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P-adic discrete semigroup of contractions

Abdelkhalek El amrani
Sidi Mohamed Ben Abdellah University, Morocco
Jawad Ettayb
Sidi Mohamed Ben Abdellah University, Morocco
and
Aziz Blali
Sidi Mohamed Ben Abdellah University, Morocco
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Abstract

Let $A \in B(X)$ be a spectral operator on a non-archimedean Banach space over $\mathbf{C}_p$. In this paper, we give a necessary and sufficient condition on the resolvent of A so that the discrete semigroup consisting of powers of A is contractions.

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1. Introduction and Preliminaries

In the archimedean operators theory, necessary and sufficient conditions on the resolvent of a densely defined closed linear operator for it to be the infinitesimal generator of a strongly continuous semigroup $(T(s))_{s\in\mathbf{R}^+}$ such that there is $M \geq 1$, $\|T(s)\| \leq M$. In particular, we have the following theorem and its corollary.

Theorem 1.1. *[7] A necessary and sufficient condition that a closed linear operator A with dense domain be the infinitesimal generator of a strongly continuous semigroup $(T(s))_{s\in\Omega_r}$ such that for all $s \in \mathbf{R}^+$, $\|T(s)\| \leq M$ is that*

$$\|R_\lambda(A)^n\| \leq \frac{M}{\lambda^n} \tag{1.1}$$

for $\lambda > 0$ and $n \in \mathbf{N}$, where $R_\lambda(A) = (\lambda I - A)^{-1}$.

Corollary 1.2. *[7] A necessary and sufficient condition that a closed linear operator A with dense domain be the infinitesimal generator of a strongly continuous semigroup $(T(s))_{s\in\Omega_r}$ such that for all $s \in \mathbf{R}^+$, $\|T(s)\| \leq 1$ is that*

$$\|R_\lambda(A)\| \leq \frac{M}{\lambda} \tag{1.2}$$

for $\lambda > 0$.

Throughout this paper, X is a non-archimedean (n.a) Banach space over a (n.a) non trivially complete valued field $\mathbf{K}$ which is also algebraically closed with valuation $|\cdot|$, $B(X)$ denote the set of all bounded linear operators on X into X, $\mathbf{Q}_p$ is the field of p-adic numbers ($p \geq 2$ being a prime) equipped with p-adic valuation $|.|_p$, $\mathbf{Z}_p$ denotes the ring of $p-$adic integers of $\mathbf{Z}_p$ is the unit ball of $\mathbf{Q}_p$. For more details and related issues, we refer to [4] and [6]. We denote the completion of algebraic closure of $\mathbf{Q}_p$ under the p-adic abolute value $|\cdot|_p$ by $\mathbf{C}_p$ (see [4]). Let $r > 0$, Ω_r be the clopen ball of $\mathbf{K}$ centred at 0 with radius $r > 0$, that is $\Omega_r = \{t \in \mathbf{K} : |t| < r\}$. Recall that a free non-archimedean Banach space X is a non-archimedean Banach space for which there exists a family $(e_i)_{i\in I}$ in $X\backslash\{0\}$ such that any element $x \in X$ can be written in the form of a convergent sum $x = \sum_{i\in I} x_i e_i$, $x_i \in \mathbf{K}$, i.e., $\lim_{i\in I} x_i e_i = 0$ (the limit is with respect to the Fréchet filter on I) and $\|x\| = \sup_{i\in I} |x_i|\|e_i\|$. Let X be a free non-archimedean Banach space, recall that every bounded linear operator A on X can be written in a unique fashion as a pointwise convergent series, that is, there exists an infinite matrix $(a_{i,j})_{(i,j)\in\mathbf{N}\times\mathbf{N}}$ with coefficients in $\mathbf{K}$ such that

$$A = \sum_{i,j\in\mathbf{N}} a_{i,j} e_j^{'} \otimes e_i, \ and\ \forall j \in \mathbf{N},\ \lim_{i\to\infty} |a_{i,j}|\|e_i\| = 0,$$

where $(\forall i \geq 1)\ e_i^{'}(u) = u_i\ \left(e_i^{'}\ is\ the\ linear\ form\ associated\ with\ e_i\right).$

Moreover, for each $j \in \mathbf{N}$, $\mathrm{A}e_j = \sum_{i\in\mathbf{N}} a_{ij}e_i$ and its norm is defined by

$$\| A \| = \sup_{i,j} \frac{|a_{ij}|\|e_i\|}{\|e_j\|}.$$

For more details see [1], Proposition 3.7.

