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On multi-symmetric functions and transportation polytopes

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Abstract

We present a study of the transportation polytopes appearing in the product rule of elementary multi-symmetric functions introduced by F. Vaccarino.

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1. Introduction

The classical transportation problems in operation research arise from the problem of transporting goods from a set of factories, and a set of consumer centers. Assuming the total supply of the set of factories equals to the total demand of consumer centers, we can optimize the cost of transporting goods (see [4, 6, 7]). Transportation polytopes have an interest in discrete mathematics and also arise naturally in optimization and statistics (see [5, 9, 15, 17]).

A transportation polytope consists of all tables of non-negative real numbers that satisfy certain equations. In this work we only consider the well-known subfamily, the classical transportation polytopes in just two indices, the 2-way transportation polytopes and we use the notation and terminology introduced by Jesus A. De Loera and Edward D. Kim in [2].

Our main motivation comes from the study of the product rule of elementary multi-symmetric functions introduced by F. Vaccarino in [14] and their relationships with transportation polytopes. The classic product rule of multi-symmetric functions and its respective generalization to the quantum case introduced by Diaz and Pariguan in [3], both have an unexplored underlying structure of transportation polytopes. The main goal of this work, see section 4, is to present a first combinatorial description of this structure in the classical case.

2. Review of multi-symmetric functions

In this section, we present a short introduction to elementary multi-symmetric functions. Fix a characteristic zero field $\mathbf{K}$. Consider the action of the symmetric group S_n on $\mathbf{K}^n$ by permutation of vector entries. The quotient space $\mathbf{K}^n/S_n$ is the configuration space of n-unlabeled points with repetitions in $\mathbf{K}$. Polynomials functions on $\mathbf{K}^n/S_n$ may be identified with the algebra $\mathbf{K}[x_1, \cdots, x_n]^{S_n}$ of S_n invariant polynomials in $\mathbf{K}[x_1, \cdots, x_n]$. It is well-known that $\mathbf{K}^n/S_n$ is an n-dimensional affine space; indeed we have an isomorphism of algebras

$$\mathbf{K}[x_1, \cdots, x_n]^{S_n} \equiv \mathbf{K}[e_1, \cdots, e_n],$$

where $\alpha \in [n] = \{1, 2, \cdots, n\}$ and e_α is the elementary symmetric polynomial determined by the identity

$$\prod_{i=1}^{n}(1 + x_i t) = \sum_{\alpha=0}^{n} e_\alpha(x_1, \cdots, x_n)t^\alpha.$$

If we consider polynomial functions over $(\mathbf{K}^d)^n/S_n$, we obtain the ring of multi-symmetric functions, also called the ring of vector symmetric functions or MacMahon's symmetric functions [8], which are given by

$$\mathbf{K}[x_{11},\cdots,x_{1d},x_{22},\cdots,x_{2d},\cdots,x_{n1},\cdots,x_{nd}]^{S_n}.$$

We will denote by $\mathbf{K}[(\mathbf{K}^d)^n]^{S_n}$ to the ring $\mathbf{K}[x_{11},\cdots,x_{nd}]^{S_n}$. The following results due to F. Vaccarino.

Fix $p,n,d\in\mathbf{N}^+$. Let $y_1,\cdots,y_d$ and $t_1,\cdots,t_d$ be independent and commutative variables in $\mathbf{K}$. For $\alpha=(\alpha_1,\cdots,\alpha_p)\in\mathbf{N}^p$ we use the following notation

$$|\alpha|=\sum_{i=1}^{p}\alpha_i,\quad t^\alpha=\prod_{i=1}^{p}t_i^{\alpha_i}.$$

Given a polynomial $f\in\mathbf{K}[y_1,\cdots,y_d]$ and $i\in[n]$, we denote by $f(i)=f(x_{i1},\cdots,x_{id})$ to the polynomial obtained by replacing each appearance of y_j in f by x_{ij}, for $j\in[d]$.

