Proyecciones (Antofagasta. On line) | vol. 40, n. 4 (2021) | pp. 939-962.

10.22199/issn.0717-6279-4430

PROYECCIONES
Journal of Mathematics

ISSN 0717-6279 (On line)

Basarab loop and the generators of its total multiplication group

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Received: 2020/08/29 | Accepted: 2021/01/29

Abstract:

A loop $(Q, \cdot)$ is called a Basarab loop if the identities: $(x{\cdot}yx^{\rho})(xz) = x{\cdot}\, yz$ and $(yx){\cdot}(x^{\lambda}z\, {\cdot}x) = yz\, {\cdot}x$ hold. It was shown that the left, right and middle nuclei of the Basarab loop coincide, and the nucleus of a Basarab loop is the set of elements x whose middle inner mapping T_x are automorphisms. The generators of the inner mapping group of a Basarab loop were refined in terms of one of the generators of the total inner mapping group of a Basarab loop. Necessary and sufficient condition(s) in terms of the inner mapping group (associators) for a loop to be a Basarab loop were established. It was discovered that in a Basarab loop: the mapping $x \mapsto T_x$ is an endomorphism if and only if the left (right) inner mapping is a left (right) regular mapping. It was established that a Basarab loop is a left and right automorphic loop and that the left and right inner mappings belong to its middle inner mapping group. A Basarab loop was shown to be an automorphic loop (A-loop) if and only if it is a middle automorphic loop (middle Aloop). Some interesting relations involving the generators of the total multiplication group and total inner mapping group of a Basarab loop were derived, and based on these, the generators of the total inner mapping group of a Basarab loop were finetuned. A Basarab loop was shown to be a totally automorphic loop (TA-loop) if and only if it is a commutative and flexible loop. These aforementioned results were used to give a partial answer to a 2013 question and an ostensible solution to a 2015 problem in the case of Basarab loop.

Keywords: Basarab loops; Inner mapping group; Automorphic loop (A-loop).

MSC (2020): 20N02, 20N05.

Cite this article as (IEEE citation style):

T. G. Jaiyéọlá and G. O. Effiong, "Basarab loop and the generators of its total multiplication group", *Proyecciones (Antofagasta, On line)*, vol. 40, no. 4, pp. 939-962, 2021, doi: 10.22199/issn.0717-6279-4430

1. Introduction

Let G be a non-empty set. Define a binary operation $(\cdot)$ on G. If $x \cdot y \in G$ for all $x, y \in G$, then the pair $(G, \cdot)$ is called a *groupoid* or *Magma*.

If each of the equations:

$$a \cdot x = b \qquad and \qquad y \cdot a = b$$

has unique solutions in G for x and y respectively, then $(G, \cdot)$ is called a *quasigroup*.

If there exists a unique element $e \in G$ called the *identity element* such that for all $x \in G$, $x \cdot e = e \cdot x = x$, $(G, \cdot)$ is called a *loop*. We write xy instead of $x \cdot y$, and stipulate that $\cdot$ has lower priority than juxtaposition among factors to be multiplied. For instance, $x \cdot yz$ stands for $x(yz)$.

Let x be a fixed element in a groupoid $(G, \cdot)$. The left and right translation maps of G, L_x and R_x respectively are defined by

$$yL_x = x \cdot y \qquad and \qquad yR_x = y \cdot x.$$

It can now be seen that a groupoid $(G, \cdot)$ is a quasigroup if its left and right translation mappings are permutations. Since the left and right translation mappings of a quasigroup are bijective, then the inverse mappings L_x^{-1} and R_x^{-1} exist. Let

$$x\backslash y = yL_x^{-1} = xM_y \qquad and \qquad x/y = xR_y^{-1} = yM_x^{-1}$$

and note that

$$x\backslash y = z \Leftrightarrow x \cdot z = y \qquad and \qquad x/y = z \Leftrightarrow z \cdot y = x.$$

In a loop $(G, \cdot)$ with identity element e, the *left inverse element* of $x \in G$ is the element $xJ_\lambda = x^\lambda \in G$ such that

$$x^\lambda \cdot x = e$$

while the *right inverse element* of $x \in G$ is the element $xJ_\rho = x^\rho \in G$ such that

$$x \cdot x^\rho = e.$$

If $x^\lambda = x^\rho$ for any $x \in G$, then we simply write $x^\lambda = x^\rho = x^{-1}$ or $J_\lambda = J_\rho = J$. Let a, b and c be three elements of a loop G. The loop associator of a, b and c is the unique element (a, b, c) of G which satisfies $(ab)c = \{a(bc)\}(a, b, c)$. The loop commutator of a and b is the unique element (a, b) of G which satisfies $(ab) = (ba)(a, b)$.

The right nucleus of G is defined by $N_\rho(G, \cdot) = \{a \in G \mid zy \cdot a = z \cdot ya \ \forall \ y, z \in G\}$. The left nucleus of G is defined by $N_\lambda(G, \cdot) = \{a \in G : ax \cdot y = a \cdot xy \ \forall \ x, y \in G\}$. The middle nucleus of G is defined by $N_\mu(G, \cdot) = \{a \in G : ya \cdot x = y \cdot ax \ \forall \ x, y \in G\}$. The nucleus of G is defined by $N(G, \cdot) = N_\lambda(G, \cdot) \cap N_\rho(G, \cdot) \cap N_\mu(G, \cdot)$. The centrum of G is defined by $C(G, \cdot) = \{a \in G : ax = xa \ \forall \ x \in G\}$. The center of G is defined by $Z(G, \cdot) = N(G, \cdot) \cap C(G, \cdot)$. $N_\rho(G, \cdot), N_\lambda(G, \cdot), N_\mu(G, \cdot), N(G, \cdot), Z(G, \cdot)$ are subgroups of $(G, \cdot)$.

The group of all permutations on G is called the permutation group of G and denoted by $SYM(G)$. The group $\mathcal{M}(G, \cdot) = \Big\langle \{R_x, L_x, \ : \ x \in G\} \Big\rangle$ is called the multiplication group of $(G, \cdot)$ and $\mathcal{M}(G, \cdot) \leq SYM(G)$.

If $e\alpha = e$ in a loop G such that $\alpha \in \mathcal{M}(G)$, then α is called an inner mapping and they form a group $Inn(G)$ called the inner mapping group. The right, left and middle inner mappings

$$R_{(x,y)} = R_x R_y R_{xy}^{-1} \ , L_{(x,y)} = L_x L_y L_{yx}^{-1} \ and \ T_x = R_x L_x^{-1}$$

respectively generate the right inner mapping group $Inn_\rho(G)$, left inner mapping group $Inn_\lambda(G)$ and the middle inner mapping group $Inn_\mu(G)$.

The triple (A, B, C) of bijections of a loop $(G, \cdot)$ is called an autotopism if

$$xA \cdot yB = (x \cdot y)C \ \forall \ x, y \in G. \tag{1.1}$$

Such triples form a group $AUT(G, \cdot)$ called the autotopism group of $(G, \cdot)$. Furthermore, if $A = B = C$, then A is called an automorphism of $(G, \cdot)$. Such bijections form a group $AUM(G, \cdot)$ called the automorphism group of $(G, \cdot)$. If

$$Inn_\lambda(G) \leq AUM(G), \ Inn_\rho(G) \leq AUM(G), \ and \ Inn_\mu(G) \leq AUM(G),$$

then G is called a left A-loop(A$_\lambda$-loop), right A-loop(A$_\rho$-loop) and middle A-loop(A$_\mu$-loop) respectively. It is well known that
$Inn(G, \cdot) = \Big\langle R_{(x,y)}, L_{(x,y)}, T_x \mid x, y \in G \Big\rangle$, but this was later improved

to be $Inn(G,\cdot) = \Big\langle L_{(x,y)}, T_x \mid x,y \in G \Big\rangle = \Big\langle R_{(x,y)}, T_x \mid x,y \in G \Big\rangle$ in Vojtěchovský [32].

