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Local vertex antimagic chromatic number of some wheel related graphs

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Abstract

Let $G = (V, E)$ be a graph of order p and size q having no isolated vertices. A bijection $f : E \to \{1, 2, 3, ..., q\}$ is called a local antimagic labeling if for all $uv \in E$ we have $w(u) \neq w(v)$, the weight $w(u) = \sum_{e \in E(u)} f(e)$, where $E(u)$ is the set of edges incident to u. A graph G is local antimagic if G has a local antimagic labeling. The local antimagic chromatic number $\chi_{la}(G)$ is defined to be the minimum number of colors taken over all colorings of G induced by local antimagic labelings of G. In this paper, we determine the local antimagic chromatic number for some wheel related graphs.

Keywords: *Local antimagic labeling, Local antimagic chromatic number, Helm graph.*

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1. Introduction

The graph $G = (V, E)$ we mean a finite, undirected graph with neither loops nor multiple edges. The order and size of G are denoted by $|V| = p$ and $|E| = q$ respectively. For graph-theoretic terminology, we refer to Chartrand and Lesniak [4].

Hartsfield and Ringel [7] was first introduced an antimagic labeling, which is defined as a bijection $f : E \to \{1, 2, ..., |E|\}$. For each vertex $u \in V(G)$, the weight $w(u) = \sum_{e \in E(u)} f(e)$, where $E(u)$ is the set of edges incident to u. If $w(u) \neq w(v)$ for any two distinct vertices u and $v \in V(G)$, then f is called an antimagic labeling of G. A graph G is called antimagic if G has antimagic labeling. Hartsfield and Ringel's [7] conjectured that every connected graph with at least three vertices admits antimagic labeling. They also made a weak conjecture that every tree with at least three vertices admits an antimagic labeling. These two conjectures were partially shown to be true by several authors, but they are still unsolved. For the best and most interesting results were obtained so far, one can see [10] for trees and [5] for general graphs. Also, for a detailed and interesting review on these conjectures, one can see chapter 6 of [6].

Arumugam et al.[1] posed a new definition as a relaxation of the notion of antimagic labeling. They called a bijection $f : E \to \{1, 2, ..., |E|\}$ is a *local antimagic labeling* of G if for any two adjacent vertices u and v in $V(G)$, the condition $w(u) \neq w(v)$ holds. They conjectured that every connected graph with at least three vertices admits a local antimagic labeling. This conjecture was solved partially in [3]. Finally, Haslegrave proved this conjecture by means of probabilistic tools [8]. In 2017, Bensmail, Senhaji, and Szabo Lyngsie [3] obtained the results on trees answers positively to Conjecture 2.3 raised by Arumugam et al.[1] using another aspect of neighbour-sum-distinguishing.

Based on the notion of local antimagic labeling, Arumugam et al.[1] introduced a new graph coloring parameter. We call *local antimagic chromatic number* $\chi_{la}(G)$, which is defined as the minimum number of colors taken over all colorings of G induced by local antimagic labelings of G.

In [1], they proved the local chromatic number of cycle on n vertices is 3 and they observed that, if the connected graph G contains a triangle C_3, then $\chi_{la}(G) \geq 3$. Also, they proved that the local chromatic number of complete graph on p vertices is p colors and the local chromatic number of wheel graph W_n is 4, where n is odd. For n is even case they proved that $\chi_{la}(W_n) = 3$, where $n \equiv 2 (\mathrm{mod}\ 4)$. Then they obtained lower and upper bounds $3 \leq \chi_{la}(W_n) \leq 5$ for $n \equiv 0 (\text{mod } 4)$. Recently, in [11], the authors obtained exact value for $\chi_{la}(W_n) = 4$ for $n \equiv 0 (\text{mod } 4)$.

Frucht and Harary [15] introduced the corona product of two graphs, which is defined as: The corona product of two graphs G and H is the graph $G \odot H$ obtained by taking one copy of G along with $|V(G)|$ copies of H, and putting extra edges making the i-th vertex of G adjacent to every vertex of the i-th copy of H, where $1 \leq i \leq |V(G)|$. In [2] Arumugam, Lee, Premalatha and Wang completely determined the local antimagic chromatic number $\chi_{la}(G \odot \overline{K_m})$ for the corona product of a graph G with the null graph $\overline{K_m}$ on $m \geq 1$ vertices (or complement graph of K_m), when G is path P_n, cycle C_n, and complete graph K_n.

Arumugam et al.[1], Shaebani [9] and Lau et al.[11, 14] studied independently, the local antimagic chromatic number for the join graphs, which is defined as: Let G_1 and G_2 be two vertex disjoint graphs. The *join graph* of G_1 and G_2, denoted by $G_1 \vee G_2$, is the graph whose vertex set is $V(G_1) \cup V(G_2)$ and its edge set equals $E(G_1) \cup E(G_2) \cup \{ab : a \in V(G_1) \text{ and } b \in V(G_2)\}$.

Arumugam et al.[1] obtained lower and upper bounds for the join graph $G_1 \vee G_2$. Recently, the author Shaebani [9] obtained counterexamples to a Theorem 2.16 [1] which asserts that if a graph G has at least four vertices, then $\chi_{la}(G) + 1 \leq \chi_{la}(G \vee \overline{K_2})$, where $\overline{K_2}$ is the complement graph of a complete graph with two vertices. In this regard, the author Shaebani [9] proved that if n is odd and $n+1$ is not divisible by 3, then $\chi_{la}(K_{1,n} \vee \overline{K_2}) = 3$. Moreover, Lau et al.[11, 14] also studied the local antimagic chromatic number for join graphs independently.