Definition 1. *Fix $\alpha\in\mathbf{N}^p$ such that $|\alpha|\le n$ and $f=(f_1,\cdots,f_p)\in\mathbf{K}[y_1,\cdots,y_d]^p$. The multisymmetric functions $e_\alpha(f)\in\mathbf{K}[(\mathbf{K}^d)^n]^{S_n}$, are given by the identity*

$$\prod_{i=1}^{n}(1+f_1(i)t_1+f_2(i)t_2+\cdots+f_p(i)t_p)=\sum_{|\alpha|\le n}e_\alpha(f)t^\alpha.$$

For $p,q\in\mathbf{N}^+$, we denote by $\mathrm{Map}(\{0\}\cup[p]\times\{0\}\cup[q],\mathbf{N})$ the set of matrices of size $(p+1)\times(q+1)$ which entries are elements of $\mathbf{N}$. The following result provide an explicit formula for the product rule of multi-symmetric functions

Theorem 2. *Fix $p,q,n\in\mathbf{N}^+$, $f\in\mathbf{K}[y_1,\cdots,y_d]^p$ and $g\in\mathbf{K}[y_1,\cdots,y_d]^q$. Let $\alpha\in\mathbf{N}^p$ and $\beta\in\mathbf{N}^q$ be such that $|\alpha|,|\beta|\le n$, then we have*

$$e_\alpha(f)e_\beta(g)=\sum_{\gamma\in L(\alpha,\beta,n)}e_\gamma(f,g,fg),$$

where:

1. *$(f,g,fg)=(f_1,\cdots,f_p,g_1,\cdots,g_q,f_1g_1,\cdots,f_1g_q,f_2g_1,\cdots,f_2g_q,\cdots,f_pg_1,\cdots,f_pg_q)$.*
2. *$L(\alpha,\beta,n)$ is the set of matrices $\gamma\in\mathrm{Map}(\{0\}\cup[p]\times\{0\}\cup[q],\mathbf{N})$ such that*

- $\gamma_{00} = 0$,
- $|\gamma| = \sum_{i=0}^{p}\sum_{j=0}^{q} \gamma_{ij} \leq n$,
- $\sum_{j=0}^{q} \gamma_{ij} = \alpha_i$ *for* $i \in [p]$.
- $\sum_{i=0}^{p} \gamma_{ij} = \beta_j$ *for* $j \in [q]$.

Graphically, a matrix γ is represented as

$$\begin{array}{cc}
\begin{array}{cccccc}
 & \beta_1 & \beta_2 & \beta_3 & & \beta_q \\
 & \uparrow & \uparrow & \uparrow & \cdots & \uparrow
\end{array} & \\
\left[\begin{array}{cccccc}
0 & \gamma_{01} & \gamma_{02} & \gamma_{03} & \cdots & \gamma_{0q} \\
\gamma_{10} & \gamma_{11} & \gamma_{12} & \gamma_{13} & \cdots & \gamma_{1q} \\
\gamma_{20} & \gamma_{21} & \gamma_{22} & \gamma_{23} & \cdots & \gamma_{2q} \\
\vdots & & & & & \vdots \\
\gamma_{p0} & \gamma_{p1} & \gamma_{p2} & \gamma_{p3} & \cdots & \gamma_{pq}
\end{array}\right] &
\begin{array}{c}
 \\ \rightarrow \; \alpha_1 \\ \rightarrow \; \alpha_2 \\ \vdots \\ \rightarrow \; \alpha_p
\end{array}
\end{array}$$

where the arrows $\rightarrow\uparrow$ represent, respectively, row and column sums and the matrix γ will be identify with the vector

$$\vec{\gamma} = (\gamma_{10}, \cdots, \gamma_{p0}, \gamma_{01}, \cdots, \gamma_{0q}, \gamma_{11}, \cdots, \gamma_{1q}, \gamma_{21}, \cdots, \gamma_{2q}, \cdots, \gamma_{p1}, \cdots, \gamma_{pq}).$$

The main goal of this work is the study of the combinatorial structure underlying in the set of matrices $L(\alpha, \beta, n)$ introduced in Theorem 2.