If $Inn(G,\cdot) \le AUM(G,\cdot)$, then $(G,\cdot)$ is called an automorphic loop (A-loop)

The group $\mathcal{TM}(G,\cdot) = \Big\langle \{R_x, L_x, M_x \ : \ x \in G\} \Big\rangle$ is called the total multiplication group of $(G,\cdot)$ and an element $\alpha \in \mathcal{TM}(G,\cdot)$ such that $e\alpha = e$ is called a total inner mapping of $(G,\cdot)$. It has been established in Stanovský et al. [25] that the total inner mapping group

$$TInn(G,\cdot) = \Big\langle R_{(x,y)}, L_{(x,y)}, T_x, M_{(x,y)}, U_x \mid x,y \in G \Big\rangle, \ where$$

$$M_{(x,y)} = M_y M_x M_{y\backslash x}^{-1}$$

and $U_x = M_x R_x^{-1}$. Although, Syrbu [28] showed that

$$TInn(G,\cdot) = \Big\langle R_{(x,y)}, L_{(x,y)}, T_x, P_{(x,y)}, P'_{(x,y)}, U_x^{-1}, V_x \mid x,y \in G \Big\rangle, \ where$$

$$P_{(x,y)} = M_x M_y L_x R_y^{-1},$$

$P'_{(x,y)} = M_y^{-1} M_x^{-1} R_y L_x^{-1}$ and $V_x = R_x M_x$ but this result was later on improved to $TInn(G,\cdot) = \Big\langle R_{(x,y)}, L_{(x,y)}, T_x, P_{(x,y)}, V_x \mid x,y \in G \Big\rangle$ by Syrbu and Grecu [30]. In an inverse property loop $(G,\cdot)$,

TM(G,·) $= \Big\langle \{L_x, J \ : \ x \in G\} \Big\rangle =< \{M_x \ : \ x \in G\} > \ and$

$TInn(G,\cdot) = \Big\langle Inn(G,\cdot), J \ \Big| \ x,y \in G \Big\rangle = \Big\langle M_{(x,y)} \ \Big| \ x,y \in G \Big\rangle.$

If $TInn(G,\cdot) \le AUM(G,\cdot)$, then $(G,\cdot)$ is called a totally automorphic loop (TA-loop). TA-loops are commutative. Some works in the direction of structure of A-loops and TA-loops can be found in [6, 16, 18, 19, 20, 22, 25, 32]. According to Stanovský et al. [25], $TInn(G,\cdot)$ characterizes normal subloops.

Theorem 1.1. *(Kinyon [18]) A loop is a TA-loop if and only if it is a commutative Moufang loop.*

The above fact is gotten from the result below.

Theorem 1.2. *(Kinyon [18]) Let $(G,\cdot)$ be a loop. The following are equivalent:*

1. $\left\langle L_{x\backslash y}^{-1}M_yM_x, R_{y/x}^{-1}M_y^{-1}M_x^{-1}\ |x,y\in G\right\rangle \le AUM(G,\cdot)$.

2. $(G,\cdot)$ *is both an A-loop and a Moufang loop.*

And Theorem 1.2 led to the following Question 1.1.

Question 1.1. *(Kinyon [18]) What other interesting varieties of loops can be characterized by specifying that some group of total inner mappings acts as automorphisms?*

Question 1.2. *(Stanovský et al. [25]) Are the generating sets of $TInn(G)$ for a loop G minimal, i.e, can any of the five types of mappings be removed? Is there a generating set for $\mathcal{TM}(Q)$ with only two types of inner mappings?*

Remark 1.1. *Syrbu and Grecu [30] were able to provide answers to Question 1.2 by showing that:*

1. *If Q is a power associative loop, then*
   $TInn(Q) = \left\langle R_{(x,y)}, L_{(x,y)}, P_{(x,y)}, V_x \mid x,y\in Q\right\rangle$.

2. *If Q is a middle Bol loop, then*
   $TInn(Q) = \left\langle R_{(x,y)}, P_{(x,y)}, V_x \mid x,y\in Q\right\rangle$.

The authors furthermore established the importance of the total inner mapping group by showing that if Q is a middle Bol loop, then $Inn(Q) \trianglelefteq TInn(Q)$. But this result is not necessarily true for a left or right Bol loop (note that, every middle Bol loop corresponds to a right or left Bol loop and vice versa). It must be recalled that Grecu and Syrbu [13], Syrbu [27], Syrbu and Grecu [29] established isostrophy invariance in several structural properties between middle Bol loops and their corresponding right or left Bol loop.

For an overview of the theory of loops, readers may check [7, 8, 9, 12, 15, 21, 23, 31].

Definition 1.1. *Let $(G,\cdot)$ be a quasigroup. Then*

1. *$U\in SYM(G)$ is called λ-regular if there exists $(U,I,U)\in AUT(G,\cdot)$; the set of all such mappings forms a group $\Lambda(G,\cdot)$ called the group of left regular mappings.*

2. *a bijection U is called ρ-regular if there exists $(I,U,U)\in AUT(G,\cdot)$; the set of all such mappings forms a group $\mathcal{P}(G,\cdot)$ called the group of right regular mappings.*

Definition 1.2. *A loop $(G,\cdot)$ is called a cross inverse property loop(CIPL) if it obeys the identity* $xy\cdot x^\rho = y$ *or* $x\cdot yx^\rho = y$ *or* $x^\lambda\cdot(yx) = y$ *or* $x^\lambda y\cdot x = y$
for all $x,y,\in G$.

A loop $(G,\cdot)$ is called an automorphic inverse property loop(AIPL) if it obeys the identity
$(xy)^\rho = x^\rho y^\rho$ *or* $(xy)^\lambda = x^\lambda y^\lambda$
for all $x,y,\in G$.

A loop satisfying the identical relation $xy\cdot zx = (x\cdot yz)x$ is called a Moufang loop.

A loop $(Q,\cdot)$ is called a Basarab loop (or K-loop), if the identities:

$$\underbrace{(x\cdot yx^\rho)\cdot xz = x\cdot yz}_{BK1},\quad \underbrace{yx\cdot(x^\lambda z\cdot x) = yz\cdot x}_{BK2} \tag{1.2}$$

hold for all $x,y,z\in Q$.

Example 1.1. *(Basarab [4])*

Let $\mathcal{F}$ be a field, $\mathcal{F}'$ be the set of non-zero elements of $\mathcal{F}$. Define on the set $\mathcal{Q}=\mathcal{F}'\times\mathcal{F}$ the operation $(\cdot)$ as follows:

$$(a,x)\cdot(b,y) = \Big(a\cdot b,\ (a^{-1}-1)\cdot(b^{-1}-1)+b^{-1}x+y\Big).$$

Then, $(\mathcal{Q},\cdot)$ is a Basarab loop.

In recent years, many works have been published with the name K-loop(also called Bruck-loop, Bol-Bruck-loop). Kerby and Wefelscheid investigated the additive structure of a near-domain with extra axioms and then they called the new structure a K-loop, but according to Kiechle [14], they used the term K-loop only in talks in 1970s and the beginning of 1980s. Ungar [26] and Kiechle [14] continued to study the K-loop introduced by Kerby and Wefelscheid.

On the other hand, the term ‘K-loop’ was used for different purposes by Soikis [24] in 1970, and latter but independently by Basarab [2] in 1992. Thus, the notion of ‘K-loop’ as used by Soikis, Ungar and Basarab has different meaning respectively, and may be confusing. For example, the book titled *theory of k-loops* written by Kiechle [14] has completely different meaning from the paper titled *K-loops* published by Basarab [2].

Basarab loops (also called K-loops) are non-associative generalizations of groups. In this paper, we shall adopt the name 'Basarab loop', to refer to 'K-loop' of Basarab, as recommended by R.Artzy in the review of Basarab [4].

The first publications introducing the class of loop called Basarab loop are the two prominent papers of Basarab [1, 2] in 1992. Basarab [4] used the result published by Belousov [5] to construct an example of a Basarab loop whose nucleus is an abelian group. After his first publications in 1992 on Basarab loops, no other author researched on properties of Basarab loops until in 1996 and 1997, that he, Basarab studied the relationship between a generalized Moufang loop, Osborn loop, VD-loop, and a Basarab loop, and a special type of a Basarab loop, known as IK-loop respectively. In Jaiyéọlá and Effiong [17], the authors considered the Basarab loop and its invariance with inverse properties. Effiong et al. [10, 11] respectively investigated the connections between Basarab loop and Buchsteiner loop, and the holomorphy of Basarab loop. In this current study, an IK-loop is an automorphic inverse property Basarab loop.

Here are some existing results on Basarab loop.

