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On generalized ∗−reverse derivable maps

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Abstract:

Let R be a ring with involution containing a nontrivial symmetric idempotent element e and $\delta: R \to R$ be a generalized ∗−reverse derivable map. In this paper, our aim is to show that under some suitable restrictions imposed on R every generalized ∗−reverse derivable map of R is additive.

Keywords: Additivity; Generalized reverse derivable maps; Involution; Peirce decomposition.

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1. Introduction

Let R be a ring, by a *derivation* of R, we mean an additive map $\delta : R \to R$ such that $\delta(ab) = \delta(a)b + a\delta(b)$ for all $a, b \in R$. A derivation which is not necessarily additive is said to be a *multiplicative derivation* or *derivable map* of R. In addition, δ is called $n-$*multiplicative derivation* of R if $\delta(a_1a_2\cdots a_n) = \sum_{i=1}^{n} a_1a_2\cdots\delta(a_i)\cdots a_n$ for all $a_1, a_2, \cdots, a_n \in R$. A mapping $\psi : R \to R$ is said to be a *left (resp. right) centralizer* if $\psi(ab) = \psi(a)b$ (resp. $\psi(ab) = a\psi(b)$) for all $a, b \in R$. Moreover, if ψ is left and right centralizer, then it is called *centralizer* of R. A mapping $F : R \to R$ (not necessarily additive) such that $F(ab) = F(a)b + a\delta(b)$ for all $a, b \in R$ is said to be a *multiplicative generalized derivation* associated with derivation δ of R. Note that, every multiplicative left centralizer is a multiplicative generalized derivation. By *involution*, we mean a mapping $* : R \to R$ such that $(x+y)^* = x^* + y^*$, $(x^*)^* = x$ and $(xy)^* = y^*x^*$ for all $x, y \in R$. An element $s \in R$ satisfying $s^* = s$ is called a symmetric element of R. In [12], Herstein introduced a mapping "$\dagger$" satisfying $(a+b)^\dagger = a^\dagger + b^\dagger$ and $(ab)^\dagger = b^\dagger a + ba^\dagger$ called a *reverse derivation*, which is certainly not a derivation (see [16]). Moreover, a mapping $\delta : R \to R$ satisfying $\delta(ab) = \delta(b)a + b\delta(a)$ for all $a, b \in R$ is called a *multiplicative reverse derivation* or *reverse derivable map* of R. Let $*$ be an involution on R, an additive mapping $\delta : R \to R$ is called the $*-$*reverse derivation* if $\delta(ab) = \delta(a)b^* + a^*\delta(b)$ for all $a, b \in R$. If δ is not necessarily additive then it is called $*-$*reverse derivable map* of R. An additive mapping $F : R \to R$ is called *generalized* $*-$*reverse derivation* if there exists a $*-$reverse derivation d of R such that

$$F(xy) = F(y)x^* + y^*d(x) \quad \text{for all} \quad x, y \in R.$$

Moreover, if F is not necessarily additive, then it is called *generalized* $*-$*reverse derivable map.*

Let e be an idempotent element of R such that $e \neq 0, 1$. Then R can be decomposed as follows:

$$R = eRe \bigoplus eR(1-e) \bigoplus (1-e)Re \bigoplus (1-e)R(1-e)$$

This decomposition of R is called *two-sided Peirce decomposition relative to* e ([13], see pg. 48). It is easy to see that the components of this decomposition are the subrings of R and for our convenience, we denote $R_{11} = eRe, R_{12} = eR(1-e), R_{21} = (1-e)Re$ and $R_{22} = (1-e)R(1-e)$. For any $r \in R$, we denote the elements of R_{ij} by r_{ij} for all $i, j \in \{1, 2\}$.

