \in \mathcal{C}(0,T;W^{-1,1}(\Omega)) \text{ and } \|\tau_h u_n - u_n\|_{L^1(0,T-h;W^{-1,1}(\Omega))} \to 0 \text{ as } h \to 0,$$

for all $n \in \mathbf{N}$. Let $0 < t_1 < t_2 < T$. By the Jensen inequality, there exists a constant c_T which depends on T and p^- such that

$$\Big\| \int_{t_1}^{t_2} u_n(t)dt \Big\|^{p^-}_{W_0^{1,p(\cdot)}(\Omega)} \le c_T \int_{t_1}^{t_2} \|u_n(t)\|^{p^-}_{W_0^{1,p(\cdot)}(\Omega)} dt \le c_T |u_n|^{p^-}_X .$$

By virtue of the compact embedding $W_0^{1,p(\cdot)}(\Omega) \hookrightarrow\hookrightarrow L^1(\Omega)$, we deduce that $(\int_{t_1}^{t_2} u_n(t)\, dt)_n$ is relatively compact in $L^1(\Omega)$ and also in $W^{-1,1}(\Omega)$. Applying [17, Theorem 1], we conclude that (u_n) is relatively compact in $L^1(0,T; W^{-1,1}(\Omega))$. Since the imbedding $L^1(\Omega) \hookrightarrow W^{-1,1}(\Omega)$ is continuous, by Lemma 3.3, we get (u_n) is relatively compact in $L^1(Q)$. Therefore, up to a subsequence,

$$u_n \to u \text{ strongly in } L^1(Q) \text{ and a.e. in } Q.$$

Moreover, $u \in \mathcal{C}(0,T; W^{-1,1}(\Omega))$. □

As in [18] we can proof the following lemma

Lemma 3.5. *Assume that (2.2) holds true. If $u \in X \cap L^2(Q)$ with $\dfrac{\partial u}{\partial t} \in X' + L^1(Q)$, then there exists a sequence $\{u_i\}$ in $\mathcal{C}_0^\infty(\overline{Q})$ such that*

$$\begin{aligned} u_i &\to u \text{ strongly in } X \cap L^2(Q) \text{ and} \\ \frac{\partial u_i}{\partial t} &\to \frac{\partial u}{\partial t} \text{ in } X' + L^1(Q). \end{aligned}$$

We will use the following results which can be proved as in [2].

Lemma 3.6. *(Integration by parts.) Assume that $p(\cdot)$ satisfies (2.2). Let u, $v \in X$ such that u_t, $v_t \in X'$. Then, for almost every t_1, $t_2 \in [0,T]$ one has*

$$\int_{t_1}^{t_2} \int_\Omega u v_t dx dt + \int_{t_1}^{t_2} \int_\Omega u_t v dx dt = \Big[\int_\Omega u(t)v(t)dx \Big]_{t_1}^{t_2}.$$

4. Basic assumptions and the main result

We assume that $a(x,t,\xi) : Q \times \mathbf{R}^N \to \mathbf{R}^N$ is a Carathéodory function (i.e. continuous with respect to $\xi \in \mathbf{R}^N$ for a.e. $(x,t) \in Q$ and measurable with respect to $(x,t) \in Q$ for all $\xi \in \mathbf{R}^N$) which satisfies, for a.e. $(x,t) \in Q$, for every $\xi, \xi' \in \mathbf{R}^N$ with $\xi \ne \xi'$, and some $C(x,t) \in L^{p'(\cdot)}(Q)$ the following assumptions:

$$|a(x,t,\xi)| \le \Big(C(x,t) + |\xi|^{p(x)-1}\Big), \tag{4.1}$$

$$\Big(a(x,t,\xi) - a(x,t,\xi')\Big) \cdot \Big(\xi - \xi'\Big) > 0, \tag{4.2}$$

$$a(x,t,\xi) \cdot \xi \ge |\xi|^{p(x)}. \tag{4.3}$$

$g : Q \times \mathbf{R} \times \mathbf{R}^N \to \mathbf{R}$ is a Carathéodory function such that for a.e.$(x,t) \in Q$ and all $(s,\xi) \in \mathbf{R} \times \mathbf{R}^N$ with $\theta \in L^1(Q)$ and some continuous function $b : \mathbf{R}^+ \to \mathbf{R}^+$:

$$|g(x,t,s,\xi)| \le b(|s|)\Big(\theta(x,t) + |\xi|^{p(x)}\Big), \tag{4.4}$$

$$g(x,t,s,\xi)s \ge 0, \tag{4.5}$$

In what follows T_k $(k > 0)$ denotes the truncation function defined on $\mathbf{R}$ by $T_k(s) = \max(-k, \min(k,s))$. Our main result is the following existence theorem:

Theorem 4.1. *Let $f \in X'$ and $u_0 \in L^2(\Omega)$. Assume that (4.1)-(4.5) hold true. Then there exists at least one weak solution u of problem (1.1) in the following sense: $g(.,.,u,\nabla u) \in L^1(Q)$, $g(.,.,u,\nabla u)u \in L^1(Q)$ and*

$$\begin{aligned} &-\int_Q u\varphi_t dxdt + \int_Q a(x,t,\nabla u)\nabla\varphi dxdt + \int_Q g(x,t,u,\nabla u)\varphi dxdt \\ &= \langle f, \varphi\rangle + \int_\Omega u_0(x)\varphi(x,0)dx, \end{aligned}$$

for all $\varphi \in X \cap L^\infty(Q) \cap C^1([0,T]; L^2(\Omega))$ with $\varphi(\cdot,t) = 0$ in a neighborhood of T.

5. Proof of the main result

STEP I. Galerkin solutions.