Theorem 1.3. *(Basarab [4]) Let $(Q, \cdot)$ be a Basarab loop.*

1. *$N(Q, \cdot)$ contains the associator of any three elements of Q.*
2. *The quotient loop $Q/Z(Q, \cdot)$ is a group.*
3. *If $(Q, \cdot)$ is generated by one element, then it is solvable.*
4. *If $(Q, \cdot)$ has the automorphic inverse property, then it is nilpotent.*

Theorem 1.4. *(Basarab [2]) Let $(Q, \cdot)$ be a Basarab loop.*

1. *$N(Q, \cdot)$ is a nontrivial normal subloop.*
2. *The quotient loop $Q/N(Q, \cdot)$ is an abelian.*
3. *If $N(Q, \cdot)$ has an odd order, then $(Q, \cdot)$ is solvable.*

Theorem 1.5. *(Basarab [3])*

1. *Any Basarab loop (any VD-loop) is a G-loop.*

2. *Any Basarab loop (VD-loop) is an Osborn loop.*

3. *A Basarab loop $(Q,\cdot)$ is a VD-loop if $x^2 \in N(Q,\cdot)$ for any $x \in Q$.*

4. *A VD-loop $(Q,\cdot)$ is a Basarab loop if $x^2 \in N(Q,\cdot)$ for any $x \in Q$.*

In this present work, the nuclei of Basarab loops are characterized in terms of middle inner mappings, associators in Basarab loops are expressed in terms of total inner mappings and necessary and sufficient condition on generator(s) of the inner mapping group is given in order for a Basarab loop to be an A-loop. Some results on the generators of the inner mapping and total inner mapping groups of a Basarab loop are established in order to show that a class of total inner mappings act on a Basarab loop Q by automorphisms if and only if Q is an A-loop and flexible.

2. Main Results

2.1. Some Algebraic properties of Basarab loop

Lemma 2.1. *Let $(Q,\cdot)$ be a Basarab loop, then the following hold for all $x,y,z \in Q$:*

(i) $(x \cdot yx^\rho)x = xy$.

(ii) $(x \cdot yx^\rho) \cdot xy^\rho = x$.

(iii) $(x \cdot z^\lambda x^\rho) \cdot xz = x$.

(iv) $(z^\lambda x)(x^\lambda z \cdot x) = x$.

(v) $x(x^\lambda z \cdot x) = zx$.

(vi) $yx \cdot (x^\lambda y^\rho \cdot x) = x$.

(vii) $T_x^{-1} = R_{x^\rho} L_x$.

(viii) $T_x = L_{x^\lambda} R_x$.

(ix) $[L_{x^\lambda} R_x, R_{x^\rho} L_x] = I$.

(x) $x \cdot (x^\lambda y \cdot x)x^\rho = y$.

(xi) $x^\lambda(x \cdot yx^\rho) \cdot x = y$.

Proof. The proof of (i) to (vi) are gotten from (1.2) by making the following substitutions: $z = e, z = y^\rho, y = z^\lambda$ in BK1 and $y = z^\lambda, y = e, z = y^\rho$ in BK2.
(vii) follows from (i): $R_{x^\rho}L_xR_x = L_x \Leftrightarrow R_{x^\rho}L_x = L_xR_x^{-1} \Leftrightarrow T_x^{-1} = R_{x^\rho}L_x$.
(viii) follows from (v): $L_{x^\lambda}R_xL_x = R_x \Leftrightarrow L_{x^\lambda}R_x = R_xL_x^{-1} \Leftrightarrow T_x = L_{x^\lambda}R_x$. (ix) to (xi): From (i), $R_{x^\rho}L_xR_x = L_x \Rightarrow R_{x^\rho}L_x = L_xR_x^{-1} = (R_xL_x^{-1})^{-1} = T_x^{-1}$. From (v), $L_{x^\lambda}R_xL_x = R_x \Rightarrow L_{x^\lambda}R_x = R_xL_x^{-1} = T_x$. Thus, $T_xT_x^{-1} = L_{x^\lambda}R_xR_{x^\rho}L_x = I = T_x^{-1}T_x = R_{x^\rho}L_xL_{x^\lambda}R_x \Rightarrow L_{x^\lambda}R_xR_{x^\rho}L_x = R_{x^\rho}L_xL_{x^\lambda}R_x$. Hence, $[L_{x^\lambda}R_x, R_{x^\rho}L_x] = I$ and $x \cdot (x^\lambda y \cdot x)x^\rho = y = x^\lambda(x \cdot yx^\rho) \cdot x$ for all $x \in Q$. □

Lemma 2.2. *Let $(Q, \cdot)$ be a loop. $(Q, \cdot)$ is a Basarab loop if and only if* $(R_{x^\rho}L_x,\ L_x,\ L_x), (R_x,\ L_{x^\lambda}R_x,\ R_x) \in AUT(Q, \cdot)$
if and only if $(T_x^{-1},\ L_x,\ L_x), (R_x,\ T_x,\ R_x) \in AUT(Q, \cdot)$.

Proof. Simply put BK1 and BK2 of (1.2) in autotopic forms. The last part follows by (vii) and (viii) of Lemma 2.1. □

2.2. Nuclei of a Basarab loop

Theorem 2.1. *Let $(Q, \cdot)$ be a Basarab loop.*

1. *The following are equivalent:*
   (a) $x \in N_\rho(Q, \cdot)$ *for all* $x \in Q$.
   (b) $x \in N_\lambda(Q, \cdot)$ *for all* $x \in Q$.
   (c) $x \in N_\mu(Q, \cdot)$ *for all* $x \in Q$.
   (d) $(yx)(x^\lambda z) = yz$ *for all* $x, y, z \in Q$.
   (e) $(yx^\rho)(xz) = yz$ *for all* $x, y, z \in Q$.
   (f) $T_x \in AUM(Q, \cdot)$ *for all* $x \in Q$.
   (g) $Inn_\mu(Q, \cdot) \le AUM(Q, \cdot)$.
   (h) $(Q, \cdot)$ *is an* A_μ*-loop.*
   (i) $(Q, \cdot)$ *is a group.*
2. $N(Q, \cdot) = N_\lambda(Q, \cdot) = N_\rho(Q, \cdot) = N_\mu(Q, \cdot)$.
3. $N(Q, \cdot) = \{x \in Q \mid\ T_x \in AUM(Q, \cdot)\}$.

Proof.

1. Let $(Q,\cdot)$ be a Basarab loop. Then

   $x\in N_\rho(Q,\cdot)\Leftrightarrow(I,\ R_x,\ R_x)\in AUT(Q,\cdot)\Leftrightarrow(I,\ R_x^{-1},\ R_x^{-1})\in AUT(Q,\cdot)$
   $\Leftrightarrow(T_x^{-1},\ L_x,\ L_x)\cdot(I,\ R_x^{-1},\ R_x^{-1})\in AUT(Q,\cdot)\Leftrightarrow(T_x^{-1},\ L_xR_x^{-1},\ L_xR_x^{-1})$
   $\in AUT(Q,\cdot)$
   $\Rightarrow(T_x^{-1},\ T_x^{-1},\ T_x^{-1})\in AUT(Q,\cdot)\Leftrightarrow T_x^{-1}\in AUT(Q,\cdot)\Leftrightarrow T_x\in AUM(Q,\cdot).$

   Therefore, in a Basarab loop, $x\in N_\rho(Q,\cdot)$ if and only if T_x is an automorphism of $(Q,\cdot)$.

   Let $(Q,\cdot)$ be a Basarab loop. Then

   $(L_x^{-1},\ I,\ L_x^{-1})\in AUT(Q,\cdot)\Leftrightarrow(L_x,\ I,\ L_x)\in AUT(Q,\cdot)\Leftrightarrow x\in N_\lambda(Q,\cdot)$
   $\Leftrightarrow(L_x,\ I,\ L_x)\in AUT(Q,\cdot)\Leftrightarrow(L_x^{-1},\ I,\ L_x^{-1})\in AUT(Q,\cdot)$
   $\Leftrightarrow(R_x,\ T_x,\ R_x)\cdot(L_x^{-1},\ I,\ L_x^{-1})\in AUT(Q,\cdot)\Leftrightarrow(R_xL_x^{-1},\ T_x,\ R_xL_x^{-1})$
   $\in AUT(Q,\cdot)(T_x,\ T_x,\ T_x)\in AUT(Q,\cdot)\Leftrightarrow T_x\in AUM(Q,\cdot).$

   Thus, in a Basarab loop $(Q,\cdot)$, $x\in N_\lambda(Q,\cdot)$ if and only if T_x is an automorphism of $(Q,\cdot)$.

   Let $(Q,\cdot)$ be a Basarab loop. Then $x\in N_\mu(Q,\cdot)\Leftrightarrow(R_x,\ L_x^{-1},I\ )\in AUT(Q,\cdot)\Leftrightarrow(R_x^{-1},\ L_x,I\ )\in AUT(Q,\cdot)\Leftrightarrow(R_x,\ T_x,\ R_x)(R_x^{-1},\ L_x,I\ )\in AUT(Q,\cdot)\Leftrightarrow(R_xR_x^{-1},\ T_xL_x,\ R_x)\Leftrightarrow(I,\ R_x,\ R_x)\in AUT(Q,\cdot)\Leftrightarrow x\in N_\rho(Q,\cdot)$. Thus, in a Basarab loop $(Q,\cdot)$, $x\in N_\mu(Q,\cdot)$ if and only if $x\in N_\rho(Q,\cdot)$. $x\in N_\lambda(Q,\cdot)\Leftrightarrow(L_x,\ I,\ L_x)\in AUT(Q,\cdot)\Leftrightarrow(L_x^{-1},\ I,\ L_x^{-1})\in AUT(Q,\cdot)\Leftrightarrow(R_{x^\rho}L_x,\ L_x,\ L_x)\cdot(L_x^{-1},\ I,\ L_x^{-1})\in AUT(Q,\cdot)\Leftrightarrow(R_{x^\rho},\ L_x,\ I)\in AUT(Q,\cdot)\Leftrightarrow(yx)(x^\lambda z)=yz\Leftrightarrow(yx^\rho)(xz)=yz$.