In this paper, we investigate the local antimagic chromatic number for wheel related graphs.

2. Local Chromatic Number of Helm graph

In [12], Ayel and Favaron introduced a Helm graph, which is defined as: The *helm graph* is the graph obtained from a wheel graph W_n on $n + 1$ vertices by adjoining a pendant edge at each node of the cycle and H_n denotes it.

Krishnaa [13] studied some helm related graphs that admit antimagic labeling. In this section, we study the local antimagic vertex coloring of the same graph H_n.

Theorem 2.1. For the graph $H_n, 3 \leq n \leq 5$, we have

$$\chi_{la}(H_n) = \begin{cases} 6, & n = 3, 4 \\ 8, & n = 5. \end{cases}$$

Proof. Let $G \cong H_n$ be the helm graph and let $V(G) = \{c \cup v_i \cup u_i, 1 \leq i \leq n\}$ and $E(G) = \{cv_i \cup v_i u_i, 1 \leq i \leq n\} \cup \{v_n v_1 \cup v_i v_{i+1}, 1 \leq i \leq n-1\}$. Then $|V(G)| = p = 2n + 1$ and $|E(G)| = q = 3n$.

Case 1: $n = 3$
Suppose H_3 admits a local antimagic labeling f. Since every pendant vertex received a new color, it follows that the pendant vertices u_1, u_2 and u_3 have received the colors w_1, w_2 and w_3. Since $1 \leq w(u_i) \leq 9$ and the minimum possible vertex weight of v_i is 10, we get $w(u_i) \neq w(v_i), i = 1, 2, 3$ and hence the vertex v_1, v_2 and v_3 are received new colors w_4, w_5 and w_6. Thus $\chi_{la}(H_3) \geq 6$. So, for proving $\chi_{la}(H_3) = 6$, it suffices to provide a local antimagic labeling of H_3 that induces a local antimagic vertex coloring using exactly six colors. Now, we define bijection $f : E(H_3) \to \{1, 2, 3, \ldots, 9\}$ by

$f(cv_i) = i, \quad i = 1, 2, 3$
$f(v_i u_i) = 7 - i, \quad i = 1, 2, 3$
$f(v_1 v_2) = 8, \quad f(v_2 v_3) = 7, \quad f(v_1 v_3) = 9.$

In this case, we have $w(c) = w(u_1) = 6, w(u_2) = 5, w(u_3) = 4, w(v_1) = 24, w(v_2) = 22, w(v_3) = 23$. Therefore, f is a local antimagic labeling of H_3 that induces a local antimagic vertex coloring using exactly six colors.

Case 2: $n = 4$
Suppose H_4 admits a local antimagic labeling f. Since every pendant vertex received a new color, it follows that the pendant vertices u_1, u_2, u_3 and u_4 are received the colors w_1, w_2, w_3 and w_4. Let $e = cv_1$ or $e = v_1 u_1$ or $v_1 v_2$ in $E(H_4)$. For any of v_1, v_2, v_3, v_4, c to receive a color less than or equal to 12, the four incident edges must include labels 1 and 2. This means that at most one of them can receive such a color, and therefore there are some two adjacent vertices which receive new colors w_5, w_6 greater than 12. Thus $\chi_{la}(H_4) \geq 6$. So, for proving $\chi_{la}(H_4) = 6$, it suffices to provide a local antimagic labeling of H_4 that induces a local antimagic vertex coloring using exactly six colors. Now, we define bijection $f : E(H_4) \to \{1, 2, 3, \ldots, 12\}$ by

$f(cv_1) = 1, \quad f(cv_2) = 3, \quad f(cv_3) = 2, \quad f(cv_4) = 4,$
$f(v_1 u_1) = 9, \quad f(v_2 u_2) = 12, \quad f(v_3 u_3) = 10, \quad f(v_4 u_4) = 11,$
$f(v_1 v_2) = 6, \quad f(v_2 v_3) = 7, \quad f(v_3 v_4) = 5, \quad f(v_1 v_4) = 8.$

In this case, we have $w(c) = w(u_3) = 10, w(u_1) = 9, w(u_2) = 12, w(u_4) = 11, w(v_1) = w(v_3) = 24, w(v_2) = w(v_4) = 28$. Therefore, f is a local antimagic labeling of H_4 that induces a local antimagic vertex coloring using exactly six colors.

Case 3: $n = 5$

Suppose H_5 admits a local antimagic labeling f. Since every pendant vertex received a new color, it follows that the pendant vertices u_1, u_2, u_3, u_4 and u_5 are received the colors w_1, w_2, w_3, w_4 and w_5. Let $e = cv_1$ or v_1u_1 or v_1v_2 in $E(H_5)$. If $f(e) = 15$ then the incident vertex v_1 received a new color w_6.

If $f(e = cv_1) = 15$ then the vertex c received a color $w(c) = w(u_i)$ or a new color w_7. If $w(c) = w(u_i)$ for any i, then $w(c) = 15$ because the minimum possible vertex weight of c is 15 and $q = 15$. If additionally $w(v_2) = w(u_i) \in \{10, 11, 12, 13, 14\} - w(u_2)$, then $f(v_2u_2) \in \{6, 7, 8, 9\}$. If $f(v_2u_2) \in \{6, 7, 8, 9\}$ then the vertex v_2 weight is $w(v_2) \neq q$ and hence the vertex v_2 received a new color w_7. Suppose $w(v_3) = w_6$ and $w(v_4) = w_7$. Then the vertex v_5 must receive a new color w_8. If $w(c) = w_7$ then the adjacent vertex of v_1 is v_2 received a new color w_8.