The problem of when a multiplicative mapping must be additive has been studied by several authors. In this direction, Martindale [15] gave a remarkable result. He discovered a set of conditions on R such that every multiplicative isomorphism and anti-automorphism on R is additive. In 1991, inspired by Martindale's work Daif [1] extended these results to multiplicative derivations. He imposed same restrictions on R and obtained the additivity of multiplicative derivations. In a very nice paper [3], Eremita and Ilisevic discussed the additivity of multiplicative left centralizers that are defined from R into a bimodule M over R and gave a number of applications of the main result, that is stated as follows:

Let R be a ring and M be a bimodule over R. Further, let $e_1 \in R$ be a nontrivial idempotent (and $1 - e_1 = e_2$) such that for any $m \in M = \{m \in M : mZ(R) = (0)\}$, where $Z(R)$ denotes the center of R,

(i) $e_1 m e_1 R e_2 = (0)$ implies $e_1 m e_1 = 0$,

(ii) $e_1 m e_2 R e_1 = (0)$ implies $e_1 m e_2 = 0$,

(iii) $e_1 m e_2 R e_2 = (0)$ implies $e_1 m e_2 = 0$,

(iv) $e_2 m e_1 R e_2 = (0)$ implies $e_2 m e_1 = 0$,

(v) $e_2 m e_2 R e_1 = (0)$ implies $e_2 m e_2 = 0$,

(vi) $e_2 m e_2 R e_2 = (0)$ implies $e_2 m e_2 = 0$.

Then every left centralizer $\phi : R \to M$ is additive. An year later, Daif and Tammam-El-Sayiad [2] investigated the additivity of multiplicative generalized derivations. In 2009, Wang [17] extended the result of Daif and obtained the additivity of $n-$multiplicative derivation of R. In a recent paper, Jing and Lu [14] examined the additivity of multiplicative Jordan and multiplicative Jordan triple derivations. This sort of problems and their solutions are not limited only to the class associative rings. For the case of additivity of maps defined on non-associative rings and having a nontrivial idempotent, some results have already been proved. In alternative rings we can mention the works in [4], [5], [6], [7], [8], [9], [10], [11]. In light of all the cited papers, the natural question could be whether the results obtained for multiplicative derivations can also be discussed for multiplicative reverse derivations. In this paper, we consider this problem and answer it with the same set of assumptions taken by Daif and Tammam-El-Sayiad [2].

2. Main Results

In view of the paper [2], we consider now the multiplicative generalized $*-$reverse derivation and discuss their additivity.

The main result of this paper reads as follows:

Theorem 2.1. *Let R be a ring containing a nontrivial symmetric idempotent element e, which satisifies the following conditions:*

(D1) $xRe = 0$ *implies* $x = 0$ *(and hence* $xR = 0$ *implies* $x = 0$*).*

(D2) $exeR(1-e) = 0$ *implies* $exe = 0$.

(D3) $(1-e)xeR(1-e) = 0$ *implies* $(1-e)xe = 0$.

Then every generalized $*-$reverse derivable map $F : R \to R$ is additive.

The next example is of a generalized $*-$reverse derivation.

Example 2.1. *Let $a, b \in R$ be any fixed elements and $G : R \to R$ be a mapping such that $x \mapsto ax^* + x^*b$. Then we note that $G(xy){=}G(y)x^* + y^*g(x)$ for all $x, y \in R$, where g is the associated reverse derivation defined by $g(x) = [x^*, b]$ for all $x \in R$.*

Fact 2.1. *Clearly, $F(0) = 0$. Let $d(e) = a_{11} + a_{12} + a_{21} + a_{22}$. Then by using the expression $d(e) = d(e^2) = d(e)e + ed(e)$, it is easy to see that $d(e) = a_{12} + a_{21}$. We now consider $F(e) = F(e^2) = F(e)e + ed(e)$, where e is the nontrivial symmetric idempotent element of R. Also $F(e) = b_{11} + b_{12} + b_{21} + b_{22}$. The both expressions of $F(e)$ imply $b_{11} + b_{12} + b_{21} + b_{22} = b_{11} + b_{21} + a_{12}$. That yields $b_{12} + b_{22} = a_{12}$ and hence $b_{22} = 0$ and $b_{12} = a_{12}$. In this view, we have $F(e) = b_{11} + b_{21} + a_{12}$.*