2. In a Basarab loop $(Q,\cdot),x\in N_\lambda(Q,\cdot)$ if and only if T_x is an automorphism of $(Q,\cdot)$. Also, $x\in N_\rho(Q,\cdot)$ if and only if T_x is an automorphism of $(Q,\cdot)$. This means $x\in N_\lambda(Q,\cdot)\subset N_\rho(Q,\cdot)$ if and only if T_x is an automorphism of $(Q,\cdot)$. And $x\in N_\rho(Q,\cdot)\subset N_\lambda(Q,\cdot)$ if and only if T_x is an automorphism of $(Q,\cdot)$. Hence, $N_\lambda(Q,\cdot)=N_\rho(Q,\cdot)$.

   $$x\in N_\mu(Q,\cdot)\Leftrightarrow(R_x,\ L_x^{-1},\ I)\in AUT(Q,\cdot)\Leftrightarrow$$
   $$(L_xR_x^{-1},\ L_x,\ L_x)(R_x,\ L_x^{-1},\ I)\in AUT(Q,\cdot)\Leftrightarrow(L_x,\ I,\ L_x)\in AUT(Q,\cdot)$$
   $$\Leftrightarrow x\in N_\lambda(Q,\cdot).$$

   Hence, $N_\lambda(Q,\cdot)=N_\mu(Q,\cdot)$.

3. Use 1.

$\square$

2.3. Inner mappings and associators of Basarab loop

Theorem 2.2. *In any loop $(Q,\cdot)$, for all $x,y,z\in Q$:*

1. $zL_{(x,y)}=(yx)M_{(y\cdot xz)}$.
2. $(y,x,z)=\Big[yx\cdot zL_{(x,y)}\Big]M_{(yx\cdot z)}$.
3. $zR_{(x,y)}=(xy)M^{-1}_{(zx\cdot y)}$.
4. $(z,x,y)=(z\cdot yx)M_{[zR_{(x,y)}\cdot xy]}$.
5. $yT_x=xM_{(yx)}$.
6. $(y,x)=(xy)M_{(x\cdot yT_x)}$.

Proof.

1. $zL_{(x,y)}=zL_xL_yL_{yx}^{-1}=(y\cdot xz)L_{yx}^{-1}=(yx)M_{(y\cdot xz)}$.
2. Recall that $yx\cdot z=(y\cdot xz)(y,x,z)\Rightarrow y\cdot xz=(yx\cdot z)/(y,x,z)$. So, from 1, $zL_{(x,y)}=(y\cdot xz)L_{yx}^{-1}\Rightarrow y\cdot xz=yx\cdot zL_{(x,y)}\Rightarrow (yx\cdot z)/(y,x,z)=yx\cdot zL_{(x,y)}\Rightarrow (y,x,z)M^{-1}_{(yx\cdot z)}=yx\cdot zL_{(x,y)}\Rightarrow (y,x,z)=\Big[yx\cdot zL_{(x,y)}\Big]M_{(yx\cdot z)}$.
3. $zR_{(x,y)}=zR_xR_yR_{xy}^{-1}=(zx\cdot y)R_{xy}^{-1}=(xy)M^{-1}_{(zx\cdot y)}$.
4. Recall that $zx\cdot y=(z\cdot xy)(z,x,y)$. So, from 3, $zR_{(x,y)}=(zx\cdot y)R_{xy}^{-1}\Rightarrow zx\cdot y=zR_{(x,y)}\cdot xy\Rightarrow (z\cdot xy)(z,x,y)=zR_{(x,y)}\cdot xy\Rightarrow (z,x,y)=\Big[zR_{(x,y)}\cdot xy\Big]L^{-1}_{(z\cdot xy)}\Rightarrow (z,x,y)=(z\cdot yx)M_{[zR_{(x,y)}\cdot xy]}$.
5. $yT_x=yR_xL_x^{-1}=(yx)L_x^{-1}=xM_{(yx)}$.
6. Recall that $yx=(xy)(y,x)$. So, from 5, $yT_x=[(xy)(y,x)]L_x^{-1}\Rightarrow (xy)(y,x)=x\cdot yT_x\Rightarrow (y,x)=\Big[x\cdot yT_x\Big]L^{-1}_{(xy)}=(xy)M_{[x\cdot yT_x]}\Rightarrow (y,x)=(xy)M_{[x\cdot yT_x]}$.

□

Theorem 2.3. *Let $(Q,\cdot)$ be a loop. The following are equivalent:*

1. $(Q,\cdot)$ *is a Basarab loop.*
2. $(x,y,z)=\Big[(x\cdot yx^{\rho})(xz)\Big]\backslash(xy\cdot z)$ *and* $(x,y,z)=(x\cdot yz)\backslash\Big[(xz)(z^{\lambda}y\cdot z)\Big]$ *for all* $x,y,z\in Q$
3. $Inn(Q,\cdot)=\Big\langle L_xL_{x\cdot yx^{\rho}}L_{xy}^{-1},\ T_x\ |x,y\in Q\Big\rangle$ *and* $Inn(Q,\cdot)=\Big\langle R_xR_{x^{\lambda}z\cdot x}R_{zx}^{-1},\ T_x\ |x,z\in Q\Big\rangle$.

Proof.

1⇔2 This is achieved by simply using (1.2) and the fact that $xy\cdot z=(x\cdot yz)(x,y,z)$.

1⇔3 BK1 of (1.2) is true if and only if $L_yL_x=L_xL_{x\cdot yx^{\rho}}\Leftrightarrow L_yL_xL_{xy}^{-1}=L_xL_{x\cdot yx^{\rho}}L_{xy}^{-1}\Leftrightarrow L_{(y,\,x)}=L_xL_{x\cdot yx^{\rho}}L_{xy}^{-1}\Leftrightarrow Inn(Q,\cdot)=\Big\langle L_xL_{x\cdot yx^{\rho}}L_{xy}^{-1},\ T_x\ |x,y\in Q\Big\rangle$. BK2 of (1.2) is true if and only if $R_zR_x=R_xR_{(x^{\lambda}z\cdot x)}\Leftrightarrow R_zR_xR_{zx}^{-1}=R_xR_{(x^{\lambda}z\cdot x)}R_{zx}^{-1}\Leftrightarrow R_{(z,\,x)}=R_xR_{(x^{\lambda}z\cdot x)}R_{zx}^{-1}\Leftrightarrow Inn(Q,\cdot)=\Big\langle R_xR_{x^{\lambda}z\cdot x}R_{zx}^{-1},\ T_x\ |x,z\in Q\Big\rangle$.

□

Theorem 2.4. *In a Basarab loop $(Q,\cdot)$, for all $x,y,z\in Q$:*

1. $(y,x,z)=\Big[zL_{yx}J_{\lambda}\cdot zL_{(x,y)}L_{yx}\Big]J_{\lambda}$.
2. $(z,x,y)=(z\cdot yx)^{\lambda}\Big[zR_{(x,y)}\cdot xy\Big]$.
3. $(x,y,z)=[yT_x^{-1}\cdot xz]^{\lambda}(xy\cdot z)$.
4. $(x,y,z)=(x\cdot yz)^{\lambda}\cdot(xz)(yT_z)$.

Proof. Note that in a Basarab loop $(Q,\cdot)$, $(x,y,z)\in N(Q,\cdot)$.

1. From the proof of 2 of Theorem 2.2, $(yx\cdot z)/(y,x,z)=yx\cdot zL_{(x,y)}\Rightarrow(yx\cdot z)(y,x,z)^{-1}=yx\cdot zL_{(x,y)}\Rightarrow(y,x,z)^{-1}=(yx\cdot z)^{\lambda}\Big[yx\cdot zL_{(x,y)}\Big]\Rightarrow(y,x,z)=\Big[zL_{yx}J_{\lambda}\cdot zL_{(x,y)}L_{yx}\Big]J_{\lambda}$.
2. From the proof of 4 of Theorem 2.2, $(z\cdot xy)(z,x,y)=zR_{(x,y)}\cdot xy\Rightarrow(z,x,y)=(z\cdot yx)^{\lambda}\Big[zR_{(x,y)}\cdot xy\Big]$.