If $f(e = v_1u_1) = 15$ then the vertex c received a color $w(u_1) = 15$ and hence $w(c) = w(u_1) = 15$. Therefore, the edges $\{cv_i, 1 \leq i \leq 5\}$ are received the labels from $\{1, 2, 3, 4, 5\}$. Then the vertex v_2 adjacent vertices are received the minimum possible labels are $\{1, 6, 7, 8\}$, this sum gives the minimum possible weight of a vertex v_2. Therefore, the vertex v_2 received a new color w_7. Suppose $w(v_3) = w_6$ and $w(v_4) = w_7$. Then the vertex v_5 must receive a new color w_8.

If $f(e = v_1v_2) = 15$ then the vertex v_2 received a new color w_7. Suppose $w(v_3) = w_6$ or $w(u_i), i \neq 3$ and $w(v_4) = w_7$ or $w(u_i), i \neq 4$. Then the vertex v_5 must receive a new color w_8. Hence $\chi_{la}(H_5) \geq 8$.

So, for proving $\chi_{la}(H_5) = 8$, it suffices to provide a local antimagic labeling of H_5 that induces a local antimagic vertex coloring using exactly eight colors. Now, we define bijection $f : E(H_5) \to \{1, 2, 3, \ldots, 15\}$ by

$$\begin{array}{lllll} f(cv_1) = 1, & f(cv_2) = 4, & f(cv_3) = 2, & f(cv_4) = 5, & f(cv_5) = 3, \\ f(v_1u_1) = 11, & f(v_2u_2) = 13, & f(v_3u_3) = 12, & f(v_4u_4) = 14, & f(v_5u_5) = 15, \\ f(v_1v_2) = 7, & f(v_2v_3) = 9, & f(v_3v_4) = 6, & f(v_4v_5) = 8, & f(v_1v_5) = 10. \end{array}$$

In this case, we have $w(c) = w(u_5) = 15, w(u_1) = 11, w(u_2) = 13, w(u_3) = 12, w(u_4) = 14, w(v_1) = w(v_3) = 29, w(v_2) = w(v_4) = 33, w(v_5) = 36$. Therefore, f is a local antimagic labeling of H_5 that induces a local antimagic vertex coloring using exactly eight colors. □

Theorem 2.2. Let H_n, $n \geq 6$ be a helm graph. Then $\chi_{la}(H_n) = n+3$.

Proof. Let $H_n, n \geq 6$ be a helm graph and let $V(H_n) = \{c \cup v_i \cup u_i, 1 \leq i \leq n\}$ and $E(H_n) = \{cv_i \cup v_iu_i, 1 \leq i \leq n\} \cup \{v_nv_1\} \cup \{v_iv_{i+1}, 1 \leq i \leq n-1\}$. Then $|V(H_n)| = p = 2n+1$ and $|E(H_n)| = q = 3n$.

Suppose H_n admits a local antimagic labeling f. Since every pendant vertex received a new color, it follows that the pendant vertices $u_i, 1 \leq i \leq n$ are received the colors $w_i, 1 \leq i \leq n$. Clearly, the minimum possible weight of a vertex c is $w(c) \geq \frac{n(n+1)}{2} > q$. Therefore, the vertex c received a new color w_{n+1}. Let $e = cv_1$ or v_1u_1 or v_1v_2 in $E(H_n)$. If $f(e = v_1u_1) = q$ or $f(e = cv_1) = q$ then the vertex v_1 received a new color w_{n+2}. Let $e' = v_1v_2 \in E(H_n)$. Then the vertex v_2 weight $w(v_2) \in W$ or $W' = W - \{q\}$ and $w(v_2) \notin W$ or $W' = W - \{q\}$, where $W = \{10, 11, 12, \ldots, q-1, q\}$. If the vertex v_2 weight $w(v_2) \notin W$ or $W' = W - \{q\}$ then the vertex v_2 received a new color w_{n+3} and hence $\chi_{la}(H_n) \geq n+3$.

Let S be the set of v_i that have weight w_{n+2}, and T be the remainder. Since no two vertices in S are adjacent, T contains $(n+k)/2$ vertices for some $k \geq 0$, and covers all the edges of the main cycle together with at least $n+k$ other edges. So the sum of weights of vertices in T is at least

$$1+2+3+\ldots+2n+k = \frac{(2n+k)(2n+k+1)}{2} \geq (n+k)(2n+1).$$

However, the maximum possible sum of weights of vertices in T if all these are in W or W' is $3n(n+k)/2$. Therefore at least one of these vertices has a new weight w_{n+3}. Thus $\chi_{la}(H_n) \geq n+3$.

So, for proving $\chi_{la}(H_n) = n+3$ it suffices to provide a local antimagic labeling of H_n that induces a local antimagic vertex coloring using exactly $n+3$ colors.