Example 3. *For* $n = 3, \alpha = (2,1), \beta = (1,2), f = (y_1, y_2)$ *and* $g = (y_1y_3, y_2)$, *we have the following identity*

$$e_{(2,1)}(y_1, y_2)e_{(1,2)}(y_1y_3, y_2) = \sum_{\gamma} e_\gamma(y_1, y_2, y_1y_3, y_2, y_1^2y_3, y_1y_2, y_1y_2y_3, y_2^2)$$

where $\gamma = (\gamma_{10}, \gamma_{20}, \gamma_{01}, \gamma_{02}, \gamma_{11}, \gamma_{12}, \gamma_{21}, \gamma_{22}) \in \mathbf{N}^8$ *is such that* $|\gamma| \leq 3$ *and:*

$$\begin{array}{ll}
\gamma_{10} + \gamma_{11} + \gamma_{12} = 2, & \gamma_{01} + \gamma_{11} + \gamma_{21} = 1, \\
\gamma_{20} + \gamma_{21} + \gamma_{22} = 1, & \gamma_{20} + \gamma_{12} + \gamma_{22} = 2.
\end{array}$$

Finding the solutions we obtain the vectors

$$(0,0,0,0,1,1,0,1), (0,0,0,0,0,2,1,0)$$

then we have that

$$\begin{aligned}
e_{(2,1)}(y_1, y_2)e_{(1,2)}(y_1y_3, y_2) &= e_{(1,1,1)}(y_1^2y_3, y_1y_2, y_2^2) \\
&+ e_{(2,1)}(y_1y_2, y_1y_2y_3).
\end{aligned}$$

3. Classical transportation polytopes

In this section, we review a few needed notions on classical 2-way transportation polytopes and we assume the reader to be somewhat familiar with De Loera and Kim's work [2].

Definition 4. *Fix $p, q \in \mathbf{N}$ and let $u \in \mathbf{R}^p_{\geq 0}$, $v \in \mathbf{R}^q_{\geq 0}$ be two vectors. The transportation polytope P of size $p \times q$ defined by the vectors u and v is the convex polytope on $p \times q$ variables $x_{ij} \in \mathbf{R}_{\geq 0}$, where $i \in [p]$ and $j \in [q]$, which satisfy the $p + q$ equations given by:*

$$\sum_{j=1}^{q} x_{ij} = u_i \text{ and } \sum_{i=1}^{p} x_{ij} = v_j. \tag{3.1}$$

The vectors u and v are called marginals vectors or margins vectors of the polytope P.

These polytopes are called transportation polytopes because they model the transportation of goods from p supply locations to q demand locations.

Example 5. *Let us consider the transportation of goods for 3-supply locations to 3-demand location with suppliyng vector $u = (5, 4, 3)$ and demanding vector $v = (6, 2, 4)$. A point x in the transportation polytope P of size 3×3 defined by the margins u and v is given by*

$$x^* = \begin{bmatrix} x^*_{11} & x^*_{12} & x^*_{13} \\ x^*_{21} & x^*_{22} & x^*_{23} \\ x^*_{31} & x^*_{32} & x^*_{33} \end{bmatrix} = \begin{array}{c} \begin{matrix} 6 & 2 & 4 \\ \uparrow & \uparrow & \uparrow \end{matrix} \\ \begin{bmatrix} 4 & 0 & 1 \\ 1 & 2 & 1 \\ 1 & 0 & 2 \end{bmatrix} \end{array} \begin{array}{l} \\ \\ \rightarrow 5 \\ \rightarrow 4 \\ \rightarrow 3 \end{array}$$

where the horizontal and vertical arrows represent, respectively, row and column sums.

Lemma 6. *Let P be a 2-way transportation polytope of size $p \times q$ defined by the margins $u \in \mathbf{R}^p_{\geq 0}$ and $v \in \mathbf{R}^q_{\geq 0}$. The polytope P is not empty if and only if*

$$\sum_{i \in [p]} u_i = \sum_{j \in [q]} v_j.$$

This proof uses the northwest corner rule algorithm (see [10]).