Definition 1.3. *[1] Let $\omega = (\omega_i)_i$ be a sequence of non-zero elements of $\mathbf{K}$. We define $\mathbf{E}_\omega$ by*
$\mathbf{E}_\omega = \{x = (x_i)_i : \forall i \in \mathbf{N},\ x_i \in \mathbf{K},\ and\ \lim_{i\to\infty} |\omega_i|^{\frac{1}{2}}|x_i| = 0\}$,
it is equipped with the norm
$\forall x \in \mathbf{E}_\omega : x = (x_i)_i,\ \|x\| = \sup_{i\in\mathbf{N}}(|\omega_i|^{\frac{1}{2}}|x_i|)$.

Remark 1.4. *(1) [1], Exemple 2.21. The space $\left(\mathbf{E}_\omega,\ \|\cdot\|\right)$ is a non archimedean Banach space.*

(2) If

$$\begin{array}{rcl}\langle\cdot,\cdot\rangle : \mathbf{E}_\omega \times \mathbf{E}_\omega & \longrightarrow & \mathbf{K} \\ (x,y) & \mapsto & \displaystyle\sum_{i=0}^{\infty} x_i y_i \omega_i,\end{array}$$

where $x = (x_i)_i$ and $y = (y_i)_i$. Then, the space $\left(\mathbf{E}_\omega, \|\cdot\|, \langle\cdot,\cdot\rangle\right)$ is called a p-adic (or non archimedean) Hilbert space.

(2) The orthogonal basis $\{e_i,\ i \in \mathbf{N}\}$ is called the canonical basis of $\mathbf{E}_\omega$, where for all $i \in \mathbf{N}, \|e_i\| = |\omega_i|^{\frac{1}{2}}$.

Definition 1.5. *[6] Let $A \in B(X)$, set $\nu(A) = \inf_n \|A^n\|^{\frac{1}{n}} = \lim_n \|A^n\|^{\frac{1}{n}}$, A is said to be a spectral operator if $\sup\{|\lambda| : \lambda \in \sigma(A)\} = \nu(A)$. For $A \in B(X)$, set $U_A = \{\lambda \in \mathbf{K} : (I - \lambda A)^{-1}$ exists in $B(X)\}$. $\Big(U_A$ is open and $0 \in U_A\Big)$ and*

$$C_A = \{\alpha \in \mathbf{K} : B(0, |\beta|) \subset U_A \text{ for some } \beta \in \mathbf{K},\ |\beta| > |\alpha|\}.$$

Proposition 1.6. *[6] Let $A \in B(X)$, the following are equivalent.*

(i) A is a spectral operator.

(ii) For all $\lambda \in C_A$, $(I - \lambda A)^{-1} = \sum_{n=0}^{\infty} \lambda^n A^n$.

(iii) For each $\alpha \in C_A^*$, the function $\lambda \mapsto (I - \lambda A)^{-1}$ is analytic on $B(0, |\alpha|)$.

2. Discrete semigroups of bounded linear operators on non-archimedean Banach space

We begin with the following definition.

Definition 2.1. *Let X be a non-archimedean Banach space over $\mathbf{K}$. A family $(T(n))_{n\in\mathbf{N}}$ of bounded linear operators from X into X is said to be a discrete semigroup of bounded linear operators on X if*

(i) $T(0) = I$, where I is the unit operator of X;

(ii) For all $m, n \in \mathbf{N}$, $T(m + n) = T(m)T(n)$.