For $n \geq 6$, define $f_1 : E(H_n) \to \{1, 2, 3, \ldots, q\}$ by

$$f_1(cv_i) = \begin{cases} \frac{n+3}{2}, & i = 1, n \text{ is odd} \\ \frac{n+2}{2}, & i = 1, n \text{ is even} \\ \frac{3(2n+1)-i}{2}, & i \text{ is odd}, 3 \leq i \leq n \\ \frac{2(n+2)+i}{2}, & i \text{ is even}, 2 \leq i \leq n \end{cases}$$

$$f_1(v_iv_{i+1}) = \begin{cases} \frac{i+1}{2}, & i \text{ is odd}, 1 \leq i \leq n-1 \\ \frac{2(n+3)-i}{2}, & i \text{ is even}, 2 \leq i \leq n-1 \end{cases}$$

$$f_1(v_nv_1) = \begin{cases} \frac{n+1}{2}, & n \text{ is odd} \\ \frac{n+6}{2}, & n \text{ is even} \end{cases}$$

$$f_1(v_iu_i) = \begin{cases} \frac{n+5}{2}, & i = 1, n \text{ is odd} \\ \frac{n+4}{2}, & i = 1, n \text{ is even} \\ \frac{4n+i+1}{2}, & i \text{ is odd, } n \text{ is odd}, 3 \le i \le n \\ \frac{4(n+1)-i}{2}, & i \text{ is even, } n \text{ is odd}, 2 \le i \le n-1 \\ \frac{4n+i+3}{2}, & i \text{ is odd, } n \text{ is even}, 3 \le i \le n-1 \\ \frac{2(2n+3)-i}{2}, & i \text{ is even, } n \text{ is even}, 2 \le i \le n \end{cases}$$

Then the weights of vertices are,

$$w_1(v_i) = \begin{cases} \frac{3n+11}{2}, & i = 1, n \text{ is odd} \\ \frac{3n+14}{2}, & i = 1, n \text{ is even} \\ 6(n+1), & i \text{ is odd, } n \text{ is odd}, 3 \le i \le n \\ 4n+7, & i \text{ is even, } n \text{ is odd}, 2 \le i \le n-1 \\ 6n+7, & i \text{ is odd, } n \text{ is even}, 3 \le i \le n-1 \\ 4(n+2), & i \text{ is even, } n \text{ is even}, 2 \le i \le n \end{cases}$$

$$w_1(c) = \begin{cases} 2n^2, & n \text{ is odd} \\ \frac{n(4n-1)}{2}, & n \text{ is even} \end{cases}$$

$$w_1(u_i) = f_1(v_iu_i), 1 \le i \le n$$

Hence, $\chi_{la}(H_n) \le n+3$. Therefore, f_1 is a local antimagic labeling of H_n that induces a local antimagic vertex coloring using exactly $n+3$ colors. □

From Theorem 2.1 and Theorem 2.2, we obtain the following theorem.

Theorem 2.3. Let $H_n, n \ge 3$ be a helm graph. Then

$$\chi_{la}(H_n) = \begin{cases} n+2, & n = 4 \\ n+3, & n \ne 4. \end{cases}$$

3. Local Chromatic Number for Some Wheel Related Graphs

In this section, we determine the local vertex antimagic chromatic number for wheel related graphs W_n^m, where m is a fixed positive integer.

Definition 3.1. Let W_n be a wheel graph on $n+1$ vertices. A graph W_n^m is obtained from W_n by attaching m pendant vertices to any arbitrary vertex $v \ne c$ of W_n, where c is the central vertex of W_n.

Now, we consider the graph W_n^m, with two different m values $m_1 = t + \frac{n(n-3)}{2}, t \geq 0$ and less than m_1,that is, $\frac{n(n-3)}{2} - t, t \geq 1$. The following theorem gives the exact local chromatic number of $W_n^m, m = m_1$ is $m_1 + 1$,for $n \geq 7$ and for every $t \geq 0$.

Theorem 3.2. Let $W_n^{m_1}$ be a graph with $n \geq 7, t \geq 0$ and $m_1 = t + \frac{n(n-3)}{2}$. Then $\chi_{la}(W_n^{m_1}) = m_1 + 1$.

Proof. Let $W_n^{m_1}$ be a graph with $n \geq 7$ and $m_1 = t + \frac{n(n-3)}{2}$. Let $V(W_n^{m_1}) = \{c \cup v_i \cup u_k, 1 \leq i \leq n, 1 \leq k \leq m_1\}$ and $E(W_n^{m_1}) = \{cv_i \cup v_n u_k, 1 \leq i \leq n, 1 \leq k \leq m_1\} \cup \{v_i v_{i+1} \cup v_n v_1, 1 \leq i \leq n-1\}$ be the vertex set and edge set of $W_n^{m_1}$. Then $|V(W_n^{m_1})| = m_1 + n + 1$ and $|E(W_n^{m_1})| = q = m_1 + 2n$.