The equations given in (3.1) and the inequalities $x_{ij} \geq 0$ can be expressed in matrix form as follows

$$P = \{x \in \mathbf{R}^{pq} : Ax = b, x \geq 0\}, \tag{3.2}$$

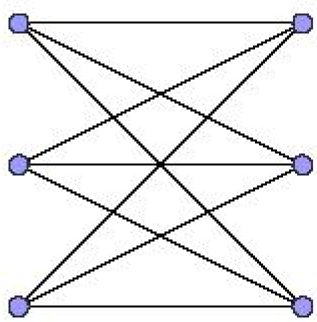
where A is a matrix of size $(p+q) \times pq$ and $b \in \mathbf{R}^{p+q}$. The matrix A is called the constraint matrix.

Transportation polytopes have a relationship with complete bipartite graph $K_{p,q}$ ([11, 16]) of two sets of vertices of U and V of cardinality p and q, respectively, when we consider U the supply and V is the demand.

Definition 7. *The graph $K_{p,q}$ is the complete bipartite graph consisting of two sets U and V of cardinality p and q, respectively such that for any $i \in U$ and $j \in V$ there is an edge e_{ij} connecting them.*

It is well known that the constraint matrix for a $p \times q$ transportation polytope is the vertex-edge incidence matrix of the complete bipartite graph $K_{p,q}$.

Example 8. *Consider the 3×3 transportation polytope P defined by $u = (5,4,3)$ and $v = (6,2,4)$, then the complete bipartite graph $K_{3,3}$ is given by:*



We also have that $P = \{x \in \mathbf{R}^9 : Ax = b, x \geq 0\}$, where the constraint matrix A is given as follows

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \text{ and } b = \begin{bmatrix} 5 \\ 4 \\ 3 \\ 6 \\ 2 \\ 4 \end{bmatrix}.$$

In the Example 5, the solution of $Ax = b$ can be expressed as $x^t = (4, 1, 1, 0, 2, 0, 1, 1, 2)$.

4. Multi-symmetric functions and transportation polytopes

The product rule of elementary multi-symmetric functions given in Theorem 2 involve a set of matrices with some remarkable properties. In this section we will provide some characterizations of the set $L(\alpha, \beta, n)$ in terms of transportation polytopes. In order to simplify our notation we will denote by L to the set $L(\alpha, \beta, n)$ (see Definition 10) and we can think of $\gamma \in L$ as natural points of transportation polytopes P.

In particular, the study of integer points of transportation polytopes is very popular in combinatorics, a lot of mathematical objects rich in combinatorial properties appear when we study integer points in polytopes such as magic squares [1], sudoku arrangements [13], and others.

Definition 9. *Fix $p, q, N \in \mathbf{N}$ and let $u \in \mathbf{N}^{p+1}$, $v \in \mathbf{N}^{q+1}$ be two vectors such that $u_0 = N - \sum_{i=1}^{p} u_i$ and $v_0 = N - \sum_{j=1}^{q} v_j$. The transportation polytope P_N of size $p + 1 \times q + 1$ defined by the vectors u and v is the convex polytope on $p + 1 \times q + 1$ variables $x_{ij} \in \mathbf{R}_{\geq 0}$, where $i \in \{0\} \cup [p]$ and $j \in \{0\} \cup [q]$, which satisfy the $p + q + 2$ equations given by:*

$$\sum_{j=0}^{q} x_{ij} = u_i \text{ and } \sum_{i=0}^{p} x_{ij} = v_j. \tag{4.1}$$

Definition 10. *Fix $p, q, n \in \mathbf{N}$, $\alpha \in \mathbf{N}^p$ and $\beta \in \mathbf{N}^q$. We denote by L the set of matrices $\gamma \in \mathrm{Map}(\{0\} \cup [p] \times \{0\} \cup [q], \mathbf{N})$ which satisfy the equations*

- $\gamma_{00} = 0$.
- $|\gamma| = \sum_{i=0}^{p} \sum_{j=0}^{q} \gamma_{ij} \leq n$.
- $\sum_{j=0}^{q} \gamma_{ij} = \alpha_i$ *for* $i \in [p]$.
- $\sum_{i=0}^{p} \gamma_{ij} = \beta_j$ *for* $j \in [q]$.