We choose a sequence of functions $\{\omega_i\}_{i=1}^\infty \subset \mathcal{C}_0^\infty(\Omega)$ orthonormal with respect to the Hilbert space $L^2(\Omega)$ such that $\bigcup_{k=1}^\infty V_k$, where we denote $V_k = \operatorname{span}\{\omega_1, \ldots, \omega_k\}$, is dense in $H_0^s(\Omega)$ with s large enough such as

$s > \frac{N}{2} + 1$ so that $H_0^s(\Omega)$ is continuously embedded in $\mathcal{C}^1(\overline{\Omega})$, (see [14]). Let

$$X_k = \Big\{ v(x,t) : \ v = \sum_{i=1}^{k} d_i(t)\omega_i(x), \ d_i(t) \in \mathcal{C}^1([0,T]) \Big\}.$$

It's easy to see that $\mathcal{C}_0^\infty(Q) \subset \overline{\bigcup_{k=1}^\infty X_k}$ with respect to the norm

$$\|v\|_{\mathcal{C}_0^{1,0}(Q)} = \sup_{(x,t)\in Q} \{|v(x,t)|, |\nabla v(x,t)|\}.$$

Since $\mathcal{C}_0^\infty(Q)$ is dense in $L^{p^+}(0,T;W_0^{1,p(\cdot)}(\Omega))$, one has $\bigcup_{k=1}^\infty X_k$ is dense in the space $L^{p^+}(0,T;W_0^{1,p(\cdot)}(\Omega))$. Then, according to Remark 3.2 we get that $\bigcup_{k=1}^\infty X_k$ is dense in X.
For every $f \in X'$, there is a sequence $\{f_n\}_n \subset \bigcup_{k=1}^\infty X_k$ such that $f_n \to f$ strongly in X'. Indeed, let $\varepsilon > 0$ be arbitrary. f can be represented as $f = -\operatorname{div} F$, where $F = (f_1, f_2, \dots, f_N) \in (L^{p'(\cdot)}(Q))^N$. Since $\mathcal{C}_0^\infty(Q)$ is dense in $L^{p'(\cdot)}(Q)$, for every $i \in \{1,2,\dots,N\}$ there is $\theta_i \in \mathcal{C}_0^\infty(Q)$ such that $\|f_i - \theta_i\|_{L^{p'(\cdot)}(Q)} \le \frac{\varepsilon}{2N}$. Setting $\theta = -\operatorname{div} \Theta$ with $\Theta = (\theta_1, \theta_2, \dots, \theta_N)$, one has $\theta \in \mathcal{C}_0^\infty(Q)$. By the fact that $\mathcal{C}_0^\infty(Q) \subset \overline{\bigcup_{n=1}^\infty X_n}$, there exists $h \in \bigcup_{n=1}^\infty X_n$ such that $\|\theta - h\|_{\infty,Q} \le \frac{\varepsilon}{2c_0}$, c_0 being the constant in Lemma 3.1. Therefore,

$$\begin{aligned} \|f-h\|_{X'} &= \sup_{v\in X, |v|_X \le 1} |\langle f-h, v\rangle| \\ &\le \sum_{i=1}^{N} \|f_i - \theta_i\|_{L^{p'(x)}(Q)} + \|\theta - h\|_{\infty,Q} \sup_{v\in X, |v|_X\le 1} \|v\|_{L^1(Q)} \le \varepsilon. \end{aligned}$$

We also note that there exists a sequence $u_{0_n} \subset \bigcup_{i=1}^\infty V_n$ such that $u_{0_n} \to u_0$ in $L^2(\Omega)$.

Definition 5.1. *A function $u_n \in X_n$ is called Galerkin solution of (1.1) if and only if*

$$\int_\Omega \frac{\partial u_n}{\partial t} v dx + \int_\Omega a(x,t,\nabla u_n)\cdot \nabla v dx + \int_\Omega g(x,t,u,\nabla u_n) v dx = \int_\Omega f_n(t) v dx, \tag{5.1}$$

for all $v \in V_n$ and all $t \in [0,T]$ with $u_n(x,0) = u_{0_n}(x)$.

Setting $u_n(x,t) = \sum_{i=1}^{n} d_i(t)\omega_i(x)$, we then try to look for the coefficients d_i. To do this, we define a vector valued function $y_n(t,d) : [0,T]\times \mathbf{R}^n \to \mathbf{R}^n$

for $d = (d_1, \dots, d_n)$ by

$$
\begin{aligned}
(y_n(t,d))_i &= \int_\Omega a\Big(x,t,\sum_{i=1}^{n} d_{ni}(t)\nabla\omega_i(x)\Big)\cdot\nabla\omega_i(x)dx \\
&\quad + \int_\Omega g\Big(x,t,\sum_{i=1}^{n} d_{ni}(t)\omega_i(x),\sum_{i=1}^{n} d_{ni}(t)\nabla\omega_i(x)\Big)\omega_i(x)dx,
\end{aligned}
$$

for $i = 1, \dots, n$.

Note that the function $y_n(t,d)$ is continuous because a and g are Carathéodory functions. We obtain the following system of ordinary differential equations

$$
\left\{\begin{aligned} d' + y_n(t,d) &= F_n, \\ d(0) &= v_n, \end{aligned}\right. \tag{5.2}
$$

where $(F_n(t))_i = \displaystyle\int_\Omega f_n(t)\omega_i\,dx$ and $(v_n)_i = \displaystyle\int_\Omega u_{0_n}\omega_i\,dx$, $i = 1, \dots, n$. Multiplying the first equation of (5.2) by $d(t)$ and using (4.3) and (4.5) one has $F_n(t,d)d \geq 0$, we apply the Young inequality to obtain

$$
\frac{1}{2}\frac{\partial}{\partial t}|d(t)|^2 \leq |F_n(t)||d(t)| \leq \frac{1}{2}|F_n(t)|^2 + \frac{1}{2}|d(t)|^2.
$$

By virtue of Gronwall's lemma one has

$$
|d(t)| \leq C_n(T).
$$

Thus, we get $|d(t) - d(0)| \leq 2C_n(T)$. Let $M_n = \max_{t\in[0,T]} |F_n - y_n(t,d(t))|$ and $q = \min\{T, \frac{2C_n(T)}{M_n}\}$. By the Cauchy-Peano theorem (see for instance [1]) we obtain a local solution in $[0,q]$. Starting with the initial value q, we obtain a local solution in $[q, 2q]$ and so on we get a local solution d_n in $\mathcal{C}^1([0,T])$. By construction, we know that the function $u_n(x,t) = \sum_{i=1}^{n} d_{n,i}(t)\omega_i(x)$ which belongs to X_n is a Galerkin solution for (1.1) which satisfies

$$
\begin{aligned}
\int_0^\tau \int_\Omega \frac{\partial u_n}{\partial t} v dx + \int_{Q_\tau} a(x,t,\nabla u_n)\cdot\nabla v\,dx\,dt + \int_{Q_\tau} g(x,t,u,\nabla u_n) v\,dx\,dt \\
= \int_{Q_\tau} f_n v\,dx\,dt,
\end{aligned} \tag{5.3}
$$

for all $v \in X_n$ and all $\tau \in [0,T]$ with $u_n(x,0) = u_{0_n}(x)$.