3. By 2 of Theorem 2.3, $xy \cdot z = x(yz) \cdot (x, y, z) \Rightarrow xy \cdot z = (x \cdot yx^\rho)(xz) \cdot (x, y, z) \Rightarrow [(x \cdot yx^\rho)(xz)]^\lambda (xy \cdot z) = (x, y, z) \Rightarrow [yT_x^{-1} \cdot xz]^\lambda (xy \cdot z) = (x, y, z)$.

4. By 2 of Theorem 2.3, $xy \cdot z = x(yz) \cdot (x, y, z) \Rightarrow (x \cdot yz)(x, y, z) = (xz)(z^\lambda y \cdot z) \Rightarrow (x, y, z) = (x \cdot yz)^\lambda \cdot (xz)(z^\lambda y \cdot z) \Rightarrow (x, y, z) = (x \cdot yz)^\lambda \cdot (xz)(yT_z)$.

□

Corollary 2.1. *In a Basarab loop $(Q, \cdot)$:*

1. $Inn(Q, \cdot) = \left\langle L_x L_{yT_x^{-1}} L_{xy}^{-1},\ T_x \mid x, y \in Q \right\rangle$.

2. $Inn(Q, \cdot) = \left\langle R_x R_{zT_x} R_{zx}^{-1},\ T_x \mid x, z \in Q \right\rangle$.

3. $R_{(y,\, x)} = L_{(y,\, x)}$ *if and only if* $R_x R_{yT_x} R_{yx}^{-1} = L_x L_{yT_x^{-1}} L_{xy}^{-1}$ *for all* $x, y \in Q$.

Proof. 1 and 2 follow by 3 of Theorem 2.3. For 3, in a Basarab loop $(Q, \cdot)$, $R_{(z,\, x)} = R_x R_{zT_x} R_{zx}^{-1}, \forall\, x, z \in Q$ and $L_{(y,\, x)} = L_x L_{yT_x^{-1}} L_{xy}^{-1}$, $\forall\, x, y \in Q$. This implies, $R_{(y,\, x)} = R_x R_{yT_x} R_{yx}^{-1}$, $\forall\ x, y, z \in Q$. Thus, $R_{(y,\, x)} = L_{(y,\, x)} \Leftrightarrow R_x R_{yT_x} R_{yx}^{-1} = L_x L_{yT_x^{-1}} L_{xy}^{-1}\ \ \forall\, x, y \in Q$. □

Theorem 2.5. *In a Basarab loop $(Q, \cdot)$ the following are equivalent for all $x, y \in Q$:*

(i) $T_{xy} = T_x T_y$.

(ii) $L_{(x,\, y)} \in \mathcal{P}(Q, \cdot)$.

(iii) $R_{(x,\, y)} \in \Lambda(Q, \cdot)$.

Proof. Let $(Q, \cdot)$ be a Basarab loop and let $B_1(x) = (T_x^{-1},\ L_x,\ L_x) \in AUT(Q, \cdot)$,

$$B_1(y) = (T_y^{-1},\ L_y,\ L_y) \in AUT(Q, \cdot), \text{ and } B_1(yx)^{-1} = (T_{yx},\ L_{yx}^{-1},\ L_{yx}^{-1}) \in AUT(Q, \cdot)$$

$$\text{then } B_1(x) B_1(y) B_1(yx)^{-1} \in AUT(Q, \cdot) \Rightarrow T_x^{-1} T_y^{-1} T_{yx} = I$$

$$\Leftrightarrow T_{yx} = T_y T_x \Leftrightarrow (I,\ L_{(y,\, x)},\ L_{(y,\, x)}) \in AUT(Q, \cdot) \Leftrightarrow L_{(y,\, x)} \in \mathcal{P}(Q, \cdot),$$

$$\forall\, x, y \in Q.$$

Also, let $B_2(x) = (R_x,\ T_x,\ R_x) \in AUT(Q,\cdot), B_2(y) = (R_y,\ T_y,\ R_y) \in AUT(Q,\cdot)$

$$\text{and } B_2(xy)^{-1} = (R_{xy}^{-1},\ T_{xy}^{-1},\ R_{xy}^{-1}) \in AUT(Q,\cdot)$$

$$\text{then } B_2(x)B_2(y)B_2(xy)^{-1} \in AUT(Q,\cdot) \Rightarrow T_xT_yT_{xy}^{-1} = I$$

$$\Leftrightarrow T_{xy} = T_xT_y \Leftrightarrow (R(x,\ y),\ I,\ R(x,\ y) \in AUT(Q,\cdot)$$

$$\Leftrightarrow R(x,\ y) \in \Lambda(Q,\cdot)\ \forall\, x, y \in Q.$$

□

2.4. Relationship between Basarab loop and automorphic loop

Theorem 2.6. *Let $(Q,\cdot)$ be a Basarab loop. Then:*

(i) *$(Q,\cdot)$ is an A_λ-loop if and only if $L_{(x,\ y)} = T_x^{-1}T_y^{-1}T_{yx}$ for all $x, y \in Q$ if and only if $R_{yx} = T_yR_xL_y$ if and only if $Inn_\lambda(Q,\cdot) = Inn_\mu(Q,\cdot)$.*

(ii) *$(Q,\cdot)$ is an A_ρ-loop if and only if $R_{(x,\ y)} = T_xT_yT_{xy}^{-1}$ for all $x, y \in Q$ if and only if $L_{xy} = T_y^{-1}L_xR_y$ if and only if $Inn_\rho(Q,\cdot) = Inn_\mu(Q,\cdot)$.*

(iii) *$(Q,\cdot)$ is an A-loop if and only if $Inn(Q,\cdot) = \left\langle T_x : x \in Q \right\rangle \leq AUM(Q,\cdot) \Leftrightarrow R_{yx}L_y^{-1}R_x^{-1} = T_y = L_xR_yL_{xy}^{-1} \in AUM(Q,\cdot)$ for all $x, y \in Q$ if and only if $Inn(Q,\cdot) = Inn_\mu(Q,\cdot) \leq AUM(Q,\cdot)$.*

Proof.

(i) From the Basarab laws,
let $B_1(x) = (T_x^{-1},\ L_x,\ L_x) \in AUT(Q,\cdot), B_1(y) = (T_y^{-1},\ L_y,\ L_y) \in AUT(Q,\cdot)$, and $B_1(yx)^{-1} = (T_{yx},\ L_{yx}^{-1},\ L_{yx}^{-1}) \in AUT(Q,\cdot)$

It follows that, $B_1(x)B_1(y)B_1(yx)^{-1} = (T_x^{-1}T_y^{-1}T_{yx},\ L_{(x,\ y)},\ L_{(x,\ y)}) \in AUT(Q,\cdot)$. Then, a Basarab loop $(Q,\cdot)$ is an A_λ-loop $\Leftrightarrow L_{(x,\ y)} = T_x^{-1}T_y^{-1}T_{yx}$, $\forall\, x, y \in Q$. $L_{(x,\ y)} = T_x^{-1}T_y^{-1}T_{yx} \Leftrightarrow L_xL_yL_{yx}^{-1} = T_x^{-1}T_y^{-1}T_{yx} \Leftrightarrow L_xR_x^{-1}T_y^{-1}R_{yx}L_{yx}^{-1} = L_xL_yL_{yx}^{-1} \Leftrightarrow R_x^{-1}T_y^{-1}R_{yx} = L_y \Leftrightarrow T_yR_xL_y = R_{yx}$.

(ii) Also, from the Basarab laws, let $B_2(x) = (R_x,\ T_x,\ R_x) \in AUT(Q,\cdot)$, $B_2(y) = (R_y,\ T_y,\ R_y) \in AUT(Q,\cdot)$, and $B_2(xy)^{-1} = (R_{xy}^{-1},\ T_{xy}^{-1},\ R_{xy}^{-1}) \in AUT(Q,\cdot)$. Thus, $B_2(x)B_2(y)B_2(xy)^{-1}$

$= (R_{(x,\, y)},\ T_xT_yT_{xy}^{-1},\ R_{(x,\, y)}) \in AUT(Q, \cdot)$. This implies, a Basarab loop $(Q, \cdot)$ is an A_ρ-loop $\Leftrightarrow R_{(x,\, y)} = T_xT_yT_{xy}^{-1}$. $R_{(x,\, y)} = T_xT_yT_{xy}^{-1} \Leftrightarrow R_xR_yR_{xy}^{-1} = T_xT_yT_{xy}^{-1} \Leftrightarrow T_xT_yL_{xy}R_{xy}^{-1} = R_xR_yR_{xy}^{-1} \Leftrightarrow T_xT_yL_{xy} = R_xR_y \Leftrightarrow R_xL_x^{-1}T_yL_{xy} = R_xR_y \Leftrightarrow L_x^{-1}T_yL_{xy} = R_y \Leftrightarrow L_{xy} = T_y^{-1}L_xR_y$.