Fact 2.2. *If Δ is a $*-$reverse derivable map, then $\Delta(R_{ij}) \subset R_{ji}$, where $i, j \in \{1, 2\}$.*

Let $G : R \to R$ and $\wp : R \to R$ be the mappings such that $G(x) = (b_{11} + b_{21})x^* + x^*(a_{12} - a_{21})$ and $\wp(x) = [x^*, a_{12} - a_{21}]$. Then it is easy to observe that G is a generalized $*-$reverse derivation with associated reverse derivation $\wp$. Further, we note that $G(e) = b_{11} + b_{21} + a_{12}$ and $\wp(e) = a_{12} + a_{21}$. Now, we set $\Lambda = F - G$ and $\Delta = d - \wp$. Then it is difficult to check that Λ is a generalized $*-$reverse derivable map associated with $*-$reverse derivation Δ and Λ is additive if and only if F is so.

We shall use the following proposition very frequently in the sequel.

Proposition 2.1. *Let $s \in R$ ($s_{ij} \in R_{ij}$, where $i, j \in \{1, 2\}$). Then $s_{ij}^* = r_{ji}$, where $r = s^* \in R$. Moreover, $s_{ij} = r_{ji}^*$.*

Proof. Let $s \in R$ be any element. Then for $es(1-e) = s_{12} \in R_{12}$, we have $(es(1-e))^* = (1-e)^* s^* e^* = (1-e)s^* e$. It gives that $s_{12}^* = r_{21}$, where $r = s^*$. Similarly, one can easily observe that $s_{21}^* = r_{21},\ s_{11}^* = r_{11}$ and $s_{22}^* = r_{22}$. Moreover, since "$*$" is bijective, for each $s_{ij} \in R$ there exists unique $r \in R$ such that $r_{ji}^* = s_{ij}$. $\square$

Lemma 2.1. $\Lambda(e) = 0$.

Proof. In view of our settings, it is trivial. $\square$

Lemma 2.2. *(i).* $\Lambda(R_{i1}) \subset R_{1i}$ *for all* $i \in \{1,2\}$.

(ii). Λ *is additive on* R_{i1} *for all* $i \in \{1,2\}$.

(iii). $\Lambda(x_{11} + x_{21}) = \Lambda(x_{11}) + \Lambda(x_{21})$ *for all* $x_{11} \in R_{11}$ *and* $x_{21} \in R_{21}$.

(iv). $\Lambda(R_{11} + R_{21}) \subset R_{11} + R_{12}$.

(v). $\Lambda(R_{11} + R_{12}) \subset R_{11} + R_{21}$.

(vi). $\Lambda(R_{12}) \subset R_{11} + R_{21}$.

(vii). $\Lambda(R_{22}) \subset R_{12} + R_{22}$.

Proof. Let $x_{i1} \in R_{i1}$ be any element, where $i \in \{1,2\}$. Then, we have

$$\begin{aligned} \Lambda(x_{i1}) &= \Lambda(x_{i1}e) \\ &= \Lambda(e)x_{i1}^* + e^*\Delta(x_{i1}) \\ &= \Delta(x_{i1}). \end{aligned}$$

$$\Lambda(x_{i1}) = \Delta(x_{i1}) \ \text{ for all } \ i \in \{1,2\}. \tag{2.1}$$

Since $\Delta(R_{i1}) \subset R_{1i}$, we get $\Lambda(R_{i1}) \subset R_{1i}$ for all $i \in \{1,2\}$, which proves our claim (i). Moreover, claim (ii) easily follows from equation 2.1.