We multiply the equation (5.1) by the coefficients $d_{n,i}(t)$, $i = 1, 2, \cdots, n$ and we integrate the equation over $[0,\tau]$ for an arbitrarily $\tau \in [0,T]$

$$\tfrac{1}{2}\int_\Omega \left[(u_n(t))^2\right]_0^T dx + \int_{Q_\tau} a(x,t,\nabla u_n)\cdot \nabla u_n dxdt + \int_{Q_\tau} g(x,t,u_n,\nabla u_n)u_n dxdt$$

$$= \int_{Q_\tau} f u_n dxdt \tag{5.4}$$

By (4.3) and (4.5) we can write

$$\frac{1}{2}\|u_n(\tau)\|^2_{L^2(\Omega)} + \int_{Q_\tau} |\nabla u_n|^{p(x)} dxdt \le \int_{Q_\tau} f u_n dxdt + \frac{1}{2}\|u_0\|^2_{L^2(\Omega)}. \tag{5.5}$$

Then we get

$$\int_{Q_\tau} |\nabla u_n|^{p(x)} dxdt \le \|f\|_{X'_\tau} |u_n|_{X_\tau} + \frac{1}{2}\|u_0\|^2_{L^2(\Omega)}. \tag{5.6}$$

We distinguish two cases. If $\|\nabla u_n\|_{L^{p(\cdot)}(\Omega)} > 1$, we get

$$\|\nabla u_n\|^{p^-}_{L^{p(\cdot)}(\Omega)} \le \int_\Omega |\nabla u_n|^{p(x)} dx.$$

Thus, we obtain

$$\|u_n\|_{L^{p^-}(0,T;W_0^{1,p(\cdot)}(\Omega))} \le \left(\int_{Q_\tau} |\nabla u_n|^{p(x)} dxdt\right)^{\frac{1}{p^-}},$$

which together with (5.6) imply

$$\int_{Q_\tau} |\nabla u_n|^{p(x)} dxdt$$
$$\le \|f_n\|_{X'_\tau}\left(\int_{Q_\tau} |\nabla u_n|^{p(x)} dxdt\right)^{\frac{1}{p^-}} + \|f_n\|_{X'_\tau}\|\nabla u_n\|_{L^{p(\cdot)}(Q_\tau)} + \frac{1}{2}\|u_0\|^2_{L^2(\Omega)}$$

By the Young inequality we arrive at

$$\int_{Q_\tau} |\nabla u_n|^{p(x)} dxdt$$
$$\le 2\frac{p^- - 1}{p^-}\left(\frac{p^-}{2}\right)^{-\frac{1}{p^- - 1}} \|f_n\|^{\frac{p^-}{p^- - 1}}_{X'_\tau} + 2\|f_n\|_{X'_\tau}\|\nabla u_n\|_{L^{p(\cdot)}(Q_\tau)} + \|u_0\|^2_{L^2(\Omega)}$$

Since $\int_{Q_\tau} |\nabla u_n|^{p(x)} dxdt$ is greater than either $\|\nabla u_n\|^{p^+}_{L^{p(\cdot)}(Q_\tau)}$ or $\|\nabla u_n\|^{p^-}_{L^{p(\cdot)}(Q_\tau)}$ according to whether the norm $\|\nabla u_n\|_{L^{p(\cdot)}(Q_\tau)}$ is greater or less than the unity, one can use again the Young inequality in the right-hand side of the above inequality with either the exponent p^+ or p^- to see that $\|\nabla u_n\|_{L^{p(\cdot)}(Q_\tau)}$ is bounded uniformly in n.

While if $\|\nabla u_n\|_{L^{p(\cdot)}(\Omega)} \leq 1$, then

$$\int_\Omega |\nabla u_n|^{p(x)} dx \leq 1.$$

Thus, we get

$$\|\nabla u_n\|_{L^{p(\cdot)}(Q_\tau)} \leq 1 + T.$$

Therefore, in both cases having in mind that $\|f_n\|_{X'_\tau}$ is uniformly bounded in n, we conclude that there is a constant $c > 0$, not depending on n, such that

$$|u_n|_X \leq c. \tag{5.7}$$

From now on, we denote by c various positive real numbers, not depending on n, which may vary from line to line. Thanks to (5.5), we get

$$\|u_n\|_{L^\infty(0,T;L^2(\Omega))} \leq c.$$

Going back to (5.4), one has

$$\int_{Q_\tau} a(x, t, \nabla u_n) \cdot \nabla u_n \, dx \, dt \leq c \tag{5.8}$$

and

$$0 \leq \int_{Q_\tau} g(x, t, u_n, \nabla u_n) u_n \, dx \, dt \leq c. \tag{5.9}$$

Using (4.4) and (4.5) we can write

$$\begin{aligned}
&\int_{Q_\tau} |g(x, t, u_n, \nabla u_n)| dxdt \\
&\leq \int_{Q_\tau \cap \{|u_n| \leq 1\}} |g(x, t, u_n, \nabla u_n)| dxdt + \int_{Q_\tau} g(x, t, u_n, \nabla u_n) u_n dxdt \\
&\leq b(1)\Big(\|\theta\|_{L^1(Q_\tau)} + \int_{Q_\tau} |\nabla u_n|^{p(x)} dxdt\Big) + c \leq c.
\end{aligned}$$

Hence, the sequence $\{g(\cdot,\cdot,u_n,\nabla u_n)\}_n$ remains bounded in $L^1(Q)$.
Let us define the operator $A : X \to X'$ by

$$Au = -diva(x,t,\nabla u)$$

then,

$$\langle Au, \psi\rangle = \int_Q a(x,t,\nabla u)\nabla\psi \, dx\, dt,$$

for all u, ψ in X.
By (4.1) and (5.7) we have

$$\begin{aligned}&\int_{Q_\tau} |a(x,t,\nabla u_n)|^{p'(x)}\, dx\, dt \\ &\leq 2^{(p')^+-1}\int_Q \Big(C(x,t)^{p'(x)} + |\nabla u_n|^{p(x)}\Big) dxdt,\end{aligned}$$

thus, it follows that

$$\|a(x,t,\nabla u_n)\|_{(L^{p'(\cdot)}(Q))^N} \leq c,$$

which allows us to assert that $\{Au_n\}$ is bounded in X' since for all $\psi \in X$
—$\langle Au_n, \psi\rangle| \leq \int_Q |a(x,t,\nabla u_n)\cdot\nabla\psi|$
$\leq \|a(x,t,\nabla u_n)\|_{(L^{p'(\cdot)}(Q))^N}\|\nabla\psi\|_{(L^{p(\cdot)}(Q))^N}$
$\leq c|\psi|_X$.
Therefore,

$$\frac{\partial u_n}{\partial t} = f - Au_n - g(x,t,u_n,\nabla u_n) \text{ is bounded in } X' + L^1(Q).$$