Since every pendant vertex receives a different color, and v_n receives a higher color than any of them. We get $\chi_{la}(W_n^{m_1}) \geq m_1 + 1$. So, for proving $\chi_{la}(W_n^{m_1}) = m_1 + 1$ it suffices to provide a local antimagic labeling of $W_n^{m_1}$ that induces a local antimagic vertex coloring using exactly $m_1 + 1$ colors. Now, we define $f_2 : E(W_n^{m_1}) \to \{1, 2, 3, ..., m_1 + 2n\}$ by

$$f_2(cv_i) = \begin{cases} n+1-i, & n \text{ is even}, i \text{ is odd}, 1 \leq i \leq n-1 \\ n-1-i, & n \text{ is even}, i \text{ is even}, 2 \leq i \leq n-2 \\ n-1, & n \text{ is even}, i = n \\ n-1-i, & n \text{ is odd}, i \text{ is odd}, 1 \leq i \leq n-2 \\ n, & n \text{ is odd}, i = n \\ n+1-i, & n \text{ is odd}, i \text{ is even}, 1 \leq i \leq n-1 \end{cases}$$

$$f_2(v_n v_1) = n + 1$$

$$f_2(v_n u_k) = 2n + k, 1 \leq k \leq m_1$$

$$f_2(v_i v_{i+1}) = \begin{cases} \frac{3n+i+1}{2}, & n \text{ is even}, i \text{ is odd}, 1 \leq i \leq n-1 \\ \frac{2n+2+i}{2}, & n \text{ is even}, i \text{ is even}, 2 \leq i \leq n-2 \\ \frac{3n+2+i}{2}, & n \text{ is odd}, i \text{ is odd}, 1 \leq i \leq n-2 \\ \frac{2n+2+i}{2}, & n \text{ is odd}, i \text{ is even}, 1 \leq i \leq n-1. \end{cases}$$

Then the vertex weights are

$$w_2(c) = w_2\left(u_{\frac{n(n-3)}{2}}\right) = \frac{n(n+1)}{2}$$
$$w_2(v_i) = \begin{cases} \frac{7n+4}{2}, & n \text{ is even}, i \text{ is odd}, 1 \le i \le n-1 \\ \frac{7n}{2}, & n \text{ is even}, i \text{ is even}, 2 \le i \le n-2 \\ \frac{7n+1}{2}, & n \text{ is odd}, i \text{ is odd}, 1 \le i \le n-2 \\ \frac{7n+5}{2}, & n \text{ is odd}, i \text{ is even}, 2 \le i \le n-1 \\ \frac{8n+4nm_1+m_1(m_1+1)}{2}, & n \text{ is even}, i = n \\ \frac{7n+3+4nm_1+m_1(m_1+1)}{2}, & n \text{ is odd}, i = n \end{cases}$$
$$w_2(u_k) = 2n + k, 1 \le k \le m_1.$$

Hence, $\chi_{la}(W_n^{m_1}) \le m_1 + 1$. Therefore, f_2 is a local antimagic labeling of $W_n^{m_1}$ that induces a local antimagic vertex coloring using exactly $m_1 + 1$ colors. □

From Theorem 3.2, we observe the local chromatic number for the graph W_n^m, with $m < m_1$ is greater than $m + 1$. We consider $m = \frac{n(n-3)}{2} - t < m_1, t \ge 1$. Since all the pendant vertices of W_n^m received new color say $w_i, 1 \le i \le m$, and these pendant vertices adjacent to a vertex v also received a new color w_{m+1}. Clearly, the minimum possible central vertex weight is $w(c) = \frac{n(n+1)}{2} > q = m + 2n$, $n \ge 4$. Therefore, the central vertex c received a new color w_{m+2}. Hence $\chi_{la}(W_n^m) \ge m+2$ and gives the following remark.

Remark 3.3. *Let W_n^m, $n \ge 4$ be a graph with $m = \frac{n(n-3)}{2} - t, t \ge 1$. Then $\chi_{la}(W_n^m) \ge m + 2$.*

Definition 3.4. Let W_n^m, $n \ge 5$ be a wheel graph on $n+1$ vertices. A graph $\widehat{W_n^{m_2}}$ is obtained from W_n by attaching $m_2 = \lceil \frac{n-4}{2} \rceil$, pendant vertices in every vertex $v \ne c$ of W_n.

Theorem 3.5. *Let $\widehat{W_n^{m_2}}$ be a graph, where $n \ge 5$ and $m_2 = \lceil \frac{n-4}{2} \rceil$. Then $\chi_{la}(\widehat{W_n^{m_2}}) = m_2 n + 3$.*

Proof. Let $\widehat{W_n^{m_2}}$ be the graph and let $V(\widehat{W_n^{m_2}}) = \{c \cup v_i \cup u_i^k, 1 \le i \le n, 1 \le k \le m_2\}$ and $E(\widehat{W_n^{m_2}}) = \{cv_i \cup v_i v_{i+1} \cup v_n v_1 \cup v_i u_i^k, 1 \le i \le n, 1 \le k \le m_2\}$. Then $|V(\widehat{W_n^{m_2}})| = n(m_2+1) + 1$ and $|E(\widehat{W_n^{m_2}})| = n(m_2+2)$.

Since every pendant vertex received a new color it follows that, the vertices u_i^k are received the colors $w_t, 1 \le t \le m_2 n$. If n is even, then the minimum possible weight of the central vertex c is $w(c) = \frac{n(n+1)}{2} > q$ and hence the vertex c received a new color $w_{m_2 n+1}$. Let $e \in E(\widehat{W_n^{m_2}})$. If

$f(e = v_1v_2) = q$ then its incident vertex v_1 received a new color w_{m_2n+2} and hence the vertex v_2 received a new color w_{m_2n+3}. A similar argument applies for the case of $f(e = v_1u_1) = q$ or $f(e = cv_1) = q$ in Theorem 2.2 with substituting u_1 by u_1^k and $q = n(m_2n+2)$. Thus the vertex v_2 received a new color w_{m_2n+3}.