A similar argument implies that if $x_{11} \in R_{11}$ and $x_{21} \in R_{21}$, then

$$\begin{aligned} \Lambda(x_{11} + x_{21}) &= \Lambda((x_{11} + x_{21})e) \\ &= e^*\Delta(x_{11} + x_{21}) \\ &= \Delta(x_{11}) + \Delta(x_{21}) \\ &= \Lambda(x_{11}) + \Lambda(x_{21}) \ (\textit{using } (2.1)) \end{aligned}$$

and hence $\Lambda(R_{11}+R_{21}) \subset R_{11}+R_{12}$. It proves our claim (iii) and (iv). For any $x_{11} \in R_{11}$ and $x_{12} \in R_{12}$, we have $\Lambda(x_{11}+x_{12}) = r_{11}+r_{12}+r_{21}+r_{22}$. And $\Lambda(x_{11}+x_{12}) = \Lambda(e(x_{11}+x_{12})) = \Lambda(x_{11}+x_{12})e^* = r_{11}+r_{21}$. It proves claim (v). Now for any $x_{12} \in R_{12}$, we have $\Lambda(x_{12}) = k_{11}+k_{12}+k_{21}+k_{22}$. But $\Lambda(x_{12}) = \Lambda(ex_{12}) = \Lambda(x_{12})e^* + x_{12}^*\Delta(e) = \Lambda(x_{12})e = k_{11}+k_{21}$. It implies that $\Lambda(x_{12}) \in R_{11}+R_{21}$ for all $x_{12} \in R_{12}$. It proves our claim (vi). Finally, let $x_{22} \in R_{22}$. Then $\Lambda(x_{22}) = \ell_{11}+\ell_{12}+\ell_{21}+\ell_{22}$. Clearly, $0 = \Lambda(ex_{22}) = \Lambda(x_{22})e = \ell_{11}+\ell_{21}$. It implies that $\ell_{11} = 0 = \ell_{21}$, hence $\Lambda(x_{22}) = \ell_{12}+\ell_{22}$. That is $\Lambda(x_{22}) \in R_{12}+R_{22}$ for all $x_{22} \in R_{22}$, it proves claim (vii). □

The following lemmas have the same hypotheses of Theorem 2.1 and we need these lemmas for the proof of this theorem.

Lemma 2.3. $\Lambda(x_{12}+x_{21}) = \Lambda(x_{12})+\Lambda(x_{21})$ *for all* $x_{12} \in R_{12}$ *and* $x_{21} \in R_{21}$.

Proof. For any $x_{12} \in R_{12}, x_{21} \in R_{21}$ and $u_{1i} \in R_{1i}$, where $i \in \{1,2\}$.

$$\begin{aligned}
(\Lambda(x_{12})+\Lambda(x_{21}))u_{1i} &= \Lambda(x_{12})u_{1i} \\
&= \Lambda(x_{12})v_{i1}^* \\
&= \Lambda(v_{i1}x_{12}) - x_{12}^*\Delta(v_{i1}) \\
&= \Lambda(v_{i1}(x_{12}+x_{21})) - x_{12}^*\Delta(v_{i1}) \\
&= \Lambda(x_{12}+x_{21})v_{i1}^* + (x_{12}+x_{21})^*\Delta(v_{i1}) - x_{12}^*\Delta(v_{i1}) \\
&= \Lambda(x_{12}+x_{21})u_{1i}.
\end{aligned}$$

It implies that $(\Lambda(x_{12}+x_{21}) - \Lambda(x_{12})+\Lambda(x_{21}))u_{1i} = 0$. That is

$$(\Lambda(x_{12}+x_{21}) - \Lambda(x_{12})+\Lambda(x_{21}))R_{1i} = (0) \quad \text{for all} \quad i \in \{1,2\}. \tag{2.2}$$

In a similar way, we obtain

$$(\Lambda(x_{12}+x_{21}) - \Lambda(x_{12})+\Lambda(x_{21}))R_{2i} = (0) \quad \text{for all} \quad i \in \{1,2\}. \tag{2.3}$$

Combining (2.2) and (2.3), we obtain

$$(\Lambda(x_{12}+x_{21}) - \Lambda(x_{12})+\Lambda(x_{21}))R = (0)$$

By hypothesis (D1), we have $\Lambda(x_{12}+x_{21}) = \Lambda(x_{12})+\Lambda(x_{21})$. □

Lemma 2.4. $\Lambda(x_{11}+x_{12}) = \Lambda(x_{11})+\Lambda(x_{12})$ *for all* $x_{11} \in R_{11}$ *and* $x_{12} \in R_{12}$.