(iii) This follows from (i) and (ii).

□

Corollary 2.2. *Let $(Q, \cdot)$ be a Basarab loop. Then:*

1. *$(Q, \cdot)$ is an A_λ-loop, $L_{(x,\, y)} = T_x^{-1}T_y^{-1}T_{yx}$, $R_{yx} = T_yR_xL_y$ and $Inn_\lambda(Q, \cdot) = Inn_\mu(Q, \cdot)$ for all $x, y \in Q$.*
2. *$(Q, \cdot)$ is an A_ρ-loop, $R_{(x,\, y)} = T_xT_yT_{xy}^{-1}$, $L_{xy} = T_y^{-1}L_xR_y$ and $Inn_\rho(Q, \cdot) = Inn_\mu(Q, \cdot)$ for all $x, y \in Q$.*
3. *$(Q, \cdot)$ is an A-loop if and only if $Inn(Q, \cdot) = \Big\langle T_x : x \in Q \Big\rangle \le AUM(Q, \cdot) \Leftrightarrow T_y = R_{yx}L_y^{-1}R_x^{-1} = L_xR_yL_{xy}^{-1} \in AUM(Q, \cdot)$ if and only if $Inn(Q, \cdot) = Inn_\mu(Q, \cdot) \le AUM(Q, \cdot)$ for all $x, y \in Q$.*

Proof. In a Basarab loop $(Q, \cdot)$, $(T_x^{-1}T_y^{-1}T_{yx},\ L_{(y,\, x)},\ L_{(y,\, x)}), (R_{(x,\, y)}, T_xT_yT_{xy}^{-1},\ R_{(x,\, y)}) \in AUT(Q, \cdot)$. Since $eL_{(y,\, x)} = e = eR_{(x,\, y)}$, then $L_{(y,\, x)}, R_{(x,\, y)} \in AUM(Q, \cdot)$. Other conclusions follow from Theorem 2.6. □

2.5. Basarab loop and generators of its total multiplication group

Theorem 2.7. *Let $(Q, \cdot)$ be a Basarab loop and let $U_x = M_xR_x^{-1}, V_x = M_x^{-1}L_x^{-1}, W_x = R_xM_x$ for any arbitrarily fixed $x \in Q$.*

1. *$T_x^{-1} = J_\rho L_xM_x^{-1}$.*
2. *$T_x = J_\lambda R_xM_x$.*
3. *The following are true*
    1. *$T_x = W_xV_x$.*
    2. *$M_x^2V_x = U_xT_x$.*
    3. *$U_xT_x^2 = T_xW_x$.*
    4. *$R_x = L_x \Leftrightarrow M_x^2 = T_xJ_\rho^2T_x$.*

5. $|M_x| = 2 \Leftrightarrow T_xJ_\lambda = J_\rho T_x^2$.
6. $|T_x| = 2 \Leftrightarrow J_\rho^2 = W_xU_xT_x$.
7. $J_\rho U_xT_x = W_xJ_\rho$.
8. $J_\rho R_x = R_xT_xJ_\rho$.
9. $T_xJ_\rho = J_\rho T_x \Leftrightarrow U_xT_x = W_x$.
10. $R_x^nT_xJ_\rho L_x^n = R_x^{n-1}J_\rho T_xL_x^{n-1}\ \forall\ n \in \mathbf{N}$.
11. $J_\lambda T_x = T_xV_x$.
12. $W_x = J_\rho T_x$.
13. $U_xT_x = T_xJ_\rho$.
14. $U_xW_x = M_x^2$.
15. $W_xU_xR_x = R_xU_xW_x$.
16. $M_xU_xR_x = U_xW_x$.
17. $M_x = U_xR_x$ and $W_x = R_xM_x$.

4. *The following are equivalent:*

   1. $J_\rho = J_\lambda$.
   2. $T_x^2 = U_xT_xR_xM_x$.
   3. $T_x^2 = U_xT_xW_x$.
   4. $U_xT_x^2 = T_x^2V_x$.

   Hence, $\mathcal{TM}(Q,\cdot) = \Big\langle\{R_x, L_x, U_x\}|x \in Q\Big\rangle = \Big\langle\{R_x, L_x, V_x\}|x \in Q\Big\rangle = \Big\langle\{R_x, L_x, W_x\}|x \in Q\Big\rangle$.

5. $T_x^2 = U_x^nT_x^2V_x^n$ and $T_x^{-2} = V_x^nT_x^{-2}U_x^n$ *for all* $n \in \mathbf{N}$. *If* $|T_x| = 2$, *then* $U_x^nV_x^n = I$ *for all* $x \in \mathbf{N}$.

6. *The following are true for any* $n \in \mathbf{N}$*:*

   1. $J_\rho^n = T_x^{-1}U_x^nT_x$.
   2. $|J_\rho| = n \Leftrightarrow |U_x| = n$.
   3. $J_\lambda^n = T_xV_x^{n-1}W_x^{-1}$.
   4. $|J_\lambda| = n \Leftrightarrow T_xV_x^{n-1} = W_x$.
   5. $J_\rho^n = J_\lambda^n \Leftrightarrow T_x^2V_x^n = U_x^nT_x^2$.

Proof.

1. From property (ii) of Lemma 2.1, $x \cdot yx^\rho = x/(xy^\rho) = (xy^\rho)M_x^{-1} \Rightarrow yR_{x^\rho}L_x = yJ_\rho L_x M_x^{-1} \Rightarrow T_x^{-1} = J_\rho L_x M_x^{-1}$.

2. From property (iv) of Lemma 2.1, $x^\lambda z \cdot x = z^\lambda x\backslash x \Rightarrow zL_{x^\lambda} \cdot x = (z^\lambda x)M_x \Rightarrow zL_{x^\lambda}R_x = zJ_\lambda R_x M_x \Rightarrow L_{x^\lambda}R_x = J_\lambda R_x M_x \Rightarrow L_{xJ_\lambda}R_x = J_\lambda R_x M_x \Rightarrow T_x = J_\lambda R_x M_x$.