If n is odd, the minimum possible weight of the central vertex is q. If it receives a higher weight, we proceed as above. If it receives weight q then we necessarily have $f(cv_1), \ldots, f(cv_n)$ being $1, 2, \ldots, n$ in some order. It follows that $w(v_i) > (d(v_i) - 1)(n + 1) + 1 > q$, so all colors used on $\{v_1, \ldots, v_n\}$ are new. Since it is an odd cycle, there are at least three new colors used. Thus, where $d(v_i)$ is degree of the vertex v_i. $\chi_{la}(\widehat{W_n^{m_2}}) \geq m_2n + 3$.

So, for proving $\chi_{la}(\widehat{W_n^{m_2}}) = m_2n + 3$ it suffices to provide a local antimagic labeling of $\widehat{W_n^{m_2}}$ that induces a local antimagic vertex coloring using exactly $m_2n + 3$ colors. If $m_2 = 1$ then $n = 5$ and 6. Clearly, the graph $\widehat{W_5^1} \cong H_5$ and $\widehat{W_6^1} \cong H_6$ and by Theorem 2.3, we get $\chi_{la}(\widehat{W_n^{m_2}}) = n + 3$, where $n = 5, 6$.

If $m_2 = 2$ then we get $n = 7$ and 8. For $n = 7$, we define $f : E(\widehat{W_7^2}) \to \{1, 2, 3, ..., 28\}$ by

$$f(cv_i) = i, 1 \leq i \leq 7$$
$$f(v_iv_{i+1}) = 15 - i, 1 \leq i \leq 6$$
$$f(v_7v_1) = 8, \quad f(v_1u_1^1) = 27, \quad f(v_1u_1^2) = 28,$$
$$f(v_iu_i^k) = \begin{cases} 14 + \frac{i}{2}, & i = 2,4,6 \text{ and } k = 1 \\ 17 + \frac{i}{2}, & i = 2,4,6 \text{ and } k = 2 \end{cases}$$
$$f(v_iu_i^k) = \begin{cases} 20 + \frac{i-1}{2}, & i = 3,5,7 \text{ and } k = 1 \\ 23 + \frac{i-1}{2}, & i = 3,5,7 \text{ and } k = 2. \end{cases}$$

In this case, we have $w(c) = w(u_1^2) = 28, w(u_i^k) = f(v_iu_i^k), w(v_1) = 78, w(v_3) = w(v_5) = w(v_7) = 73, w(v_2) = w(v_4) = w(v_6) = 62$. Therefore, f is a local antimagic labeling of $\widehat{W_7^2}$ that induces a local antimagic vertex coloring using exactly 17 colors.

For $n = 8$, we define $f : E(\widehat{W_8^2}) \to \{1, 2, 3, ..., 32\}$ by

$$f(cv_i) = i, 1 \leq i \leq 8$$
$$f(v_iv_{i+1}) = 17 - i, 1 \leq i \leq 7$$
$$f(v_8v_1) = 9 \quad f(v_1u_1^1) = 20, \quad f(v_1u_1^2) = 24$$
$$f(v_iu_i^k) = \begin{cases} 25 + \frac{i-2}{2}, & i = 2,4,6,8 \text{ and } k = 1 \\ 29 + \frac{i-2}{2}, & i = 2,4,6,8 \text{ and } k = 2 \end{cases}$$
$$f(v_iu_i^k) = \begin{cases} 16 + \frac{i-1}{2}, & i = 3,5,7 \text{ and } k = 1 \\ 20 + \frac{i-1}{2}, & i = 3,5,7 \text{ and } k = 2. \end{cases}$$

In this case, we have $w(c) = 36, w(u_i^k) = f(v_i u_i^k), w(v_1) = w(v_3) = w(v_5) = w(v_7) = 70, w(v_2) = w(v_4) = w(v_6) = w(v_8) = 87$. Therefore, f is a local antimagic labeling of $\widehat{W_8^2}$ that induces a local antimagic vertex coloring using exactly 19 colors.

For $n \geq 9$, we define a labeling $f_3 : E(\widehat{W_n^{m_2}}) \to \{1, 2, 3, \dots, nm_2 + 2n\}$ as follows:

Case 1: $m_2 \geq 3$ is odd

$$f_3(cv_i) = \begin{cases} \frac{i+1}{2}, & n \text{ is odd}, i \text{ is odd}, 1 \leq i \leq n-2 \\ \frac{2n-i}{2}, & n \text{ is odd}, i \text{ is even}, 2 \leq i \leq n-1 \\ n, & n \text{ is odd}, i = n \\ \frac{i+1}{2}, & n \text{ is even}, i \text{ is odd}, 1 \leq i \leq n-1 \\ \frac{2n+2-i}{2}, & n \text{ is even}, i \text{ is even}, 2 \leq i \leq n \end{cases}$$

$$f_3(v_i v_{i+1}) = \begin{cases} \frac{3n+i}{2}, & n \text{ is odd}, i \text{ is odd}, 1 \leq i \leq n-2 \\ \frac{3n+1-i}{2}, & n \text{ is odd}, i \text{ is even}, 2 \leq i \leq n-1 \\ \frac{2n+1+i}{2}, & n \text{ is even}, i \text{ is odd}, 1 \leq i \leq n-1 \\ \frac{4n-i}{2}, & n \text{ is even}, i \text{ is even}, 2 \leq i \leq n-2 \end{cases}$$