Remark 2.2. *Let $A \in B(X)$, $T(n) = A^n$ is a discrete semigroup of bounded linear operators on X, its generator A.*

Definition 2.3. *Let X be a non-archimedean Banach space over $\mathbf{K}$. A discrete semigroup $(T(n))_{n\in\mathbf{N}}$ is said to be uniformly bounded if $\sup_{n\in\mathbf{N}} \|T(n)\|$ is finite.*

In contrast with the classical setting, we have the following example.

Example 2.4. *Let* $\mathbf{K} = \mathbf{Q}_p$, *if*

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

then A *generate a discrete semigroup of bounded linear operators* $(T(n))_{n\in\mathbf{N}}$ *given by:*

$$\forall n \in \mathbf{N},\ T(n) = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}.$$

In fact, it is easy to check that:

(i) $T(0) = I$ *where* I *is the unit operator on* $\mathbf{Q}_p^2$.

(ii) For all $m, n \in \mathbf{N},\ T(m+n) = T(m)T(n)$.

(iii) For all $z = (x,\ y) \in \mathbf{Q}_p^2$, $n \in \mathbf{N}$, *we have*

$$\begin{aligned} \|T(n)z\| &= \left\| \begin{pmatrix} x + ny \\ y \end{pmatrix} \right\|, \\ &= \max\{|x+ny|_p, |y|_p\}, \\ &\leq \max\{|x|_p,\ |ny|_p,\ |y|_p\}, \\ &\leq \max\{|x|_p,\ |y|\} \textit{ with } |n|_p \leq 1, \\ &\leq \|z\|. \end{aligned}$$

Then $(T(n))_{n\in\mathbf{N}}$ *is an uniformly bounded discrete semigroup of bounded linear operators on* $\mathbf{Q}_p^2$.

We have the following definition.

Definition 2.5. *Let* X *be a non-archimedean Banach space over* $\mathbf{K}$, *let* $(T(n))_{n\in\mathbf{N}}$ *be a discrete semigroup of bounded linear operators on* X, $(T(n))_{n\in\mathbf{N}}$ *is said to be semigroup of contractions if for all* $n \in \mathbf{N},\ \|T(n)\| \leq 1$.

Example 2.6. *Let* $X = \mathbf{E}_\omega$ *where for all* $i \in \mathbf{N},\ \omega_i = p^i$. *Let* A *be unilateral shift given by*

for all $i \in \mathbf{N},\ Ae_i = e_{i+1}$.

Then, for all $n \in \mathbf{N},\ A^n e_i = e_{n+i}$, *hence, for all* $n \in \mathbf{N}$, $\frac{\|A^n e_i\|}{\|e_i\|} = p^{\frac{-n}{2}} \leq 1$. *Consequently, for all* $n \in \mathbf{N},\ \|A^n\| \leq 1$. *Moreover,* $(A^n)_{n\in\mathbf{N}}$ *is a discrete semigroup of contractions on* $\mathbf{E}_\omega$.

We start with the following statements.

Lemma 2.7. *Let* $(T(n))_{n\in\mathbf{N}}$ *be a discrete semigroup of bounded linear operators on* X *such that* $\sup_{n\in\mathbf{N}}\|T(n)\| \leq M$. *Then there exists an equivalent norm on* X *such that* $(T(n))_{n\in\mathbf{N}}$ *becomes a contraction.*

Proof. Set: $|x|_1 = \sup_{n\in\mathbf{N}}\|T(n)x\|$. We have $\|T(0)\| = 1$ and $(\forall n \in \mathbf{N},)\ \|T(n)\| \leq M$, then $(\forall x \in X)\ \|x\| \leq |x|_1 \leq M\|x\|$ Hence $|\cdot|_1$ is a norm on X which is equivalent to the original norm $\|\cdot\|$ on X Furthermore, for all $x \in X$, all $n \in \mathbf{N}$, $|T(n)x|_1 = \sup_{m\in\mathbf{N}}\|T(n)T(m)x\| = \sup_{m\in\mathbf{N}}\|T(n+m)x\| = \sup_{m\geq n}\|T(m)x\| \leq \sup_{m\in\mathbf{N}}\|T(m)x\| = |x|_1$. □

In the next Proposition, we assume that $\mathbf{Q}_p \subset \mathbf{K}$.