By virtue of Lemma 3.4, we conclude that there exists a subsequence of (u_n), still indexed by n, a function $u \in X \cap \mathcal{C}(0,T;W^{-1,1}(\Omega))$ and a function $\overline{a} \in (L^{p'(\cdot)}(Q))^N$ such that

$$\begin{aligned}\nabla u_n &\rightharpoonup \nabla u \text{ weakly in } (L^{p(\cdot)}(Q))^N,\\ u_n &\to u \text{ strongly in } L^1(Q) \text{ and a.e. in } Q,\\ u_n &\overset{*}{\rightharpoonup} u \text{ weakly} - * \text{ in } L^\infty(0,T;L^2(\Omega)),\\ a(x,t,\nabla u_n) &\rightharpoonup \overline{a} \text{ weakly in } (L^{p'(x)}(Q))^N.\end{aligned} \tag{5.10}$$

STEP II. Almost everywhere convergence of ∇u_n
Our aim is to prove that

$$\nabla u_n \longrightarrow \nabla u \quad \text{a.e. in } Q.$$

For that, we need apply the following lemma (see [4] for a similar result):

Lemma 5.2. *Assume that (4.1),(4.2) and (4.3) are satisfied and let* (u_n) *be a sequence in* X *such that* $u_n \longrightarrow u$ *weakly in* X *and*

$$n \longrightarrow \infty \lim \int_Q [a(x,t,\nabla u_n) - a(x,t,\nabla u)].\nabla(u_n - u)\, dx\, dt = 0.$$

Then $u_n \longrightarrow u$ *strongly in* X.

We shall prove that for all $k > 0$,

$$\nabla T_k(u_n) \to \nabla T_k(u) \text{ strongly in } (L^{p(\cdot)}(Q))^N. \tag{5.11}$$

For $k > 0$, we define $S_k(t) = \int_0^t T_k(s)ds$, for every $s \in \mathbf{R}$. It's easy to see that $0 \leq S_k(r) \leq k|r|$. Note that $T_k(u_n) \in X \hookrightarrow L^1(0,T;W_0^{1,1}(\Omega))$ and hence $\nabla T_k(u_n) = \nabla u_n \chi_{\{|u_n|<k\}}$, a.e. in Q. Choosing $T_k(u_n)$ as test function in the equation (5.3) and then using (4.5), we obtain

$$\int_\Omega S_k(u_n(T)) - S_k(u_n(0))dx + \int_Q a(x,t,\nabla u_n)\nabla T_k(u_n)dxdt \leq \langle f, T_k(u_n)\rangle,$$

Using (4.3) and (5.8) we get

$$\begin{aligned}
\int_Q |\nabla T_k(u_n)|^{p(x)} &\leq \int_Q a(x,t,\nabla u_n)\cdot \nabla T_k(u_n)dxdt \\
&= \int_{\{|u_n|\leq k\}} a(x,t,\nabla u_n)\cdot \nabla u_n dxdt \\
&\leq c,
\end{aligned}$$

the sequence $\{\nabla T_k(u_n)\}_n$ is uniformly bounded in $(L^{p(\cdot)}(Q))^N$ so that $\nabla T_k(u_n) \rightharpoonup v_k$ weakly in $(L^{p(\cdot)}(Q))^N$. Moreover, an application of Lebesgue's dominated convergence theorem gives $T_k(u_n) \to T_k(u)$ strongly in $L^{p(\cdot)}(Q)$.
Let $\Phi = (\phi_1, \phi_2, \ldots, \phi_N)$ with $\phi_i \in \mathcal{C}_0^\infty(Q)$ for every $i = 1, 2, \ldots, N$. Setting $\phi = -$ div Φ one has $\Phi \in X'$. On one hand

$$\langle \phi, T_k(u_n)\rangle_{X',X} = \int_Q \Phi \cdot \nabla T_k(u_n)dxdt \to \int_Q \Phi \cdot v_k dxdt.$$

On the other hand

$$\begin{aligned}
&\langle \phi, T_k(u_n)\rangle_{X',X} = -\sum_{i=1}^N \int_Q \frac{\partial \phi}{\partial x_i} T_k(u_n)dxdt \\
&\to -\sum_{i=1}^N \int_Q \frac{\partial \phi}{\partial x_i} T_k(u)dxdt = \int_Q \Phi \cdot \nabla T_k(u)dxdt = \langle \phi, T_k(u)\rangle_{X',X}
\end{aligned}$$

Therefore, we get $v_k = \nabla T_k(u)$ a.e. in Q and for all $k > 0$

$$\begin{aligned} \nabla T_k(u_n) &\rightharpoonup \nabla T_k(u) \text{ weakly in } (L^{p(\cdot)}(Q))^N, \\ T_k(u_n) &\rightharpoonup T_k(u) \text{ weakly in } X. \end{aligned}$$

By virtue of (5.10) and Lebesgue's dominated convergence theorem we have

$$T_k(u_n) \to T_k(u) \text{ strongly in } L^{p(\cdot)}(Q)\text{a.e. in } Q,\ \forall k > 0.$$

Fix $k > 0$ and let $\varphi(s) = se^{\delta s^2}$, $(\delta > 0)$. It's easy to check that if $\delta \geq \left(\frac{b(k)}{2}\right)^2$, one has

$$\varphi'(s) - b(k)|\varphi(s)| \geq \frac{1}{2}, \quad \forall s \in \mathbf{R}.$$