Example 11. *For $\alpha = (2,1), \beta = (1,2)$ and $n = 3$, the set L is given by:*

$$L = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix} \right\}.$$

We denote by L_N the subset of L given by:

$$L_N = \{\gamma \in L : |\gamma_{ij}| = N, \text{ for some } N \leq n\} \tag{4.2}$$

The following result provides some combinatorial properties of L_N.

Theorem 12. *The following identities holds*

1. $L_N \neq \emptyset$ *if* $\max\{|\alpha|, |\beta|\} \leq N \leq |\alpha| + |\beta|$.
2. $L = \displaystyle\bigsqcup_{N=\max\{|\alpha|,|\beta|\}}^{n} L_N$, *if* $n < |\alpha| + |\beta|$.
3. $L = \displaystyle\bigsqcup_{N=\max\{|\alpha|,|\beta|\}}^{|\alpha|+|\beta|} L_N$, *if* $n \geq |\alpha| + |\beta|$.

Proof. Fix N such that $\max\{|\alpha|, |\beta|\} \leq N \leq |\alpha| + |\beta|$. We are going to construct an element γ such that $\gamma \in L_N$ as follows: Let $\gamma \in L$ such that $(\gamma_{01}, \gamma_{02}, \cdots, \gamma_{0q})$ be a q-weak composition of α_0 which satisfy $\gamma_{0j} \leq \beta_j$ $\forall j \in [q]$, and let $(\gamma_{10}, \gamma_{20}, \cdots, \gamma_{p0})$ be a p-weak composition of β_0 which satisfy $\gamma_{i0} \leq \alpha_i$ $\forall i \in [p]$.

Denote by $\beta_j^{(k)} := \beta_j - \displaystyle\sum_{i=1}^{k-1} \gamma_{ij}$, for $(k, j) \in [p] \times [q]$ and let $(\gamma_{11}, \gamma_{12}, \cdots, \gamma_{1q})$ be a q-weak composition of $\alpha_1 - \gamma_{10}$ which satisfy $\gamma_{1j} \leq \beta_j^{(1)}$. Analogously we consider $(\gamma_{21}, \gamma_{22}, \cdots, \gamma_{2q})$ a q-weak composition of $\alpha_2 - \gamma_{20}$ such that $\gamma_{2j} \leq \beta_j^{(2)}$. Let's go through this process until we get $(\gamma_{p1}, \gamma_{p2}, \cdots, \gamma_{pq})$ a q-weak composition of $\alpha_p - \gamma_{p0}$ with $\gamma_{pj} \leq \beta_j^{(p)}$ and finally under this construction the reader can check that $\gamma_{ij} = \gamma \in L_N$, therefore $L_N \neq \emptyset$.

It is not difficult to check statements 2 and 3. $\square$

Example 13. *The set L defined by vectors $\alpha = (1,1)$, $\beta = (2,1)$ and $n = 4$ is given by*

$$L = \left\{ \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \right\}$$

We have that $L = \bigsqcup_{N=3}^{4} L_N$*, where* L_3 *and* L_4 *are given by*

$$L_3 = \left\{ \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}.$$

and

$$L_4 = \left\{ \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \right\}.$$

The following result shows that L_N is a set of natural points in some transportation polytope.

Theorem 14. *There is a transportation polytope* P_M *such that* $L_N \subset P_M$*.*

Proof. Let $\gamma \in L_N$, then γ satisfy the equations given in Definition 10. Under the assumptions of Definition 9, consider the transportation polytope P_M defined by margins $\overline{\alpha} \in \mathbf{N}^{p+1}$ and $\overline{\beta} \in \mathbf{N}^{q+1}$ such that

- $\overline{\alpha} = (\alpha_0, \alpha_1, \cdots, \alpha_p)$ with $\alpha_0 = M - \sum_{i=1}^{p} \alpha_i$.
- $\overline{\beta} = (\beta_0, \beta_1, \cdots, \beta_q)$ with $\beta_0 = M - \sum_{i=1}^{q} \beta_i$.