3.
   1. $W_xV_x = R_xM_xM_x^{-1}L_x^{-1} = T_x \Rightarrow T_x = W_xV_x$.
   2. $M_x^2V_xT_x^{-1} = M_x^2M_x^{-1}L_x^{-1}L_xR_x^{-1} = M_xR_x^{-1} = U_x \Rightarrow M_x^2V_x = U_xT_x$.
   3. From 1 and 2: $J_\lambda = T_xM_x^{-1}R_x^{-1}$ and $J_\lambda = L_xM_x^{-1}T_x$. So, $T_xM_x^{-1}R_x^{-1} = L_xM_x^{-1}T_x \Rightarrow T_x(R_xM_x)^{-1} = L_xM_x^{-1}T_x \Rightarrow T_xW_x^{-1} = L_xM_xM_x^{-2}T_x = V_x^{-1}M_x^{-2}T_x \Rightarrow T_xW_x^{-1} = V_x^{-1}M_x^{-2}T_x \Rightarrow V_xT_x = M_x^{-2}T_xW_x \Rightarrow M_x^2V_xT_x = T_xW_x \Rightarrow U_xT_x^2 = T_xW_x$.
   4. From 1 and 2: $L_x = J_\lambda T_x^{-1}M_x$ and $R_x = J_\rho T_xM_x^{-1}$. So, $L_x = R_x \Leftrightarrow J_\lambda T_x^{-1}M_x = J_\rho T_xM_x^{-1} \Leftrightarrow M_x^2 = T_xJ_\rho^2T_x$. Thus, $R_x = L_x \Leftrightarrow M_x^2 = T_xJ_\rho^2T_x$.
   5. From 1 and 2: $M_x = R_x^{-1}J_\rho T_x$ and $M_x^{-1} = L_x^{-1}J_\lambda T_x^{-1}$. So, $|M_x| = 2 \Leftrightarrow M_x^{-1} = M_x \Leftrightarrow R_x^{-1}J_\rho T_x = L_x^{-1}J_\lambda T_x^{-1} \Leftrightarrow J_\rho T_x^2 = T_xJ_\lambda$. Thus, $|M_x| = 2 \Leftrightarrow T_xJ_\lambda = J_\rho T_x^2$.
   6. From 1 and 2: $|T_x| = 2 \Leftrightarrow T_x^{-1} = T_x \Leftrightarrow J_\rho L_xM_x^{-1} = J_\lambda R_xM_x \Leftrightarrow J_\rho^2 = R_xM_xM_xL_x^{-1} = W_xM_xL_x^{-1} = W_xM_x^2M_x^{-1}L_x^{-1} = W_xM_x^2V_x = W_xU_xT_x \Leftrightarrow J_\rho^2 = W_xU_xT_x$. Thus, $|T_x| = 2 \Leftrightarrow J_\rho^2 = W_xU_xT_x$.
   7. From 1 and 2: $I = T_xT_x^{-1} = J_\lambda R_xM_xJ_\rho L_xM_x^{-1} \Rightarrow R_x^{-1}J_\rho M_x = M_xJ_\rho L_x \Rightarrow J_\rho M_xL_x^{-1} = R_xM_xJ_\rho \Rightarrow J_\rho M_x^2M_x^{-1}L_x^{-1} = W_xJ_\rho \Rightarrow J_\rho M_x^2V_x = W_xJ_\rho \Rightarrow J_\rho U_xT_x = W_xJ_\rho$.
   8. From 1 and 2: $M_x = R_x^{-1}J_\rho T_x$ and $M_x = T_xJ_\rho L_x$. So, $R_x^{-1}J_\rho T_x = T_xJ_\rho L_x \Rightarrow J_\rho R_x = R_xT_xJ_\rho$.
   9. From 1 and 2: $T_x^{-1}J_\lambda = M_x^{-1}R_x^{-1}$ and $J_\lambda T_x^{-1} = L_xM_x^{-1}$. So, $T_xJ_\rho = J_\rho T_x \Leftrightarrow M_x^{-1}R_x^{-1} = L_xM_x^{-1} \Leftrightarrow (R_xM_x)^{-1} = L_xM_xM_x^{-2} \Leftrightarrow W_x^{-1} = V_x^{-1}M_x^{-2} \Leftrightarrow U_xT_x = W_x$. Thus, $T_xJ_\rho = J_\rho T_x \Leftrightarrow U_xT_x = W_x$.
   10. From 1 and 2: $M_x = R_x^{-1}J_\rho T_x$ and $M_x^{-1} = L_x^{-1}J_\lambda T_x^{-1}$. So, $I = M_xM_x^{-1} = R_x^{-1}J_\rho T_xL_x^{-1}J_\lambda T_x^{-1} \Rightarrow R_xT_xJ_\rho = J_\rho T_xL_x^{-1} \Rightarrow R_xT_xJ_\rho L_x = J_\rho T_x$. So, $R_x^nT_xJ_\rho L_x^n = R_x^{n-1}J_\rho T_xL_x^{n-1}\ \forall\ n \in \mathbf{N}$.

11. From 2: $M_x^{-1} = T_x^{-1} J_\lambda R_x$. So, $M_x^{-1} L_x^{-1} = V_x = T_x^{-1} J_\lambda R_x L_x^{-1} = T_x^{-1} J_\lambda T_x \Rightarrow V_x T_x^{-1} = T_x^{-1} J_\lambda \Rightarrow J_\lambda T_x = T_x V_x$.
12. From 2: $M_x = R_x^{-1} J_\rho T_x$. So, $W_x = R_x M_x = R_x R_x^{-1} J_\rho T_x = J_\rho T_x \Rightarrow W_x = J_\rho T_x$.
13. From 1: $M_x = T_x J_\rho L_x$. Now, $U_x = M_x R_x^{-1} = T_x J_\rho L_x R_x^{-1} = T_x J_\rho T_x^{-1} \Rightarrow U_x T_x = T_x J_\rho$.
14. $U_x W_x = M_x R_x^{-1} R_x M_x = M_x^2 \Rightarrow U_x W_x = M_x^2$.
15. $W_x U_x = R_x M_x^2 R_x^{-1} \Rightarrow W_x U_x R_x = R_x M_x^2 = R_x U_x W_x \Rightarrow W_x U_x R_x = R_x U_x W_x$.
16. $M_x U_x = M_x^2 R_x^{-1} = U_x W_x R_x^{-1} \Rightarrow M_x U_x R_x = U_x W_x$.
17. Trivial.

4. We shall use 1 and 2.

(a)⇔ (b) $J_\lambda = J_\rho \Leftrightarrow T_x M_x^{-1} R_x^{-1} = T_x^{-1} M_x L_x^{-1} \Rightarrow T_x M_x^{-1} R_x^{-1} R_x = T_x^{-1} M_x L_x^{-1} R_x \Rightarrow T_x^2 M_x^{-1} = M_x L_x^{-1} R_x = M_x R_x^{-1} R_x L_x^{-1} R_x \Rightarrow T_x^2 M_x^{-1} = U_x T_x R_x \Rightarrow T_x^2 = U_x T_x R_x M_x$.

(a)⇔ (c) $J_\lambda = J_\rho \Leftrightarrow T_x^2 = U_x T_x R_x M_x \Leftrightarrow T_x^2 = U_x T_x W_x$.

(a)⇔ (d) $J_\lambda = J_\rho \Leftrightarrow T_x^2 = U_x T_x R_x M_x \Leftrightarrow U_x = T_x^2 (T_x R_x M_x)^{-1} \Leftrightarrow U_x = T_x^2 M_x^{-1} R_x^{-1} T_x^{-1} \Leftrightarrow U_x = T_x^2 M_x^{-1} L_x^{-1} L_x R_x^{-1} T_x^{-1} \Leftrightarrow U_x = T_x^2 V_x T_x^{-2} \Leftrightarrow U_x T_x^2 = T_x^2 V_x$.

$\mathcal{TM}(Q,\cdot) = \Big\langle \{R_x, L_x, U_x\} | x \in Q \Big\rangle = \Big\langle \{R_x, L_x, V_x\} | x \in Q \Big\rangle = \Big\langle \{R_x, L_x, W_x\} | x \in Q \Big\rangle$ follows from the fact that $T_x^2 = U_x T_x R_x M_x \Leftrightarrow M_x = \Big(U_x T_x R_x \Big)^{-1} T_x^2$.

5. From 1 and 2, $J_\rho = T_x^{-1} M_x L_x^{-1}$ and $J_\lambda = T_x M_x^{-1} R_x^{-1}$. Hence:

$$
\begin{gathered}
T_x^{-1} M_x L_x^{-1} T_x M_x^{-1} R_x^{-1} = I \textit{ and } T_x M_x^{-1} R_x^{-1} T_x^{-1} M_x L_x^{-1} = I \Rightarrow \\
T_x^{-1} M_x R_x^{-1} R_x L_x^{-1} T_x M_x^{-1} L_x^{-1} L_x R_x^{-1} = I \textit{ and } \\
T_x M_x^{-1} L_x^{-1} L_x R_x^{-1} T_x^{-1} M_x R_x^{-1} R_x L_x^{-1} = I \Rightarrow \\
T_x^{-1} U_x T_x^2 V_x T_x^{-1} = I \textit{ and } T_x V_x T_x^{-2} U_x T_x = I \Rightarrow \\
\underbrace{U_x T_x^2 V_x = T_x^2}_{a} \textit{ and } \underbrace{V_x T_x^{-2} U_x = T_x^{-2}}_{b}
\end{gathered}
\tag{2.1}
$$

By multiplying (2.1)(a) and (2.1)(b), we get

$$U_x^2 T_x^2 V_x^2 = T_x^2 \tag{2.2}$$

Substituting eq:bs2 in (2.1)(a), we get $U_x^3 T_x^2 V_x^3 = T_x^2$. Making this substitution inductively, we have $T_x^2 = U_x^n T_x^2 V_x^n$ for all $\mathbf{N}$. Doing a similar thing with eq:bs2 and (2.1)(b), we get $T_x^{-2} = V_x^n T_x^{-2} U_x^n$ for all $\mathbf{N}$.

6. We shall use 1, 2 and some results in 3.

   (a) $J_\rho^2 = T_x^{-1} M_x L_x^{-1} T_x^{-1} M_x L_x^{-1} = T_x^{-1} M_x L_x^{-1} L_x R_x^{-1} M_x L_x^{-1} = T_x^{-1} M_x R_x^{-1} M_x L_x^{-1} = T_x^{-1} U_x M_x^2 M_x^{-1} L_x^{-1} = T_x^{-1} U_x M_x^2 V_x = T_x^{-1} U_x^2 T_x$. Hence, $J_\rho^3 = J_\rho^2 J_\rho = T_x^{-1} U_x^2 T_x T_x^{-1} U_x T_x = T_x^{-1} U_x^3 T_x$. Continuing by induction, we have $J_\rho^n = T_x^{-1} U_x^n T_x$.