Proof. Let $x_{11} \in R_{11}, x_{12} \in R_{12}, z_{12} \in R_{12}$ and $u_{1i} \in R_{1i}$, where $i \in \{1,2\}$. Then we have

$$(\Lambda(x_{11}+x_{12}) - \Lambda(x_{11}) - \Lambda(x_{12}))z_{12}u_{1i} = 0.$$

It implies that

(2.4) $(\Lambda(x_{11}+x_{12}) - \Lambda(x_{11}) - \Lambda(x_{12}))z_{12}R_{1i} = 0$ for all $i \in \{1,2\}$.

For any $u_{2i} \in R_{2i}$ for all $i \in \{1,2\}$, we find

$$\begin{aligned}
\Lambda(x_{11}+x_{12})z_{12}u_{2i} &= \Lambda(x_{11}+x_{12})w_{21}^*v_{i2}^* \\
&= \Lambda(x_{11}+x_{12})(v_{i2}w_{21})^* \\
&= \Lambda((v_{i2}w_{21})(x_{11}+x_{12})) - (x_{11}+x_{12})^*\Delta(v_{i2}w_{21}) \\
&= \Lambda((v_{i2}w_{21}+v_{i2})(w_{21}x_{11}+x_{12})) - (x_{11}+x_{12})^*\Delta(v_{i2}w_{21}) \\
&= \Lambda(w_{21}x_{11}+x_{12})(v_{i2}w_{21}+v_{i2})^* + (w_{21}x_{11}+x_{12})^*\Delta(v_{i2}w_{21} \\
&\quad +v_{i2}) - (x_{11}+x_{12})^*\Delta(v_{i2}w_{21}) \\
&= \Lambda(w_{21}x_{11}+x_{12})(v_{i2}w_{21}+v_{i2})^* + (w_{21}x_{11})^*\Delta(v_{i2}w_{21}) + \\
&\quad x_{12}^*\Delta(v_{i2}w_{21}) + (w_{21}x_{11})^*\Delta(v_{i2}) + x_{12}^*\Delta(v_{i2}) - x_{11}^*\Delta(v_{i2} \\
&\quad w_{21}) - x_{12}^*\Delta(v_{i2}w_{21}) \\
&= \Lambda(w_{21}x_{11}+x_{12})(v_{i2}w_{21}+v_{i2})^* - x_{11}^*\Delta(w_{21})v_{i2}^* \\
&= \Lambda(w_{21}x_{11})(v_{i2}w_{21}+v_{i2})^* + \Lambda(x_{12})(v_{i2}w_{21}+v_{i2})^* \\
&\quad -x_{11}^*\Delta(w_{21})v_{i2}^* \ (\textit{using Lemma } 2.3) \\
&= \Lambda(w_{21}x_{11})v_{i2}^* + \Lambda(x_{12})(v_{i2}w_{21})^* - x_{11}^*\Delta(w_{21})v_{i2}^* \\
&= \Lambda(x_{11})w_{21}^*v_{i2}^* + x_{11}^*\Delta(w_{21})v_{i2}^* + \Lambda(x_{12})w_{21}^*v_{i2}^* - x_{11}^*\Delta(w_{21})v_{i2}^* \\
&= (\Lambda(x_{11}) + \Lambda(x_{12}))z_{12}u_{2i}.
\end{aligned}$$

It implies that

(2.5)$(\Lambda(x_{11}+x_{12}) - \Lambda(x_{11}) - \Lambda(x_{12}))z_{12}R_{2i} = (0)$ for all $i \in \{1,2\}$.

Combining (2.4) and (2.5), we obtain

$$(\Lambda(x_{11}+x_{12}) - \Lambda(x_{11}) - \Lambda(x_{12}))z_{12}R = (0).$$

Applying (D1), we get

$$(\Lambda(x_{11}+x_{12}) - \Lambda(x_{11}) - \Lambda(x_{12}))R_{12} = (0).$$

Applying hypothesis (D2) and (D3), we get $\Lambda(x_{11}+x_{12}) = \Lambda(x_{11})+\Lambda(x_{12})$, as desired. □

Lemma 2.5. $\Lambda(x_{12}+x_{22}) = \Lambda(x_{12}) + \Lambda(x_{22})$ *for all* $x_{12} \in R_{12}, x_{22} \in R_{22}$.

