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The structure of Cayley graphs of dihedral groups of Valencies 1, 2 and 3.

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Abstract

Let G be a group and S be a subset of G such that $e \notin S$ and $S^{-1} \subseteq S$. Then $Cay(G, S)$ is a simple undirected Cayley graph whose vertices are all elements of G and two vertices x and y are adjacent if and only if $xy^{-1} \in S$. The size of subset S is called the valency of $Cay(G, S)$. In this paper, we determined the structure of all $Cay(D_{2n}, S)$, where D_{2n} is a dihedral group of order $2n$, $n \geq 3$ and $|S| = 1, 2$ or 3.

Keywords: *Valency, Cayley graph, Dihedral group.*

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1. Introduction

Algebraic graph theory is one of the most important branches of mathematics, playing an essential role in other mathematical fields in which algebraic methods are applied to problems about graphs.

In recent years much attention has been paid to associate a graph with an algebraic object. The properties of the group are employed to investigate a graph invariant and vice versa. One very old and important such example that highlights the interplay between finite groups and graph theory is the notation of a Cayley graph. The definition of the Cayley graph was first introduced by Arthur Cayley in 1878 [1]. In the last 50 years, there has been a great deal of interest in the theory of Cayley graphs which has played an essential role in algebraic graph theory. It is related to problems in group theory and graph theory such as classification, group and graph isomorphisms, varieties of graph coloring, diameter problems and enumeration problems, (see [7] and [2]). Also, many interesting application of Cayley graph in computer science and biological science, for instance (see [3] and [6]). There are also some important topics of graph theory and group theory in the Cayley graphs of dihedral groups. For instance integrality [8], distance-reqular [9], locally primitive [10], degree diameter problem [4] and edge transitivity [5] of $Cay(D_{2n}, S)$. In this paper, we aim to give the graph structure of $Cay(D_{2n}, S)$ for $n \geq 3$ and $|S| = 1, 2$ or 3.

The terminology and notations used in this paper are standered. For example for a positive integer n, we use Z_n and D_{2n} to denote the cyclic group of order n and dihedral group of order $2n$, respectively. In fact they have the presentation $Z_n =< x \mid x^n = e >= \{e, x, x^2, \cdots, x^{n-1}\}$ and $D_{2n} =< a, b \mid a^n = b^2 = e, bab = a^{-1} >= \{e, a, a^2, \cdots, a^{n-1}, b, ab, \cdots, a^{n-1}b\}$. For a graph X the set of vertices and edges are denoted by $V(G)$ and $E(G)$, respectively. For two vertices $x, y \in V(G)$, we denote $x \sim y$ or $x - y$ if x and y are adjacent. A graph with no edge is called an empty graph. The complement of X, denoted by $\overline{X}$, is a graph such that $V(X) = V(\overline{X})$ and two vertices are adjacent in $\overline{X}$ if and only if there are not adjacent in X. The degree of vertex $x \in V(G)$ denoted by $deg(x)$, is the number of adjacent vertices of x. We denote by K_n, P_n and C_n the complete graph, the path graph and the cycle graph with n vertices, respectively. The union of two graphs X_1 and X_2 denoted by $X_1 \cup X_2$ is a graph with $V(X_1 \cup X_2) = V(X_1) \cup V(X_2)$ and $E(X_1 \cup X_2) = E(X_1) \cup E(X_2)$. If $X_1 = X_2$, then $X_1 \cup X_1$ will denote by $2X_1$ and similarly nX_1 stands for the union of n copies of X_1. Moreover, the Cartesian product of two graphs X_1 and X_2 denoted by $X_1 \square X_2$ is a graph with vertex set $V(X_1 \square X_2) = \{(x, y) \mid x \in V(X_1), y \in V(X_2)\}$. Two vertices (x_1, y_1) and (x_2, y_2) are adjacent if and only if $x_1 = x_2$ and y_1 is adjacent to y_2 in X_2 or $y_1 = y_2$ and x_1 adjacent to x_2 in X_1. For a given group G , the Cayley graph of G with respect to a subset S of G, denoted by $Cay(G, S)$ is an undirect simple graph whose vertices are all elements of G where $e \notin S$ and $S^{-1} \subseteq S$. Two vertices x and y are adjacent if and only if $xy^{-1} \in S$. The size of subset S is called the valency of the Cayley graph. Some of known properties of $Cay(G, S)$ are $|S|$-regular, vertex transitive (automorphism group of the $Cay(G, S)$ acts transitively on $V(G)$ and connected whenever S is a generating set. More background information on graph theory can be found in [12]. For the group theoretical concepts not defined here, we refer the reader to [11].