Proposition 2.8. *The set of all discrete semigroup of contractions form a* $\mathbf{Z}_p$*-subspace of* $B(X)$.

Proof. Set C denote the set of all discrete semigroup of contractions on X into X.

(1) $\|I_X\| \leq 1$, Hence $C \neq \emptyset$.

(2) Let $(T(n))_{n\in\mathbf{N}}$ and $(S(n))_{n\in\mathbf{N}}$ in C and $\lambda \in \mathbf{Z}_p$, we have

$$\begin{aligned}\|T(n)+\lambda S(n)\| &\leq \max\Big\{\|T(n)\|;\ |\lambda|\|S(n)|\Big\};\\ &\leq 1.\end{aligned}$$

Hence, for all $n \in \mathbf{N}$ and for all $\lambda \in \mathbf{Z}_p$, $T(n) + \lambda S(n) \in C$.

□

In the rest of this paper, for $A \in B(X)$ be a spectral operator such that $\sup_{n\in\mathbf{N}}\|A^n\|$ is finite, we assume that $U_A = B(0,1)$ where $\mathrm{B}(0,1) = \{\lambda \in \mathbf{K} : |\lambda| < 1\}$, and for all $\lambda \in U_A, R(\lambda, A) = (I - \lambda A)^{-1}$.

Proposition 2.9. *Let* X *be a non-archimedean Banach space over* $\mathbf{K}$, *let* A *be a spectral operator and there is* $M \geq 1$ *such that* $\sup_{n\in\mathbf{N}}\|A^n\| \leq M$. *Then,*

for all $\lambda \in C_A$, $\|R(\lambda,\ A)\| \leq M$.

Proof. By Proposition 1.6, for all $\lambda \in C_A,\ \lim_{n\to\infty} |\lambda|^n \|A^n\| = 0$, hence

$$\begin{aligned} \|R(\lambda,\ T)\| &= \left\| \sum_{n=0}^{\infty} \lambda^n A^n \right\| \\ &\leq M \max_{n\in\mathbf{N}} |\lambda^n| \\ &= M. \end{aligned}$$

□

Proposition 2.10. *Let $A \in B(X)$ be a spectral operator, let $(A^n)_{n\in\mathbf{N}}$ be a discrete semigroup of bounded linear operators on X such that $\sup_{n\in\mathbf{N}} \|A^n\|$ is finite and $U_A = B(0,1)$. Then, for all $\lambda,\ \mu \in C_A$,*

$$\lambda R(\lambda,\ A) - \mu R(\mu,\ A) = (\lambda - \mu) R(\lambda,\ A) R(\mu,\ A).$$

Proof. Let $\lambda,\ \mu \in C_A$, we have

$$\lambda R(\lambda,\ A)(I - \mu A) R(\mu,\ A) - \mu R(\lambda,\ A)(I - \lambda A) R(\mu,\ A) \tag{2.1}$$

$$\begin{aligned} (2.1) &= \lambda R(\lambda,\ A) R(\mu,\ A) - \lambda\mu R(\lambda,\ A) A R(\mu,\ A) - \mu R(\lambda,\ A) R(\mu,\ A) \\ &+ \lambda\mu R(\lambda,\ A) A R(\mu,\ A); \\ &= \lambda R(\lambda,\ A) R(\mu,\ A) - \mu R(\lambda,\ A) R(\mu,\ A); \\ &= (\lambda - \mu) R(\lambda,\ A) R(\mu,\ A). \end{aligned}$$

□

Proposition 2.11. *Let $A \in B(X)$ be a spectral operator such that $U_A = B(0,1)$, let $(A^n)_{n\in\mathbf{N}}$ be a discrete semigroup of contractions on X, then for all $\lambda \in C_A,\ \|R(\lambda, A) - I\| \leq |\lambda|$.*