Proof. Let $x_{12} \in R_{12}, x_{22} \in R_{22}$ and $u_{1i} \in R_{1i}$, where $i \in \{1,2\}$. Then we have

$$\begin{aligned}
(\Lambda(x_{12}) + \Lambda(x_{22}))u_{1i} &= \Lambda(x_{12})u_{1i} \\
&= \Lambda(x_{12})v_{i1}^* \\
&= \Lambda(v_{i1}x_{12}) - x_{12}^*\Delta(v_{i1}) \\
&= \Lambda(v_{i1}(x_{12} + x_{22})) - x_{12}^*\Delta(v_{i1}) \\
&= \Lambda(x_{12} + x_{22})v_{i1}^* + (x_{12} + x_{22})^*\Delta(v_{i1}) - x_{12}^*\Delta(v_{i1}) \\
&= \Lambda(x_{12} + x_{22})u_{1i}.
\end{aligned}$$

It implies that

$$(2.6) \quad (\Lambda(x_{12} + x_{22}) - \Lambda(x_{12}) - \Lambda(x_{22}))R_{1i} = (0) \quad \text{for all} \quad i \in \{1,2\}.$$

Analogously, we obtain

$$(2.7) \quad (\Lambda(x_{12} + x_{22}) - \Lambda(x_{12}) - \Lambda(x_{22}))R_{2i} = (0) \quad \text{for all} \quad i \in \{1,2\}.$$

Combining (2.6) and (2.7), we obtain $(\Lambda(x_{12}+x_{22})-\Lambda(x_{12})-\Lambda(x_{22}))R = (0)$. In view of (D1), we get $\Lambda(x_{12} + x_{22}) = \Lambda(x_{12}) + \Lambda(x_{22})$. $\square$

Lemma 2.6. *Λ is additive on R_{12}.*

Proof. Let $x_{12}, y_{12}, z_{12} \in R_{12}$ and $u_{2i} \in R_{2i}$ where $i \in \{1,2\}$. Then we have

$$\begin{aligned}
\Lambda(x_{12} + y_{12})z_{12}u_{2i} &= \Lambda(x_{12} + y_{12})w_{21}^*v_{i2}^* \\
&= \Lambda(x_{12} + y_{12})(v_{i2}w_{21})^* \\
&= \Lambda((v_{i2}w_{21})(x_{12} + y_{12})) - (x_{12} + y_{12})^*\Delta(v_{i2}w_{21}) \\
&= \Lambda((v_{i2}w_{21} + v_{i2})(x_{12} + w_{21}y_{12})) - (x_{12} + y_{12})^*\Delta(v_{i2}w_{21}) \\
&= \Lambda(x_{12} + w_{21}y_{12})(v_{i2}w_{21} + v_{i2})^* + (x_{12} + w_{21}y_{12})^*\Delta(v_{i2}w_{21} + v_{i2}) \\
&\quad -(x_{12} + y_{12})^*\Delta(v_{i2}w_{21}) \\
&= \Lambda(x_{12})(v_{i2}w_{21})^* + \Lambda(w_{21}y_{12})v_{12}^* + (w_{21}y_{12})^*\Delta(v_{i2}) - y_{12}^* \\
&\quad \Delta(v_{i2}w_{21}) \ (\textit{using Lemma } 2.5) \\
&= \Lambda(x_{12})(v_{i2}w_{21})^* + \Lambda(y_{12})w_{21}^*v_{i2}^* \\
&= (\Lambda(x_{12}) + \Lambda(y_{12}))w_{21}^*v_{i2}^* \\
&= (\Lambda(x_{12}) + \Lambda(y_{12}))z_{12}u_{2i}.
\end{aligned}$$