2. Cayley Graphs of Dihedral Groups of Valencies 1 and 2.

In this section, we investigate and determine the Cayley graphs $\Gamma = Cay(G, S)$ on a dihedral group D_{2n} of valency 1 and 2. Let us start with the case that $\mid S \mid= 1$.

Theorem 2.1. *Let D_{2n} be a dihedral group of order $2n$. If $S \subseteq D_{2n}$ such that $e \notin S$, $S^{-1} = S$ and $|S| = 1$. Then $Cay(D_{2n}, S) = nK_2$.*

Proof. Assume that $S = \{x\}$, then by given definition of S, we should have $x \neq e$ and $x^{-1} = x$. Thus, $x^2 = e$. Now, assume that $g \in D_{2n}$ is an arbitrary element such that $g \neq x$. Then we can see that xg is adjacent to g, since,$(xg)g^{-1} = xgg^{-1} = x \in S$. Now, consider the cyclic subgroup $H =< x >= \{e, x\}$ of D_{2n}, where $|H| = 2$. It tends out that $[D_{2n} : H] = n$. Therefore there are n distinct right cosets $Hg_1 = H, Hg_2, \cdots, Hg_n$, where $g_1 = e \in H$ and $g_2, g_3, \cdots, g_n \notin H$. Moreover, we have $D_{2n} = Hg_1 \cup Hg_2 \cdots \cup Hg_n = \{e, x\} \cup \{g_2, xg_2\} \cup \{g_3, xg_3\} \cup \cdots \cup \{g_n, xg_n\}$. Hence, there are n edges $e - x, g_2 - xg_2, \cdots g_n - xg_n$ and this concludes the proof. $\square$

As an easy example, we can see that $Cay(D_8, S)$, where $S = \{a\}$ is a graph consisting of 4 edges $a - e, a^2 - a^3, b - ab, a^2b - a^3b$. In other words, $Cay(D_8, S) = 4K_2$. Now, we consider $Cay(D_{2n}, S)$, such that $\mid S \mid= 2$. According to conditions $e \notin S$, $S^{-1} = S$, we can see that $S = \{x, x^{-1}\}$ whenever $x \neq e, x^2 \neq e$ or $S = \{x, y\}$, where $x \neq e, y \neq e$ and $x^2 = y^2 = e$. In the following theorem , we give the graph structure of $Cay(D_{2n}, S)$, where $S = \{x, x^{-1}\}$ and $x \neq x^{-1}$.

Theorem 2.2. *Let D_{2n} be a dihedral group of order $2n$. If $S \subseteq D_{2n}$ such that $S = \{x, x^{-1}\}$, where $x \neq x^{-1}$ and $o(x) = m$, then $Cay(D_{2n}, S) = \frac{2n}{m}C_m$.*

Proof. Suppose that $D_{2n} = \{e, a, a^2, a^3, , \cdots, a^{n-1}, b, ab, a^2b, \cdots, a^{n-1}b\}$. Since $x \neq x^{-1}$, we have $o(x) \neq 2$. Also, we know that $o(a^ib) = 2$, for $i = 0, 1, 2, \cdots, n-1$. Therefore, $x = a^i$, for some i, where $i = 1, 2, \cdots, n-1$ and when n is even $i \neq \frac{n}{2}$. Now, assume that n is odd, then $x = a^i$, where $i = 1, 2, \cdots, n-1$. We claim that the following is a cycle of length m $e - x - x^2 - x^3 \cdots - x^{m-1} - x^m = e$. Notice that $(x^k)(x^{k+1})^{-1} = a^{ik}a^{-ik-i} = a^{-i} = x^{-1} \in S$, for $k = 0, 1, 2, \cdots m$. Moreover, let $H =< x >= \{e, x, \cdots, x^{m-1}\}$ be a cyclic subgroup of D_{2n} of order m, then we have $[D_{2n} : H] = \frac{2n}{m} = t$. Thus there are t distinct right cosets $Hg_1 = H, Hg_2, \cdots, Hg_t$, such that $g_1 = e \in H$ and $g_2, g_3, \cdots, g_t \notin H$. It tends out for each right coset $Hg_j = \{g_j, xg_j, x^2g_j, \cdots, x^{m-1}g_j\}$, we have a cycle $g_j - xg_j - x^2g_j - \cdots - x^{m-1}g_j - x^mg_j = g_j$, for $j = 1, 2, \cdots, t$. Therefore, we have t cycles of length m. Hence $Cay(D_{2n}, S) = tC_m = \frac{2n}{m}C_m$. When n is even, then $x = a^{\frac{n}{2}}$ has order 2 and does not satisfy in our assumption. □