$$f_3(v_n v_1) = 2n$$

$$f_3(v_i u_i^1) = \begin{cases} 2n+1, & n \text{ is odd}, i = 1 \\ \frac{5n+2-i}{2}, & n \text{ is odd}, i \text{ is odd}, 3 \leq i \leq n-2 \\ \frac{5n-1+i}{2}, & n \text{ is odd}, i \text{ is even}, 2 \leq i \leq n-1 \\ 3n, & n \text{ is odd}, i = n \\ \frac{5n+1-i}{2}, & n \text{ is even}, i \text{ is odd}, 1 \leq i \leq n-1 \\ \frac{5n+2+i}{2}, & n \text{ is even}, i \text{ is even}, 2 \leq i \leq n-2 \\ \frac{5n+2}{2}, & n \text{ is even}, i = n \end{cases}$$

$$f_3(v_i u_i^k) = \begin{cases} 3n + \frac{n-1}{2}(k-2) + (\frac{i+1}{2}), & n \text{ is odd}, i \text{ is odd}, k \text{ is even}, \\ & 1 \leq i \leq n-2, 2 \leq k \leq m_2 \\ 3n + \frac{n-1}{2}(k-1) - (\frac{i-1}{2}), & n \text{ is odd}, i \text{ is odd}, k \text{ is odd}, \\ & 1 \leq i \leq n-2, 3 \leq k \leq m_2 \\ 3n + \frac{n-1}{2}[(m_2-1) + (k-2)] + (\frac{i}{2}), & n \text{ is odd}, i \text{ is even}, k \text{ is even}, \\ & 1 \leq i \leq n-1, 2 \leq k \leq m_2 \\ 3n + \frac{n-1}{2}[(m_2-1) + (k-1)] - (\frac{i-2}{2}), & n \text{ is odd}, i \text{ is even}, k \text{ is odd}, \\ & 1 \leq i \leq n-1, 3 \leq k \leq m_2 \\ 3n + 2(m_2-1)\frac{n-1}{2} + (k-1), & n \text{ is odd}, i = n, 2 \leq k \leq m_2 \end{cases}$$

$$f_3(v_iu_i^k)=\begin{cases}3n+\frac{n}{2}(k-2)+(\frac{i+1}{2}), & n \text{ is even, } i \text{ is odd, } k \text{ is even,}\\ & 1\le i\le n-1, 2\le k\le m_2\\ 3n+\frac{n}{2}(k-1)-(\frac{i-1}{2}), & n \text{ is even, } i \text{ is odd, } k \text{ is odd,}\\ & 1\le i\le n-1, 3\le k\le m_2\\ 3n+\frac{n}{2}[(m_2-1)+(k-2)]+(\frac{i}{2}), & n \text{ is even, } i \text{ is even, } k \text{ is even,}\\ & 1\le i\le n, 2\le k\le m_2\\ 3n+\frac{n}{2}[(m_2-1)+(k-1)]-(\frac{i-2}{2}), & n \text{ is even, } i \text{ is even, } k \text{ is odd,}\\ & 1\le i\le n, 3\le k\le m_2.\end{cases}$$

Case 2: $m_2 \geq 4$ is even

$f_3(cv_i) = n - i + 1, 1 \leq i \leq n$

$f_3(v_iv_{i+1}) = n + i + 1, 1 \leq i \leq n - 1$

$f_3(v_nv_1) = n + 1.$

$$f_3(v_iu_i^k)=\begin{cases}\frac{7n-i-2}{2}, & k=1, n \text{ is odd}, i \text{ is odd}, 1\le i\le n-2\\ \frac{5n-i+1}{2}, & k=1, n \text{ is odd}, i \text{ is even}, 2\le i\le n-1\\ 4n-1, & k=1, n \text{ is odd}, i=n,\\ \frac{8n-i-3}{2}, & k=2, n \text{ is odd}, i \text{ is odd}, 1\le i\le n-2\\ \frac{6n-i}{2}, & k=2, n \text{ is odd}, i \text{ is even}, 2\le i\le n-1\\ 4n, & k=2, n \text{ is odd}, i=n.\end{cases}$$

$$f_3(v_iu_i^k)=\begin{cases}4n+\frac{n-1}{2}[(m_2-2)+(k-2)]-\left(\frac{i-2}{2}\right), & n \text{ is odd, } i \text{ is even, } k \text{ is even,}\\ & 2\le i\le n-1, 4\le k\le m_2\\ 4n+\frac{n-1}{2}[(m_2-2)+(k-3)]+\frac{i}{2}, & n \text{ is odd, } i \text{ is even, } k \text{ is odd,}\\ & 2\le i\le n-1, 3\le k\le m_2\\ 4n+\left(\frac{n-1}{2}\right)(k-2)-(\frac{i-1}{2}), & n \text{ is odd, } i \text{ is odd, } k \text{ is even,}\\ & 1\le i\le n-2, 4\le k\le m_2\\ 4n+\left(\frac{n-1}{2}\right)(k-3)+(\frac{i+1}{2}), & n \text{ is odd, } i \text{ is odd, } k \text{ is odd,}\\ & 1\le i\le n-2, 3\le k\le m_2\\ 4n+(n-1)(m_2-2)+k-2, & n \text{ is odd}, i=n, 3\le k\le m_2.\end{cases}$$