It implies that

$$(2.8)\ (\Lambda(x_{12}+y_{12})-\Lambda(x_{12})-\Lambda(y_{12}))z_{12}R_{2i}=(0) \quad \text{for all} \quad i\in\{1,2\}.$$

And trivially, we have

$$(2.9)\ (\Lambda(x_{12}+y_{12})-\Lambda(x_{12})-\Lambda(y_{12}))z_{12}R_{1i}=(0) \quad \text{for all} \quad i\in\{1,2\}.$$

Combining (2.8) and (2.9), we find

$$(\Lambda(x_{12}+y_{12})-\Lambda(x_{12})-\Lambda(y_{12}))z_{12}R=(0).$$

By (D1), we get

$$(\Lambda(x_{12}+y_{12})-\Lambda(x_{12})-\Lambda(y_{12}))R_{12}=(0).$$

Applying (D3), we get $\Lambda(x_{12}+y_{12})=\Lambda(x_{12})+\Lambda(y_{12})$, as desired. □

Lemma 2.7. *Λ is additive on $eR=R_{11}+R_{12}$.*

Proof. For any $x_{11},y_{11}\in R_{11}$ and $x_{12},y_{12}\in R_{12}$, we have

$$\begin{aligned}
\Lambda((x_{11}+x_{12})+(y_{11}+y_{12})) &= \Lambda((x_{11}+y_{11})+(x_{12}+y_{12}))\\
&= \Lambda(x_{11}+y_{11})+\Lambda(x_{12}+y_{12}) \ (\textit{using Lemma } 2.4)\\
&= \Lambda(x_{11})+\Lambda(y_{11})+\Lambda(x_{12})+\Lambda(y_{12})\\
&\quad (\textit{using Lemma } 2.6 \textit{ Lemma } 2.1)\\
&= \Lambda(x_{11}+x_{12})+\Lambda(y_{11}+y_{12}) \ (\textit{using Lemma } 2.4).
\end{aligned}$$

□

Proof of Theorem 2.1: Let $x,y\in R$ and $t\in Re$ be any elements. Then we see that

$$\begin{aligned}
\Lambda(x+y)t &= \Lambda(x+y)p^*\\
&= \Lambda(px+py)-(x+y)^*\Delta(p)\\
&= \Lambda(px)+\Lambda(py)-(x+y)^*\Delta(p) \ (\textit{using Lemma } 2.7)\\
&= \Lambda(x)p^*+x^*\Delta(p)+\Lambda(y)p^*+y^*\Delta(p)-(x+y)^*\Delta(p)\\
&= (\Lambda(x)+\Lambda(y))p^*\\
&= (\Lambda(x)+\Lambda(y))t.
\end{aligned}$$

It implies that

$$(\Lambda(x+y)-\Lambda(x)-\Lambda(y))Re=(0).$$

In view of (D1), we get $\Lambda(x+y)=\Lambda(x)+\Lambda(y)$ for all $x,y\in R$. It completes the proof.

Example 2.2. *It is well-known that a commutative integral domain does not contain nontrivial idempotent elements. Thus it would be a fact of interest to investigate some particular cases of our main theorem. In this view, we give a counter example showing that the prime rings admitting generalized $*-$reverse derivations are not necessarily commutative. Let* $R=\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : a,b,c,d\in Z\right\}$, *a noncommutative prime ring. Define mappings* $F,d,^*:R\to R$ *such that* $F\begin{pmatrix} a & b \\ c & d \end{pmatrix}=\begin{pmatrix} -d & 4b \\ 0 & -3a \end{pmatrix}$, $d\begin{pmatrix} a & b \\ c & d \end{pmatrix}=\begin{pmatrix} 0 & 3b \\ -3c & 0 \end{pmatrix}$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^*=\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. *One can verify that* * *is an involution and* F *is a generalized* $*-$*reverse derivation with associated* $*-$*reverse derivation* d *of* R.

We conclude with the following consequence of our main result.