In the next theorem, we deal with the second case for S. Take $S = \{x, y\}$ such that $x^2 = y^2 = e$.

Theorem 2.3. *Let D_{2n} be a dihedral group of order $2n$ and $S \subseteq D_{2n}$ such that $S = \{x, y\}$, where $x^2 = y^2 = e$. Then $Cay(D_{2n}, S) = \frac{n}{m}C_{2m}$ where $m = o(xy)$.*

Proof. Consider the case where n is odd. Since $o(x) = o(y) = 2$, then x and y cannot be of the form a^i, where $i = 1, 2, \cdots, n-1$. Thus $x = a^ib$ and $y = a^jb$, where $1 \leq i \neq j \leq n-1$. Now, since $[y(xy)^k][(xy)^{k+1}]^{-1} = y(xy)^k(xy)^{-k-1} = y(xy)^{-1} = yy^{-1}x = x \in S$ and

$[(xy)^k][y(xy)^k]^{-1} = (xy)^k(xy)^{-k}y^{-1} = y^{-1} = y \in S$, we have the following cycle of length $2m$, $e - y - xy - y(xy) - (xy)^2 - y(xy)^2 - (xy)^3 - \cdots - y(xy)^{m-2} - (xy)^{m-1} - y(xy)^{m-1} - (xy)^m = e$. We define the following subgroup $D_{2m} =< (xy), y \mid (xy)^m = y^2 = e, y(xy)y = (xy)^{-1} >= \{e, (xy), (xy)^2, \cdots,$
$(xy)^{m-1}, y, y(xy), \cdots, y(xy)^{m-1}\}$. As above, we have a cycle of length $2m$ between all elements of D_{2m}. On the other hand,

$[D_{2n} : D_{2m}] = \frac{2n}{2m} = \frac{n}{m} = t$. Therefore, we have $D_{2n} = D_{2m} \cup D_{2m}g_2 \cup D_{2m}g_3 \cup \cdots \cup D_{2m}g_t$, where each $D_{2m}g_k$ consists of a cycle of length $2m$ as

the following
$g_k - yg_k - (xy)g_k - y(xy)g_k - (xy)^2g_k - \cdots - y(xy)^{m-1}g_k - (xy)^m g_k = g_k$, for each $k = 2, 3, \cdots, t$. Hence, we have $Cay(D_{2n}, S) = tC_{2m} = \frac{n}{m}C_{2m}$ as desired. If n is even ,then we have the possibility that $x = a^{\frac{n}{2}}$ and $y = a^i b$, then we can follow the same method as above and again we get $Cay(D_{2n}, S) = tC_{2m} = \frac{n}{m}C_{2m}$. □

3. Cayley Graphs of Dihedral Groups of Valency 3.

In this section, we investigate the graph structure of $Cay(D_{2n}, S)$, whenever $\mid S \mid = 3$. Let us start with the following special case.

Theorem 3.1. *Let $D_{2n} =< a, b \mid a^n = b^2 = e, bab = a^{-1} >$ be a dihedral group of order $2n \geq 6$ and $S = \{a, a^{-1}, b\}$. Then $Cay(D_{2n}, S) = K_2 \square C_n$.*

Proof. Assume that $D_{2n} = \{e, a, a^2, a^3, , \cdots, a^{n-1}, b, ab, a^2b, \cdots, a^{n-1}b\}$. Then, we can easily see that $(a^ib)(a^{i+1}b)^{-1} = a^{-1}$, $(a^{n-i})(a^{n-i-1})^{-1} = a$ and $(a^ib)(a^{n-i})^{-1} = b$ for all $i = 0, 1, 2, \cdots, n-1$. Hence, we will get the following graph (see figure 1) which is a Cartesian product of complete graph K_2 and cycle graph C_n of length n. Thus $Cay(D_{2n}, S) = K_2 \square C_n$ as desired. □