We define the mollification with respect to time of $T_k(u)$ given by

$$T_k(u)_\mu(x,t) = \mu \int_{-\infty}^{t} T_k(u)(x,s)e^{\mu(s-t)}ds,$$

extending $T_k(u)$ by zero for $s < 0$. Observe that $T_k(u)_\mu \in X \cap L^\infty(Q)$ verifying
$|T_k(u)_\mu(x,t)| \leq k(1 - e^{-\mu t})$ and $\frac{\partial T_k(u)_\mu}{\partial t} = \mu(T_k(u) - T_k(u)_\mu(x,t))$ a.e. in Q.
Let $(\psi_i) \subset C_0^\infty(\Omega)$ such that $\psi_i \to u_0$ in $L^2(\Omega)$. Set $\omega_\mu^i = T_k(u)_\mu + e^{-\mu t}T_k(\psi_i)$. Note that ω_μ^i is a smooth function having the following properties (see [12])

$$\frac{\partial}{\partial t}(\omega_\mu^i) = \mu(T_k(u) - \omega_\mu^i), \quad \omega_\mu^i(0) = T_k(\psi_i), \quad |\omega_\mu^i| \leq k.$$

Moreover, we can easily check that

$$\omega_\mu^i \to T_k(u) \text{ a.e. in } Q, \text{weak-}* \text{ in } L^\infty(Q) \text{ and strongly in } X,\ \text{as}\mu \to \infty.$$

Using $z_n^{\mu,i} = \varphi(T_k(u_n) - \omega_\mu^i)$ which belong to $X \cap L^\infty(Q)$ as test function in (5.3) we get

$$\begin{aligned} &\langle u_n', z_n^{\mu,i}\rangle + \int_Q a(x,t,\nabla u_n)(\nabla T_k(u_n) - \nabla\omega_\mu^i)\varphi'(T_k(u_n) - \omega_\mu^i)dxdt \\ &+ \int_Q g(x,t,u_n,\nabla u_n)\varphi(T_k(u_n) - \omega_\mu^i)\,dx\,dt = \langle f, z_n^{\mu,i}\rangle, \end{aligned}$$

which implies by the fact $g(x,t,u_n,\nabla u_n)\varphi(T_k(u_n)-\omega^i_\mu)\geq 0$ on $\{(x,t)\in Q : |u_n(x,t)|>k\}$

$$\begin{aligned}
\langle u'_n, z_n^{\mu,i}\rangle &+ \int_Q a(x,t,\nabla u_n)(\nabla T_k(u_n)-\nabla\omega^i_\mu)\varphi'(T_k(u_n)-\omega^i_\mu)\,dx\,dt\\
&+ \int_{\{|u_n|>k\}} g(x,t,u_n,\nabla u_n)\varphi(T_k(u_n)-\omega^i_\mu)\,dx\,dt \leq \langle f, z_n^{\mu,i}\rangle.
\end{aligned}\tag{5.12}$$

Since $f\in X'$, $T_k(u_n)-\omega^i_\mu \rightharpoonup T_k(u)-\omega^i_\mu$ weakly in X as $n\to\infty$ and $T_k(u)_\mu + e^{-\mu t}T_k(\psi_i)\longrightarrow T_k(u)$ in X as $\mu\longrightarrow\infty$, we have

$$\langle f, z_n^{\mu,i}\rangle = \varepsilon(n,\mu).\tag{5.13}$$

By setting $G_k(s)=s-T_k(s)$, we have:

$$\begin{aligned}
\langle u'_n, z_n^{\mu,i}\rangle &= \int_Q u'_n\varphi(T_k(u_n)-\omega^i_\mu)\,dx\,dt\\
&= \int_Q \left((T_k(u_n))' + (G_k(u_n))'\right)\varphi(T_k(u_n)-\omega^i_\mu)\,dx\,dt\\
&= \int_Q (T_k(u_n)-\omega^i_\mu)'\varphi(T_k(u_n)-\omega^i_\mu)\,dx\,dt\\
&+ \int_Q (\omega^i_\mu)'\varphi(T_k(u_n)-\omega^i_\mu)\,dx\,dt\\
&+ \int_Q (G_k(u_n)'\varphi(T_k(u_n)-\omega^i_\mu)\,dx\,dt\\
&= J_1+J_2+J_3.
\end{aligned}$$

Let $\Phi(s)=\displaystyle\int_0^s \varphi(r)\,dr$, remarking that $\Phi(s)\geq 0$, one has

$$J_1=\left[\int_\Omega \Phi(T_k(u_n)-\omega^i_\mu)(t)\,dx\right]_0^T \geq -\int_\Omega \Phi(T_k(u_n)(0)-T_k(\psi_i))\,dx i\to\infty\longrightarrow 0,$$

then $J_1\geq\varepsilon(i)$. On other hand, since $(\omega^i_\mu)'=\mu(T_k(u_n)-\omega^i_\mu)$ and $\varphi(s)s\geq 0$, we have

$$J_2\geq \mu\int_Q (T_k(u)-T_k(u_n)\varphi(T_k(u_n)-\omega^i_\mu)\,dx\,dt n\to\infty\longrightarrow 0,$$

hence $J_2\geq\varepsilon(n)$. For what concerns J_3, one has by integrating by parts

$$\begin{aligned}
J_3 &= -\int_Q G_k(u_n)\varphi'(T_k(u_n)-\omega^i_\mu)(T_k(u_n)-\omega^i_\mu)'\,dx\,dt\\
&+ \left[\int_\Omega G_k(u_n)\varphi(T_k(u_n)-\omega^i_\mu)\right]_0^T,
\end{aligned}$$

if we take in consideration that $(T_k(u_n))' = 0$ on $\{|u_n| > k\}$ and $G_k(u_n) = 0$ on $\{|u_n| \leq k\}$, we have

$$\left[\int_\Omega G_k(u_n)\varphi(T_k(u_n) - T_k(u_n))\right]_0^T \geq -\int_\Omega G_k(u_0)\varphi(T_k(u_0) - T_k(\psi_i))\,dx,$$

thus

$$\begin{array}{rcl} J_3 & \geq & -\int_Q G_k(u_n)\varphi'(T_k(u_n) - \omega^i_\mu)(T_k(u_n) - \omega^i_\mu)'\,dx\,dt \\ & & +\int_\Omega G_k(u_0)\varphi(T_k(u_0) - T_k(\psi_i))\,dx \\ & = & \mu - \int_Q G_k(u_n)\varphi'(T_k(u_n) - \omega^i_\mu)(\omega^i_\mu)'\,dx\,dt \\ & & +\int_\Omega G_k(u_0)\varphi(T_k(u_0) - T_k(\psi_i))\,dx \\ & & n \to \infty \longrightarrow \mu - \int_Q G_k(u_n)\varphi'(T_k(u) - \omega^i_\mu)(\omega^i_\mu)'\,dx\,dt \\ & & +\int_\Omega G_k(u_0)\varphi(T_k(u_0) - T_k(\psi_i))\,dx \\ & \geq & -\int_\Omega G_k(u_0)\varphi(T_k(u_0) - T_k(\psi_i))\,dx i \to \infty \longrightarrow 0. \end{array}$$

by consequent $J_3 \geq \varepsilon(n,i)$. Combining all these estimates, we get

$$\langle \frac{\partial u_n}{\partial t}, \varphi(T_k(u_n) - \omega^i_\mu)\rangle \geq \varepsilon(n,i). \tag{5.14}$$