Corollary 2.1. *Let R be a prime ring containing a nontrivial symmetric idempotent element e. Then every generalized $*-$reverse derivable map $F:R\to R$ is additive.*

References

[1] M. N. Daif, "When is a multiplicative derivation additive?", *International journal of mathematics and mathematical sciences*, vol. 14, no. 3, pp. 615-618, 1991, doi: 10.1155/S0161171291000844

[2] M. N. Daif and M. S. Tammam-El-Sayiad, "Multiplicative generalized derivations which are additive", *East-West journal mathematics*, vol. 9, no. 1, pp. 31-37, 1997.

[3] D. Eremita and D. Ilisevic, "On additivity of centralisers", *Bulletin of the Australian Mathematical Society*, vol. 74, no. 2, pp. 177-184, 2006, doi: 10.1017/S0004972700035620

[4] B. L. M. Ferreira, "Jordan elementary maps on alternative division rings", *Serdica mathematical journal*, vol. 43, no. 2, pp. 161-168, 2017.

[5] R. N. Ferreira and B. L. M. Ferreira, "Jordan derivation on alternative rings", *International journal of mathematics, game theory, and algebra*, vol. 25, no. 4, pp. 435-444, 2017.

[6] R. N. Ferreira and B. L. M. Ferreira, "Jordan triple derivation on alternative rings", *Proyecciones (Antofagasta)*, vol. 37, no. 1, pp. 169-178, 2018, doi: 10.4067/S0716-09172018000100171

[7] B. L. M. Ferreira, J. C. M. Ferreira, and H. Guzzo Jr., "Jordan maps on alternative algebras", *JP Journal of algebra, number theory and applications*, vol. 31, no. 2, pp. 129-142, 2013. [On line]. Available: https://bit.ly/33TQTbs

[8] B. L. M. Ferreira, J. C. M. Ferreira, and H. Guzzo Jr., "Jordan triple elementary maps on alternative rings", *Extracta mathematicae*, vol. 29, no. 1-2, pp. 1-18, 2014. [On line]. Available: https://bit.ly/2RYt2oo

[9] B. L. M. Ferreira, J. C. M. Ferreira, and H. Guzzo Jr., "Jordan triple maps of alternative algebras", *JP Journal of algebra, number theory and applications*, vol. 33, pp. 25-33, 2014. [On line]. Available: https://bit.ly/3eUZhhj

[10] J. M. Ferreira, and B. L. M. Ferreira, "Additivity of n−multiplicative maps on alternative rings", *Communication algebra*, vol. 44, no. 4, pp. 1557-1568, 2016, doi: 10.1080/00927872.2015.1027364

[11] B. L. M. Ferreira and R. Nascimento, "Derivable maps on alternative rings", *Revista ciências exatas e naturais*, vol. 16, no. 1, pp. 9-15, 2014, doi: 10.5935/RECEN.2014.01.01

[12] I. N. Herstein, "Jordan derivations of prime rings", *Proceeding American Mathematical Society*, vol. 8, no. 6, pp. 1104-1110, 1957, doi: 10.1090/S0002-9939-1957-0095864-2

[13] N. Jacobson, *Structure of rings*, 2nd ed. Providence, RI: America Mathematical Society, 1964.

[14] W. Jing and F. Lu, "Additivity of Jordan (triple) derivations on rings", *Communication algebra*, vol. 40, no. 8, pp. 2700-2719, 2012, doi: 10.1080/00927872.2011.584927

[15] W. S. Martindale, "When are multiplicative mappings additive?", *Proceeding American Mathematical Society*, vol. 21, no. 3, pp. 695-698, 1969, doi: 10.1090/S0002-9939-1969-0240129-7

[16] G. S. Sandhu and D. Kumar, "Annihilator conditions of multiplicative reverse derivations in prime rings", *International electronic. journal of algebra*, vol. 25, pp. 87-103, 2019, doi: 10.24330/ieja.504124

[17] Y. Wang, "The additivity of multiplicative maps on rings", *Communications in algebra*, vol. 37, no. 6, pp. 2351-2356, 2009, doi: 10.1080/00927870802623369