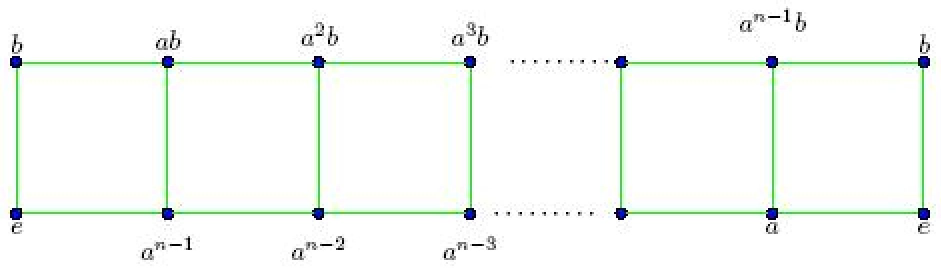

Figure 1: $Cay(D_{2n}, S) = K_2 \square C_n$

Now, we are going to consider the general case that $S = \{x, x^{-1}, y\} \subseteq D_{2n}$, where $x \neq x^{-1}$, $y^2 = e$. In the following theorem, we prove that $Cay(D_{2n}, S)$ with the above assumption is again $K_2 \square C_n$ as similar as theorem 3.1.

Theorem 3.2. *Assume that $S = \{x, x^{-1}, y\} \subseteq D_{2n}$, such that $x \neq x^{-1}$ and $y^2 = e$ and $o(x) = m$. Then $Cay(D_{2n}, S) = \dfrac{n}{m}(K_2 \square C_m)$.*

Proof. Let $D_{2n} = \{e, a, a^2, a^3, , \cdots, a^{n-1}, b, ab, a^2b, \cdots, a^{n-1}b\}$ and $S = \{x, x^{-1}, y\}$ is a subset of D_{2n} of size 3. Since ,$x \neq x^{-1}$, so $o(x) = m > 2$ which implies that x can not be as the form a^ib, for all $0 \leq i \leq n-1$. Thus $x = a^i$ for some $0 \leq i \leq n-1$ and we have $m = o(x) = o(a^i) = \frac{n}{(i,n)}$. Similarly, y is as the form a^jb for some $0 \leq j \leq n-1$ or possibly $a^{\frac{n}{2}}$ when n is even. To prove the theorem, we need the following three steps.

Step 1: Assume that $H =< x >$ is a cyclic subgroup of order m. Then $H = \{e, x, x^2, \cdots, x^{m-1}\}$ consists a cycle graph of length m as the following
$e - x - x^2 - x^3 - \cdots - x^{m-2} - x^{m-1} - e$. The proof is obvious.

Step 2 : Let Hz be a right coset of H in D_{2n}, where $z \notin H$. Then it consists the following cycle graph of length m: $z - xz - x^2z - x^3z - \cdots - x^{m-2}z - x^{m-1}z - z$. Because $(x^iz)(x^{i+1}z)^{-1} = x^{-1} \in S$ for all $i = 0, 1, 2, \cdots, n-1$.

Step 3: Suppose that Hz and Hw are two distinct right cosets of H in D_{2n}. If z is adjacent to w in $Cay(D_{2n}, S)$, then $Hz \cup Hw$ is produced an induced subgraph of $Cay(D_{2n}, S)$ isomorphic to the Cartesian product $K_2 \square C_m$. One can easily check the proof by the corresponding graph as in figure 2.