It should be clear that $\gamma \in P_M$ if $M = N$. $\square$

We make a few remarks regarding to Theorem 14. Elements $\gamma \in L_N$ are such that $|\gamma| = N$ and $\gamma \in L(\alpha, \beta, n) = L$, hence for $p, q, n \in \mathbf{N}^+$, $\alpha \in \mathbf{N}^p$ and $\beta \in \mathbf{N}^q$, γ satisfy the conditions of Theorem 2. To find

the transportation polytope P_M such that $L_N \subset P_M$, we consider the transportation polytope defined by margins $\overline{\alpha}$ and $\overline{\beta}$ which are obtained from α and β adding new inputs α_0, β_0 satisfying the condition given above. We stress that we will work with the transportation polytope P_N which follows this previous construction.
This previous considerations imply our next result which establishes an example of the transportation polytopes associated with sets L_3 and L_4 given in Example 13.

Example 15. *Fix $p = q = 2$, $N = 3$ and consider the vectors $\overline{\alpha} = (1, 1, 1)$, $\overline{\beta} = (0, 2, 1)$. The transportation polytope P_3 defined by margins $\overline{\alpha}, \overline{\beta}$ is given by*

$$P_3 = \left\{ X \in M_{3\times 3}(\mathbf{R}_{\geq 0}) : \sum_{j=1}^{3} x_{ij} = \overline{\alpha}_i \text{ and } \sum_{i=1}^{3} x_{ij} = \overline{\beta}_j \right\},$$

and we have $L_3 \subset P_3$. If we consider $X = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 \end{bmatrix}$ we have that $X \in P_3$ but $x \notin L_3$ and therefore $L_3 \neq P_3$.

On the other hand, fix $p = q = 2$, $N = 4$ and consider the vectors $\overline{\alpha} = (2, 1, 1)$, $\overline{\beta} = (1, 2, 1)$. The transportation polytope P_4 defined by margins $\overline{\alpha}, \overline{\beta}$ is given by

$$P_4 = \left\{ X \in M_{3\times 3}(\mathbf{R}_{\geq 0}) : \sum_{j=1}^{3} x_{ij} = \overline{\alpha}_i \text{ and } \sum_{i=1}^{3} x_{ij} = \overline{\beta}_j \right\},$$

and we have $L_4 \subset P_4$. If we consider $X = \begin{bmatrix} \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{bmatrix}$ we have that $X \in P_4$ but $X \notin L_4$ and therefore $L_4 \neq P_4$.

It is well known that transportation polytopes P can be represented in matrix form, therefore transportation polytopes P_N can be represented in matrix form as well (see Proposition 16). In this case we consider the graph $K'_{p,q}$ obtained from $K_{p,q}$ removing the edge e_{11}. Figure 1 shows the graph $K'_{3,3}$ associated to $K_{3,3}$.

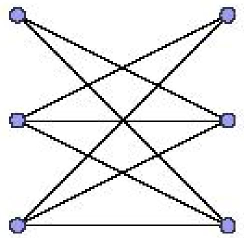


Figure 1: $K'_{3,3}$ graph.

The following result provides the matrix form associated to L_N.

Proposition 16. *For any $N \in \mathbf{N}$, each L_N can be expressed as follows:*

$$L_N = \{x_N \in \mathbf{N}^{(p+1)(q+1)-1} : Ax_N = b_N\},$$

where $b_N = (\alpha_0, \alpha_1, \cdots, \alpha_p, \beta_0, \beta_1, \cdots, \beta_q)$ is such that $\alpha_0 = N - \sum_{i=1}^{p} \alpha_i$, $\beta_0 = N - \sum_{i=1}^{q} \beta_i$, and A is the matrix is obtain by following the next construction