Let $s > 0$ and set $Q^s := \{(x,t) \in Q; |\nabla T_k(u(x,t))| \leq s\}$. Denoting by χ_s the characteristic function of Q^s, the second term of the left-hand side of (5.12) reads as

$$\begin{array}{l} \int_Q [a(x,t,\nabla T_k(u_n)) - a(x,t,\nabla T_k(u)\chi_s)][\nabla T_k(u_n) - \nabla T_k(u)\chi_s] \\ \varphi'(T_k(u_n) - \omega^i_\mu)\,dx\,dt + \int_Q a(x,t,\nabla T_k(u))(\nabla T_k(u_n) - \nabla T_k(u)\chi_s) \\ \varphi'(T_k(u_n) - \omega^i_\mu)\,dx\,dt + \int_Q a(x,t,\nabla T_k(u_n))\nabla T_k(u)\chi_s \\ \varphi'(T_k(u_n) - \omega^i_\mu)\,dx\,dt - \int_Q a(x,t,\nabla u_n)\nabla\omega^i_\mu\chi_s\varphi'(T_k(u_n) - \omega^i_\mu)\,dx\,dt \\ = I_1 + I_2 + I_3 + I_4. \end{array}$$

We have

$$a(x,t,T_k(u_n),\nabla T_k(u))\varphi'(T_k(u_n) - \omega^i_\mu) n \to \infty \to a(x,t,T_k(u),\nabla T_k(u))$$

$$\varphi'(T_k(u) - \omega^i_\mu)$$

strongly in $L^{p'(\cdot)}(Q)$.

By the fact that $\dfrac{\partial T_k(u_n)}{\partial x_i} n \to \infty \longrightarrow \dfrac{\partial T_k(u)}{\partial x_i}$ weakly in $L^{p(\cdot)}(Q)$ and that $\varphi'(T_k(u) - T_k(u)_\mu - e^{-\mu t}T_k(\psi_i)) \to 1$ a.e. in Q and is uniformly bounded by $\varphi'(2k)$ and by using Lebesgue's theorem, we can write

$$\begin{aligned} J_2 = & \int_Q a(x,t,\nabla T_k(u))(\nabla T_k(u) - \nabla T_k(u)\chi_s)\varphi'(T_k(u) - \omega_\mu^i)\,dx\,dt + \varepsilon(n) \\ & = \int_{Q\setminus Q^s} a(x,t,0)\nabla T_k(u)\,dx\,dt + \varepsilon(n,\mu), \end{aligned}$$

by letting $s \to \infty$, we conclude that $J_2 = \varepsilon(n,\mu,s)$. About j_3, we have

$$\begin{aligned} J_3 = & \int_{\{|u_n|\le k\}} a(x,t,\nabla u_n)\nabla T_k(u)\chi_s\varphi'(T_k(u_n) - \omega_\mu^i)\,dx\,dt \\ & + \int_{\{|u_n|>k\}} a(x,t,0)\nabla T_k(u)\chi_s\varphi'(T_k(u_n) - \omega_\mu^i)\,dx\,dt, \end{aligned}$$

by letting $n \to \infty$ and due to the weak convergence of $a(x,t,\nabla u_n)$ to $\overline{a}$ in $(L^{p'(\cdot)}(Q))^N$, we have

$$\begin{aligned} J_3 = & \int_{\{|u|\le k\}} \overline{a}\nabla T_k(u)\chi_s\varphi'(T_k(u) - \omega_\mu^i)\,dx\,dt \\ & + \int_{\{|u|>k\}} a(x,t,0)\nabla T_k(u).\chi_s\varphi'(T_k(u) - \omega_\mu^i)\,dx\,dt + \varepsilon(n), \end{aligned}$$

in which we can let $\mu \to \infty$ to obtain $J_3 = \int_Q \overline{a}\nabla T_k(u)\xi_s\,dx\,dt + \varepsilon(n,\mu)$, consequently, by letting $s \to \infty$,

$$J_3 = \int_Q \overline{a}\nabla T_k(u)\xi_s\,dx\,dt + \varepsilon(n,\mu,s).$$

For j_4, we have as above, by letting first n then μ go to infinity

$$\begin{aligned} J_4 &= \int_Q \overline{a}\nabla\omega_\mu^i\varphi'(T_k(u) - \omega_\mu^i)\,dx\,dt + \varepsilon(n) \\ &= -\int_Q \overline{a}\nabla T_k(u)\,dx\,dt + \varepsilon(n,\mu). \end{aligned}$$

Thus

$$\begin{aligned} & \int_Q a(x,t,\nabla u_n)[\nabla T_k(u_n) - \nabla\omega_\mu^i]\varphi'(T_k(u_n) - \omega_\mu^i)\,dx\,dt \\ (5.15) = & \int_Q [a(x,t,\nabla T_k(u_n)) - a(x,t,\nabla T_k(u)\chi_s)][\nabla T_k(u_n) - \nabla T_k(u)\chi_s] \\ & \varphi'(T_k(u_n) - \omega_\mu^i)\,dx\,dt + \varepsilon(n,\mu,s). \end{aligned}$$