   (b) This follows from (a).

   (c) $J_\lambda^2 = T_x M_x^{-1} R_x^{-1} T_x M_x^{-1} R_x^{-1} = T_x M_x^{-1} R_x^{-1} R_x L_x^{-1} M_x^{-1} R_x^{-1} = T_x M_x^{-1} L_x^{-1} (R_x M_x)^{-1} \Rightarrow J_\lambda^2 = T_x V_x W_x^{-1}$. Hence, $J_\lambda^3 = J_\lambda^2 J_\lambda = T_x V_x W_x^{-1} T_x W_x^{-1} = T_x V_x^2 W_x^{-1}$. Continuing by induction, we have $J_\lambda^n = T_x V_x^{n-1} W_x^{-1}$.

   (d) This follows from (c).

   (e) (a) and (c) answer this.

□

Lemma 2.3. *Let $(Q, \cdot)$ be a Basarab loop. The following are equivalent:*

1. *$(Q, \cdot)$ is a cross inverse property loop.*
2. *$(Q, \cdot)$ is commutative.*
3. *$(Q, \cdot)$ is an abelian group.*
4. *$L_x \in \mathcal{P}(Q, \cdot)$ for all $x \in Q$.*
5. *$R_x \in \Lambda(Q, \cdot)$ for all $x \in Q$.*

Proof. By Lemma 2.1, $x \cdot y = (x \cdot yx^\rho) \cdot x$. If $(Q, \cdot)$ has the CIP, then $x \cdot y = (x \cdot yx^\rho) \cdot x \Rightarrow x \cdot y = y \cdot x$ which implies commutativity. The converse is also true. By BK1 of (1.2), $(x \cdot yx^\rho)(xz) = x \cdot yz$. $(Q, \cdot)$ has CIP if and only if $y \cdot xz = x \cdot yz \Leftrightarrow (Q, \cdot)$ is an abelian group. By Lemma 2.2, $(T_x^{-1},\ L_x,\ L_x), (R_x,\ T_x,\ R_x) \in AUT(Q, \cdot)$. $(T_x^{-1},\ L_x,\ L_x) \in$

$AUT(Q,\cdot) \Rightarrow (L_xR_x^{-1},\ L_x,\ L_x) \in AUT(Q,\cdot)$. So, $(Q,\cdot)$ is commutative if and only if $L_x \in \mathcal{P}(Q,\cdot)$ for all $x \in Q$. $(R_x,\ T_x,\ R_x) \in AUT(Q,\cdot) \Rightarrow (R_x,\ R_xL_x^{-1},\ R_x) \in AUT(Q,\cdot)$. So, $(Q,\cdot)$ is commutative if and only if $R_x \in \Lambda(Q,\cdot)$ for all $x \in Q$. □

Lemma 2.4. *A Basarab loop is a Moufang loop if and only if it is flexible.*

Proof. By Lemma 2.2, $(T_x^{-1},\ L_x,\ L_x),(R_x,\ T_x,\ R_x) \in AUT(Q,\cdot)$. $(T_x^{-1},\ L_x,\ L_x)(R_x,\ T_x,\ R_x) = (T_x^{-1}R_x,\ L_xT_x,\ L_xR_x) = (L_x, L_xR_xL_x^{-1}, L_xR_x) \in AUT(Q,\cdot)$. So, the loop is flexible if and only if $L_xR_x = R_xL_x \Leftrightarrow (L_x, R_x, L_xR_x) \in AUT(Q,\cdot) \Leftrightarrow (xy)(zx) = (x\cdot yz)x$ if and only if the loop is a Moufang loop. □

Theorem 2.8. *A Basarab loop is a TA-loop if and only if it is a commutative and flexible loop.*

Proof. This follows from Lemma 2.4 and Theorem 1.1. □

Corollary 2.3. *A Basarab loop $(Q,\cdot)$ is a TA-loop if and only if it is a flexible loop and any of the following is true.*

1. *$(Q,\cdot)$ is a cross inverse property loop.*
2. *$(Q,\cdot)$ is commutative.*
3. *$(Q,\cdot)$ is an abelian group.*
4. *$L_x \in \mathcal{P}(Q,\cdot)$ for all $x \in Q$.*
5. *$R_x \in \Lambda(Q,\cdot)$ for all $x \in Q$.*

Proof. This follows from Lemma 2.3 and Theorem 2.8. □

Theorem 2.9. *Let $(Q,\cdot)$ be a Basarab loop. Then:*

$$\mathcal{TM}(Q,\cdot) = \left\langle \left\{ R_x, L_x, R_x^{-1}J_\rho T_x \ :\ x \in Q \right\} \right\rangle$$
$$= \left\langle \left\{ R_x, L_x, T_xJ_\rho L_x \ :\ x \in Q \right\} \right\rangle \text{ and}$$
$$TInn(Q,\cdot)$$
$$= \left\langle \left\{ T_x, T_xT_yT_{xy}^{-1}, T_x^{-1}T_y^{-1}T_{yx}, T_yJ_\rho L_yR_x^{-1}J_\rho T_xT_{y\backslash x}^{-1}J_\lambda R_{y\backslash x}, T_xJ_\rho T_x^{-1} \ \middle|\ x, y \in Q \right\} \right\rangle$$
$$= \left\langle \left\{ T_x, T_xT_yT_{xy}^{-1}, T_x^{-1}T_y^{-1}T_{yx}, T_yJ_\rho L_yR_x^{-1}J_\rho T_xL_{y\backslash x}^{-1}J_\lambda T_{y\backslash x}^{-1}, T_xJ_\rho T_x^{-1} \ \middle|\ x, y \in Q \right\} \right\rangle.$$

Proof. This is proved by Corollary 2.2 and Theorem 2.7. Recall that $\mathcal{TM}(Q,\cdot) = \Big\langle \{R_x, L_x, M_x \ : \ x \in Q\} \Big\rangle$ and
$TInn(Q,\cdot) = \Big\langle R_{(x,y)}, L_{(x,y)}, T_x, M_{(x,y)}, U_x \mid x, y \in Q \Big\rangle$ where $M_{(x,y)} = M_y M_x M_{y\backslash x}^{-1}$ and $U_x = M_x R_x^{-1}$.

$U_x = T_x J_\rho T_x^{-1}$ *and* $M_{(x,y)} = M_y M_x M_{y\backslash x}^{-1}$ *while*
$M_{(x,y)} = T_y J_\rho L_y R_x^{-1} J_\rho T_x \Big(T_{y\backslash x} J_\rho L_{y\backslash x}\Big)^{-1} = T_y J_\rho L_y R_x^{-1} J_\rho T_x L_{y\backslash x}^{-1} J_\lambda T_{y\backslash x}^{-1}$
and
$M_{(x,y)} = T_y J_\rho L_y R_x^{-1} J_\rho T_x \Big(R_{y\backslash x}^{-1} J_\rho T_{y\backslash x}\Big)^{-1} = T_y J_\rho L_y R_x^{-1} J_\rho T_x T_{y\backslash x}^{-1} J_\lambda R_{y\backslash x}$.
□

Remark 2.1. *Theorem 2.9 gives expressions for the total multiplication group and total inner mapping group of a Basarab loop in terms of finetuned generators. Hence, it is an ostensible solution to Question 1.2.*

Theorem 2.10. *1. Let $(Q,\cdot)$ be a Basarab loop. The following are equivalent.*

1. $\Bigg\langle \Bigg\{ L_{x\backslash y}^{-1} T_y J_\rho L_y R_x^{-1} J_\rho T_x, R_{y/x}^{-1} T_y^{-1} J_\lambda R_y L_x^{-1} J_\lambda T_x^{-1} \ \Bigg| \ x, y \in Q \Bigg\} \Bigg\rangle \leq AUM(Q,\cdot)$.
2. $\Big\langle T_x : x \in Q \Big\rangle \leq AUM(Q,\cdot)$ *and $(Q,\cdot)$ is flexible.*

Proof. This follows by Theorem 1.2, Theorem 2.6 and Theorem 2.7. $M_y M_x = T_y J_\rho L_y R_x^{-1} J_\rho T_x$ and $M_y^{-1} M_x^{-1} = (M_x M_y)^{-1} = \Big(T_x J_\rho L_x R_y^{-1} J_\rho T_y\Big)^{-1}$ $= T_y^{-1} J_\lambda R_y L_x^{-1} J_\lambda T_x^{-1}$. □

Remark 2.2. *Theorem 2.10 shows that a class of total inner mappings acts on a Basarab loop Q by automorphisms if and only if Q is an A-loop and flexible. Theorem 2.8 and Theorem 2.10 give partial answers to Question 1.1 in the case of Basarab loop.*

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