1. *Let B be the constraint matrix of $K'_{p+1,q+1}$, and denote by B^i the i-th column of B, for all i.*
2. *For $i \in [p]$ the i-th column A^i of matrix A is given by $A^i = B^{i(q+1)}$.*
3. *For $i \in [q]$ we have $A^{p+i} = B^i$.*
4. *Last columns of A are obtained from B after rearranging in ascended way the remaining columns.*

Proof. Let P_N be the transportation polytope such that $L_N \subset P_N$. It should be clear that P_N is an special case of 2-way transportation polytope for any $N \in \mathbf{N}$. Observe that for $\gamma \in P_N$ we have $\gamma_{00} = 0$, then P_N can be expressed in matrix form as follows (see equation (3.2))

$$P_N = \{x_N \in \mathbf{R}_{\geq 0}^{(p+1)(q+1)-1} : Bx_N = b_N\},$$

where B is the constraint matrix of the graph $K'_{p+1,q+1}$.

The matrix A obtained from B following the previous construction provides a rearrangement of x_N such that solutions of the equation $Ax_N = b_N$ are vectors $\vec{\gamma}$ which satisfy the conditions of Theorem 2, therefore we have the desired result. $\square$

Example 17. *For $N = 3$ and $b_N = (1, 1, 1, 0, 2, 1)$, we have that*

$$L_3 = \{x_3 \in \mathbf{N}^8 : Ax_N = b_3\},$$

where A is given as follows :

Let B be the constraint matrix of $K'_{3,3}$ given by:

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

Under the assumptions of Proposition 16, for $p = q = 2$, we have that

- $A^1 = B^3$ *and* $A^2 = B^6$,
- $A^3 = B^1$ *and* $A^4 = B^2$,
- *The last four columns of A are given by* $A^5 = B^4, A^6 = B^5, A^7 = B^7$ *and* $A^8 = B^8$.

Then we have

$$B = \begin{array}{c} \begin{matrix} B^1 & B^2 & B^3 & B^4 & B^5 & B^6 & B^7 & B^8 \end{matrix} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \end{array} \longrightarrow A = \begin{array}{c} \begin{matrix} B^3 & B^6 & B^1 & B^2 & B^4 & B^5 & B^7 & B^8 \end{matrix} \\ \begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} \end{array}.$$

Our next goal is to describe the structure of $\mathbf{N}$-matrix of the elements of L_N. To accomplish it, we will require some definitions due to R. Stanley (see [12]). Let $A = (a_{ij})$ be an $\mathbf{N}$-matrix with finitely many nonzero entries, that is A is an $\mathbf{N}$-matrix of finite support and we can think of A as either an infinity matrix or as an $m \times n$ matrix when $a_{ij} = 0$ for $i > m$ and $j > n$. Associate with A a generalized permutation or two-line array ω_A given by

$$\omega_A = \begin{pmatrix} i_1 & i_2 & i_3 & \cdots & i_m \\ j_1 & j_2 & j_3 & \cdots & j_m \end{pmatrix}$$

such that

1. $i_1 \leq i_2 \leq \cdots \leq i_m$.
2. If $i_r = i_s$ and $r \leq s$ then $j_r \leq j_s$.
3. For each pair (i, j), there are exactly a_{ij} values of r for which $(i_r, j_r) = (i, j)$.

A determines a unique two-line array ω_A satisfying this conditions and conversely any such array corresponds to a unique A. For instance, if $A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$, then the corresponding two-line array is $\omega_A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 2 \end{pmatrix}$.

Definition 18. *Fix A an $\mathbf{N}$-matrix and let ω_A be the two-line array associate with A. We denote by* $\mathrm{type}^1(\omega_A)$ *the vector $(u_1, \cdots, u_m)$ such that the natural number k appears exactly u_k times in the first row of ω_A and we denote by* $\mathrm{type}^2(\omega_A)$ *the vector $(v_1, \cdots, v_m)$ such that the natural number k appears exactly v_k times in the second row of ω_A.*

Example 19. *If $\omega_A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 2 \end{pmatrix}$,*
then $\mathrm{type}^1(\omega_A) = (1, 1, 1)$ *and* $\mathrm{type}^2(\omega_A) = (0, 2, 1)$.