The third term of the left-hand of (5.12) can be estimated as

$$\begin{aligned} & \left| \int_{\{|u|\le k\}} g(x,t,u_n,\nabla u_n)\varphi(T_k(u_n) - \omega_\mu^i)dxdt \right| \\ (5.16) \qquad & \le b(k)\int_Q \Big(\theta(x,t) + |\nabla u_n|^{p(x)}\Big)|\varphi(T_k(u_n) - \omega_\mu^i)|dxdt \\ & + b(k)\int_Q a(x,t,\nabla T_k(u_n))\nabla T_k(u)|\varphi(T_k(u_n) - \omega_\mu^i)|dxdt \end{aligned}$$

since $\theta(x,t) \in L^1(Q)$ and by estimation of (5.7), we have

$$b(k)\int_Q \Big(\theta(x,t) + |\nabla u_n|^{p(x)}\Big)|\varphi(T_k(u_n) - \omega_\mu^i)|\,dx\,dt = \varepsilon(n,\mu).$$

The second term of (5.16) reads as

$$\begin{aligned}
& b(k)\int_Q\Big(\theta(x,t)+|\nabla u_n|^{p(x)}\Big)|\varphi(T_k(u_n)-\omega_\mu^i)|\,dx\,dt\\
= \;& b(k)\int_Q[a(x,t,\nabla T_k(u_n))-a(x,t,\nabla T_k(u)\chi_s)][\nabla T_k(u_n)-\nabla T_k(u).\chi_s]\\
& |\varphi(T_k(u_n)-\omega_\mu^i)|\,dx\,dt+(k)\int_Q a(x,t,\nabla T_k(u)\chi_s)]\\
& [\nabla T_k(u_n)-\nabla T_k(u)\chi_s]|\varphi(T_k(u_n)-\omega_\mu^i)|\,dx\,dt\\
& +b(k)\int_Q a(x,t,\nabla T_k(u_n))\nabla T_k(u)\chi_s|\varphi(T_k(u_n)-\omega_\mu^i)|\,dx\,dt
\end{aligned}$$

As above, we can write

$$\begin{aligned}
& \left|\int_{\{|u|\le k\}} g(x,t,u_n,\nabla u_n)\varphi(T_k(u_n)-\omega_\mu^i)\,dx\,dt\right|\\
& \le b(k)\int_Q[a(x,t,\nabla T_k(u_n))-a(x,t,\nabla T_k(u)\chi_s)][\nabla T_k(u_n)-\nabla T_k(u)\chi_s]\\
& |\varphi(T_k(u_n)-\omega_\mu^i)|\,dx\,dt+\varepsilon(n,\mu).
\end{aligned}$$

Combining (5.12), (5.13), (5.14) and (5.15) we get

$$\begin{aligned}
& \int_Q\Big[a(x,t,\nabla T_k(u_n))-a(x,t,\nabla T_k(u)\chi_s)\Big]\Big[\nabla T_k(u_n)-\nabla T_k(u)\chi_s\Big]\\
& \Big[\varphi'(T_k(u_n)-\omega_\mu^i)-b(k)|\varphi(T_k(u_n)-\omega_\mu^i)|\Big]\,dx\,dt\le\varepsilon(n,\mu,i,s),
\end{aligned}$$

by fact that $\varphi'(s)-b(k)|\varphi(s)|\ge\dfrac{1}{2}\quad\forall s\in\mathbf{R}$, we have

$$\int_Q\Big[a(x,t,\nabla T_k(u_n))-a(x,t,\nabla T_k(u)\chi_s)\Big]\Big[\nabla T_k(u_n)-\nabla T_k(u)\chi_s\Big]$$

$$dx\,dt\le 2\varepsilon(n,\mu,i,s).$$

On other hand, we have with (4.2) for $r\le s$

$$\begin{aligned}
0\;& \le\int_{Q^r}\Big[a(x,t,\nabla T_k(u_n))-a(x,t,\nabla T_k(u))\Big]\Big[\nabla T_k(u_n)-\nabla T_k(u)\Big]\,dx\,dt\\
& \le\int_{Q^s}\Big[a(x,t,\nabla T_k(u_n))-a(x,t,\nabla T_k(u)\chi_s)\Big]\Big[\nabla T_k(u_n)-\nabla T_k(u)\chi_s\Big]\,dx\,dt\\
& \le\varepsilon(n,\mu,i,s).
\end{aligned}$$

which implies by passing to the limit sup over n that

$$\begin{aligned}
0\;& \le n\to\infty\limsup\int_Q\Big[a(x,t,\nabla T_k(u_n))-a(x,t,\nabla T_k(u)\chi_s)\Big]\\
& \Big[\nabla T_k(u_n)-\nabla T_k(u)\chi_s\Big]\,dx\,dt\\
& \le 2n\to\infty\lim\varepsilon(n,\mu,,i,s),
\end{aligned}$$

in which we let successively $\mu \to \infty$, $i \to \infty$ and $s \to \infty$ to obtain

$\int_{Q^r} \Bigg[a(x,t,\nabla T_k(u_n)) - a(x,t,\nabla T_k(u)) \Bigg] \Bigg[\nabla T_k(u_n) - \nabla T_k(u) \Bigg]$,
$dx\,dt \longrightarrow 0,$ as $n \to \infty$.

Hence by lemma 5.2 we have

$$T_k(u_n) \longrightarrow T_k(u) \quad \text{strongly in } X^r \ \forall k > 0.$$

We also deduce that for a subsequence denoted (u_n)

$$T_k(u_n) \longrightarrow T_k(u) \quad \text{a.e. in } \ Q^r \ \forall k > 0.$$

Since r and k are arbitrary, there exists a diagonal subsequence of (u_n) also denoted (u_n) in r and k such that

$$\nabla u_n \longrightarrow \nabla u \quad \text{a.e. in } \ Q.$$

Since $a(x,t,.)$ and $g(x,t,.,.)$ are continuous, then

$$a(x,t,\nabla u_n) \longrightarrow a(x,t,\nabla u) \quad \text{a.e. in Q.}$$

If we take in consideration that $a(x,t,\nabla u_n)$ is bounded in $\left(L^{p'(\cdot)}(Q)\right)^N$, then by lemma 2.2

$$a(x,t,\nabla u_n) \longrightarrow a(x,t,\nabla u) \quad \text{weakly in } \left(L^{p'(\cdot)}(Q)\right)^N.$$

STEP III. Equi-integrability of $g(x,t,u_n,\nabla u_n)$ on Q

Let $k > 0$ and let E a measurable subset of Q and $\varepsilon > 0$ be fixed. Since g verifies the sign condition, then by using (5.9), we have