$$f_3(v_iu_i^k)=\begin{cases}\frac{7n-i+1}{2}, & k=1, n \text{ is even, } i \text{ is odd}, 1\le i\le n-1\\ \frac{5n-i}{2}, & k=1, n \text{ is even, } i \text{ is even}, 2\le i\le n-2\\ \frac{5n}{2}, & k=1, n \text{ is even}, i=n\\ \frac{8n-i+1}{2}, & k=2, n \text{ is even, } i \text{ is odd}, 1\le i\le n-1\\ \frac{6n-i}{2}, & k=2, n \text{ is even, } i \text{ is even}, 2\le i\le n-2\\ \frac{6n}{2}, & k=2, n \text{ is even}, i=n.\end{cases}$$

$$f_3(v_iu_i^k) = \begin{cases} 4n + \frac{n}{2}[(m_2-2)+(k-2)] - \left(\frac{i-2}{2}\right), & n \text{ is even, } i \text{ is even, } k \text{ is even,} \\ & 2 \le i \le n, 4 \le k \le m_2 \\ 4n + \frac{n}{2}[(m_2-2)+(k-3)] + \frac{i}{2}, & n \text{ is even, } i \text{ is even, } k \text{ is odd,} \\ & 2 \le i \le n, 3 \le k \le m_2 \\ 4n + \frac{n}{2}(k-2) - \left(\frac{i-1}{2}\right), & n \text{ is even, } i \text{ is odd, } k \text{ is even,} \\ & 1 \le i \le n-1, 4 \le k \le m_2 \\ 4n + \frac{n}{2}(k-3) + \left(\frac{i+1}{2}\right), & n \text{ is even, } i \text{ is odd, } k \text{ is odd,} \\ & 1 \le i \le n-1, 3 \le k \le m_2. \end{cases}$$

Then the weight of the vertices are,

$w_3(c) = \frac{n(n+1)}{2}$

$$w_3(u_i^k) = \begin{cases} f_3(v_iu_i^k), & n \text{ is even}, 1 \le i \le n, 1 \le k \le m_2, \\ f_3(v_iu_i^k), & n \text{ is odd}, 1 \le i \le n-1, 1 \le k \le m_2, \\ f_3(v_iu_i^k), & n \text{ is odd}, i = n, 1 \le k \le m_2 - 1 \\ f_3(v_iu_i^k) = w_3(c), & n \text{ is odd}, i = n, k = m_2. \end{cases}$$

If $m_2 \ge 3$ is odd, then,

$$w_3(v_i) = \begin{cases} A + \frac{11n+5}{2}, & n \text{ is odd, } i \text{ is odd}, 1 \le i \le n-2 \\ B + \frac{13n-1}{2}, & n \text{ is odd, } i \text{ is even}, 2 \le i \le n-1 \\ C + 7n + 1, & n \text{ is odd}, i = n \\ D + \frac{11n+4}{2}, & n \text{ is even, } i \text{ is odd}, 1 \le i \le n-1 \\ E + \frac{13n+4}{2}, & n \text{ is even, } i \text{ is even}, 2 \le i \le n. \end{cases}$$

Where

$$\begin{aligned} A &= 3n(m_2-1) + \tfrac{1}{4}m_2(m_2-1)(n-1), \\ B &= 3n(m_2-1) + \tfrac{(m_2-1)}{4}[(3m_2-3)(n-1)+2] \\ C &= (m_2-1)\,[3n + (m_2-1)(n-1)] + \tfrac{1}{2}m_2(m_2-1) \\ D &= 3n(m_2-1) + \tfrac{1}{4}(m_2-1)(nm_2-n+2) \\ E &= 3n(m_2-1) + \tfrac{(m_2-1)}{4}[n(3m_2-3)+2]. \end{aligned}$$

If $m_2 \ge 4$ is even, then

$$w_3(v_i) = \begin{cases} P + \frac{21n-1}{2}, & n \text{ is odd, } i \text{ is odd}, 1 \le i \le n-2 \\ Q + \frac{17n+5}{2}, & n \text{ is odd, } i \text{ is even}, 2 \le i \le n-1 \\ R + 11n + 1, & n \text{ is odd}, i = n \\ S + \frac{21n+6}{2}, & n \text{ is even, } i \text{ is odd}, 1 \le i \le n-1 \\ T + \frac{17n+4}{2}, & n \text{ is even, } i \text{ is even}, 2 \le i \le n \end{cases}$$

Where

$$\begin{aligned} P &= 4n(m_2-2) + \tfrac{(m_2-2)}{4}[(n-1)m_2 - 2n + 4] \\ Q &= 4n(m_2-2) + \tfrac{(m_2-2)}{4}[(n-1)(3m_2-6)+2] \\ R &= (m_2-2)\,[4n + (m_2-2)(n-1)] + \tfrac{1}{2}(m_2-2)(m_2-1) \\ S &= 4n(m_2-2) + \tfrac{(m_2-2)}{4}[n(m_2-1) - n + 2] \\ T &= 4n(m_2-2) + \tfrac{(m_2-2)}{4}[n(3m_2-6)+2]. \end{aligned}$$

Clearly, any two adjacent vertices v_i and v_j receive different colors and hence $\chi_{la}(\widehat{W_n^{m_2}}) \leq m_2 n + 3$. Thus $\chi_{la}(\widehat{W_n^{m_2}}) = m_2 n + 3$. $\square$

4. Conclusion

We proved the local chromatic number for helm graph and wheel-related graphs W_n^m with two different m values. A natural open question is to extend this technique to more general graphs having exactly only one maximum degree vertex, other than the wheel graph.

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