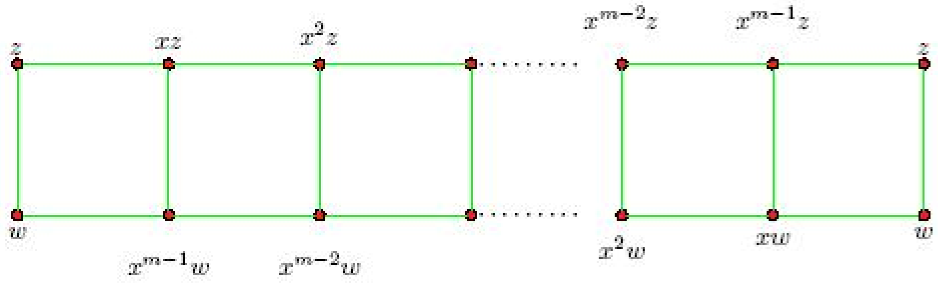

Figure 2: $K_2 \square C_m$

Now by the above three steps, we can continue our proof of the theorem. Assume that $[D_{2n} : H] = t$, then $\frac{2n}{m} = t$ and so t is even. We know that $Cay(D_{2n}, S)$ is 3-regular and it implies that every pair of cosets (Hz, Hw) will produce a Cartesian product $K_2 \Box C_m$. Since, we have $\frac{t}{2}$ disjoint pairs of such cosets, so we will have union of $\frac{n}{m}$ Cartesian product $K_2 \Box C_m$. Hence $Cay(D_{2n}, S) = \frac{n}{m}(K_2 \Box C_m)$ and the proof is complete. $\Box$

Example 3.3. *For Dihedral group D_{12} of order 12, we have $D_{12} =< a, b \mid a^6 = b^2 = e, bab = a^{-1} >= \{e, a, a^2, a^3, a^4, a^5, b, ab, a^2b, a^3b, a^4b, a^5b\}$. If $S = \{x, x^{-1}, y\} \subset D_{12}$, where $x = a^2$ and $y = ab$. Then by Theorem 3.2, we can see that $Cay(D_{12}, S) = 2(K_2 \Box C_3)$. Because, we have $o(x) = o(a^2) = 3$, $H =< x >= \{e, a^2, a^4\}$ and distinct cosets $Ha = \{a, a^3, a^5\}$, $Hb = \{b, a^2b, a^4b\}$ and $Hab = \{ab, a^3b, a^5b\}$. For more details we refer to figure 3.*

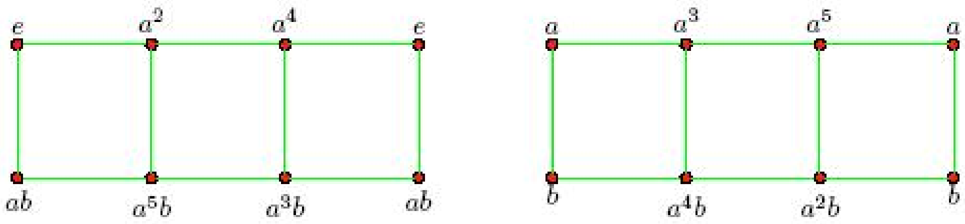

Figure 3: $Cay(D_{12}, S) = 2(K_2 \Box C_3)$

The following example is another possibility for $S \subseteq D_{2n}$ with $|S| = 3$. In fact, $S = \{x, y, z\}$, where $x^2 = y^2 = z^2 = e$.

Example 3.4. *Let D_8 be a dihedral group of order 8. Then we have $D_8 =< a, b \mid a^4 = e = b^2, bab = a^{-1} >= \{e, a, a^2, a^3, b, ab, a^2b, a^3b\}$. If $S = \{b, ab, a^2b\}$, then $Cay(D_8, S)$ is the Cartesian product $K_2 \Box C_4$ as the following:*

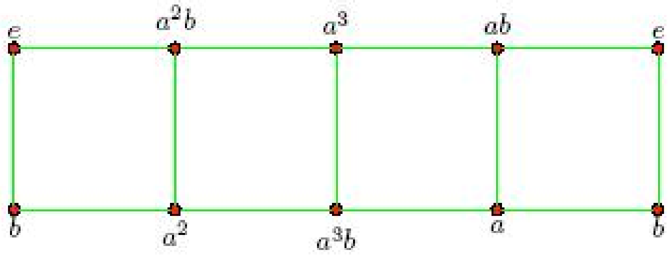

Figure 4: $K_2 \square C_4$

Now by extending Theorem 2.3 and the method used in Theorem 3.2, we can state the following theorem consisting the last case of S with $\mid S \mid = 3$. We omit the proof.

Theorem 3.5. *Let $S = \{x, y, z\}$ be a subset of D_{2n}, where $n \geq 3$, $|S| = 3$ and $x^2 = y^2 = z^2 = e$. Then $Cay(D_{2n}, S) = K_2 \square C_n$.*

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