$\int_E |g(x,t,u_n,\nabla u_n)|\,dx\,dt$
$= \int_{E\cap\{|u_n|\le k\}} g(x,t,u_n,\nabla u_n)\,dx\,dt + \int_{E\cap\{|u_n|>k\}} g(x,t,u_n,\nabla u_n)\,dx\,dt$
$\le \int_{E\cap\{|u_n|\le k\}} g(x,t,u_n,\nabla u_n)\,dx\,dt + \dfrac{1}{k}\int_E g(x,t,u_n,\nabla u_n)u_n\,dx\,dt$
$\le \int_{E\cap\{|u_n|\le k\}} g(x,t,T_k(u_n),\nabla T_k(u_n))\,dx\,dt + \dfrac{1}{k}c$
$\le b(k)\int_E \theta(x,t)\,dx\,dt + b(k)\int_E |\nabla T_k(u_n)|^{p(x)}\,dx\,dt + \dfrac{c}{k}.$

The strong convergence in (5.11) implies that there exists $\delta > 0$ such that meas(E) $< \delta \Longrightarrow \int_E |\nabla T_k(u_n)|^{p(x)}\,dx\,dt \le \dfrac{\varepsilon}{3}$. Since $\theta \in L^1(Q)$, then

$b(k)\displaystyle\int_E \theta(x,t)\,dx\,dt \leq \frac{\varepsilon}{3}$ and choosing k large nought such that $\dfrac{c}{k} \leq \dfrac{\varepsilon}{3}$.
Hence, there exists $\delta > 0$ such that meas(E) $< \delta \Longrightarrow \displaystyle\int_E |g(x,t,u_n,\nabla u_n)|\,dx\,dt \leq \varepsilon$ $\forall n \in \mathbf{N}$. Thus $|g(x,t,u_n,\nabla u_n)|$ is uniformly equi-integrable on Q.
Recall that $u_n \longrightarrow u$ a.e. in Q and $\nabla u_n \longrightarrow \nabla u$ a.e. in Q, therefore because $g(x,t,.,.)$ is continuous

$$g(x,t,u_n,\nabla u_n) \longrightarrow g(x,t,u,\nabla u) \quad \text{a.e. in } Q.$$

By Vitali's theorem, we obtain

$$g(x,t,u_n,\nabla u_n) \longrightarrow g(x,t,u,\nabla u) \quad \text{strongly in} \quad L^1(Q).$$

Since g is continuous in the two last arguments, we have

$$g(x,t,u_n,\nabla u_n)u_n \rightarrow g(x,t,u,\nabla u)u \quad \text{a.e. in } Q,$$

Moreover, we have $g(x,t,u_n,\nabla u_n)u_n \geq 0$ a.e. it follows by (5.9) and Fatou's lemma that

$$g(x,t,u,\nabla u)u \in L^1(Q).$$

STEP IV. Passage to the limit

Recall that $u \in X \cap C(0,T;W^{-1,1}(\Omega))$ and in particular $u \in X \cap L^2(Q) \cap C([0,T],L^2(\Omega))$. In addition $\frac{\partial u}{\partial t} \in X' + L^1(Q)$. Let $\tau \in (0,T]$. For all $\varphi \in C^1([0,\tau];\mathcal{C}_0^\infty(\Omega))$ with $\varphi(\cdot,t) = 0$ in a neighborhood of T, we can write

$$\int_{Q_\tau} \frac{\partial u_n}{\partial t}\varphi dx = \int_\Omega (u_n(x,\tau)\varphi(x,\tau) - u_n(x,0)\varphi(x,0))dx - \int_{Q_\tau} u_n \frac{\partial \varphi}{\partial t} dx.$$

Since $u_n \rightharpoonup u$ in $L^2(\Omega)$ and $u_n \rightharpoonup u$ in $L^2(Q)$, we obtain

$$\lim_{n\to+\infty} \int_{Q_\tau} \frac{\partial u_n}{\partial t}\varphi dx = \Big[\int_\Omega u\varphi dx\Big]_0^\tau - \int_{Q_\tau} u\frac{\partial \varphi}{\partial t} dx$$

Therefore, passing to the limit in (5.3) we get

$$-\int_{Q_\tau} u\frac{\partial \varphi}{\partial t}dx + \Big[\int_\Omega u\varphi dx\Big]_0^\tau \int_{Q_\tau} a(x,t,\nabla u)\cdot\nabla\varphi dxdt$$

$$+\int_{Q_\tau} g(x,t,u,\nabla u)\varphi dxdt = \langle f,\varphi\rangle_{X',X}. \tag{5.17}$$

Let now $v \in X \cap L^2(Q)$ with $\dfrac{\partial v}{\partial t} \in X'$. By Lemma 3.5, there exists a sequence $\{v_\epsilon\}$ in $C_0^\infty([0,T], C_0^\infty(\Omega))$ such that

$$\begin{aligned} v_\epsilon &\to v \text{ strongly in } X \cap L^2(Q) \text{ and} \\ \frac{\partial v_\epsilon}{\partial t} &\to \frac{\partial v}{\partial t} \text{ in } X' + L^1(Q). \end{aligned}$$

Let $\tau \in (0,T]$. Inserting $\phi = v_\epsilon \chi_{(0,\tau)} \in C^1([0,T]; \mathcal{C}_0^\infty(\Omega))$ as test function in (5.17) we get

$$\begin{aligned} -\int_{Q_\tau} u\frac{\partial v_\epsilon}{\partial t} dx + \left[\int_\Omega u v_\epsilon dx\right]_0^\tau + \int_{Q_\tau} a(x,t,\nabla u)\cdot \nabla v_\epsilon dxdt \\ + \int_{Q_\tau} g(x,t,u,\nabla u) v_\epsilon dxdt = \langle f, v_\epsilon\rangle_{X',X}. \end{aligned}$$

Letting $\epsilon \to 0$, we obtain

$$\begin{aligned} -\langle \frac{\partial v}{\partial t}, u\rangle_{X',X} + \left[\int_\Omega uvdx\right]_0^\tau + \int_{Q_\tau} a(x,t,\nabla u)\cdot \nabla v\, dx\, dt \\ + \int_{Q_\tau} g(x,t,u,\nabla u) v\, dx\, dt = \langle f, v\rangle_{X',X}. \end{aligned}$$

The proof of Theorem 4.1 is completed.

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