Fix $N \in \mathbf{N}$, we denote by ω_N the set of two-line array given by

$$\omega_N = \left\{ \omega_A = \begin{pmatrix} i_1 & i_2 & i_3 & \cdots & i_N \\ j_1 & j_2 & j_3 & \cdots & j_N \end{pmatrix} : (i_1, j_1) \neq (1, 1) \right\}.$$

Theorem 20. *There is a bijection between elements of L_N and elements $\omega_A \in \omega_N$ such that* $\mathrm{type}^1(\omega_A) = \overline{\alpha}$ *and* $\mathrm{type}^2(\omega_A) = \overline{\beta}$.

Proof. Let $\gamma \in L_N$. Using Stanley's construction, we can think of $\gamma \in L_N$ as an $(p+1)\times(q+1)$ matrix when $\gamma_{ij} = 0$ for $i > p+1$ and $j > q+1$, then γ determines a unique two-line array ω_γ satisfying the previous conditions. It should be clear that there is a injective map $L_N \to \omega_N$. On the other hand, note that since the elements $\omega_A \in \omega_N$ are such that $(i_1, j_1) \neq (1,1)$, it follows that $a_{11} = 0$, moreover $\mathrm{type}^1(\omega_A) = \overline{\alpha}$ and $\mathrm{type}^2(\omega_A) = \overline{\beta}$, then $A = (a_{ij})$ is an element of L_N. Thus we conclude that there is a injective map $\omega_N \to L_N$. $\square$

Example 21. *Under the assumptions of Example 13 consider L_4 given by*

$$L_4 = \left\{ \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \right\}.$$

The set ω_4 is given by

$$w_4 = \left\{ \begin{pmatrix} 1 & 1 & 2 & 3 \\ 2 & 2 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 2 & 3 \\ 2 & 2 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 2 & 3 \\ 2 & 3 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 2 & 3 \\ 2 & 3 & 2 & 1 \end{pmatrix} \right\}$$

It is well known that we can associated with an $\mathbf{N}$-matrix A of finite support a pair (P, Q) of semistandard Young tableau (SSYT) of the same shape using the RSK algorithm. The RSK algorithm is a bijection between $\mathbf{N}$-matrices of finite support and ordered pairs (P, Q) of SSYTs of the same shape.

On the other hand, we know that any $\gamma \in L_N$ is an $\mathbf{N}$-matrix of finite support such that $\mathrm{row}(\gamma) = \overline{\alpha}$ and $\mathrm{col}(\gamma) = \overline{\beta}$. Using Theorem 20 we can see that RSK algorithm is a bijection between elements $\gamma \in L_N$ and ordered pairs (P, Q) of SSYTs of the same shape such that $\mathrm{type}(P) = \mathrm{col}(\gamma) = \overline{\beta}$, $\mathrm{type}(Q) = \mathrm{row}(\gamma) = \overline{\alpha}$ and the first box of the last row of P and Q is not equal to 1 simultaneously. Therefore, we can summarize it as follows

Corollary 22. *There is a bijection between L_N and ordered pairs (P, Q) of SSYTs of the same shape such that* $\mathrm{type}(P) = \mathrm{col}(\gamma) = \overline{\beta}$, $\mathrm{type}(Q) = \mathrm{row}(\gamma) = \overline{\alpha}$ *and the first box of the last row of P and Q is not equal to* 1 *simultaneously.*

Example 23. *Let* $\omega_\gamma = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 2 \end{pmatrix}$ *be the two-line array associated with*

$\gamma = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$. *The ordered pairs* (P, Q) *of SSYTs are the following*

$$\left(\begin{array}{|c|c|} \cline{1-1} 3 & \multicolumn{1}{c}{} \\ \hline 2 & 2 \\ \hline \end{array} \, , \, \begin{array}{|c|c|} \cline{1-1} 2 & \multicolumn{1}{c}{} \\ \hline 1 & 3 \\ \hline \end{array} \right)$$

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