Proof. Let $A \in B(X)$ be a spectral operator, then for all $\lambda \in C_A,\ R(\lambda, A) = \sum_{n=0}^{\infty} \lambda^n A^n$. Hence, for all $\lambda \in C_A$,

$$\|R(\lambda, A) - I\| = \|\sum_{n=1}^{\infty} \lambda^n A^n\|, \tag{2.2}$$

$$\leq \sup_{n\geq 1} \|\lambda^n A^n\|, \tag{2.3}$$

$$\leq |\lambda|. \tag{2.4}$$

□

Proposition 2.12. *Let $A \in B(X)$ be a spectral operator such that for all $n \in \mathbf{N}$, $\|A^n\| \leq 1$, then for all $n \in \mathbf{N}$, $\alpha \in C_A^*$, $\lambda \in \Omega_{|\alpha|}$,*
$R^{(n)}(\lambda, A) = \frac{n!(R(\lambda,A)-I)^n R(\lambda,A)}{\lambda^n}$.

Proof. Using Proposition 2.10, for all $\lambda, \mu \in \Omega_{|\alpha|}$ whith $\alpha \in C_A^*$,

$$\Big(\lambda I + (\mu - \lambda)I + (\lambda - \mu)R(\lambda, A)\Big)R(\mu, A) = \lambda R(\lambda, A). \tag{2.5}$$

Then

$$\Big(I - \frac{1}{\lambda}(\mu - \lambda)(R(\lambda, A) - I)\Big)R(\mu, A) = R(\lambda, A). \tag{2.6}$$

The quantity in square brackets on the left of this equation is invertible for $|\lambda|^{-1}|\mu - \lambda|\|R(\lambda, A) - I\| < 1$. Thus

$$R(\mu, A) = \sum_{n=0}^{\infty} \frac{(R(\lambda, A) - I)^n R(\lambda, A)}{\lambda^n}(\mu - \lambda)^n. \tag{2.7}$$

But it follows from Proposition 1.6 that $R(\mu, A)$, is analytic on $B(\lambda, |\alpha|)$. From $A \in B(X)$ is spectral operator, then for all $s, t \in \Omega_{|\alpha|}$, $R(\mu, A)$ can be written as follows:
R$(\mu, A) = \sum_{n=0}^{\infty} \frac{R^{(n)}(\lambda,A)}{n!}(\mu - \lambda)^n$.
Hence, for all $n \in \mathbf{N}$, $\lambda \in \Omega_{|\alpha|}$,
R$^{(n)}(\lambda, A) = \frac{n!(R(\lambda,A)-I)^n R(\lambda,A)}{\lambda^n}$. □
We have the following theorem.

Theorem 2.13. *Let X be a non-archimedean Banach space over $\mathbf{C}_p$, and $A \in B(X)$ be a spectral operator, then for all $n \in \mathbf{N}$, $\|A^n\| \leq 1$ if and only if*

$$\|\Big(R(\lambda, A) - I\Big)^n R(\lambda, A)\| \leq |\lambda|_p^n, \tag{2.8}$$

for all $\lambda \in \Omega_{|\alpha|}$ where $\alpha \in C_A^*$ and $R(\lambda, A) = (I - \lambda A)^{-1}$.

Proof. Assume that for all $n \in \mathbf{N}$, $\|A^n\| \leq 1$, let $\alpha \in C_A^*$, by Proposition 1.6, $R(\lambda, A) = (I-\lambda A)^{-1} = \sum_{k=0}^{\infty} \lambda^k A^k$ is analytic on $\Omega_{|\alpha|}$. Using Proposition 2.12, for all $n \in \mathbf{N}$, $\lambda \in \Omega_{|\alpha|}$,

$$R^{(n)}(\lambda, A) = \frac{n!(R(\lambda, A) - I)^n R(\lambda, A)}{\lambda^n}, \tag{2.9}$$

and

$$\mathrm{R}^{(n)}(\lambda, A) = \sum_{k=n}^{\infty} k(k-1)\cdots(k-n+1)\lambda^{k-n}A^k == \sum_{k=n}^{\infty} n!\binom{k}{n}\lambda^{k-n}A^k,$$

then for all $n \in \mathbf{N}$ and $\lambda \in \Omega_{|\alpha|}$,

$$\begin{aligned}
\left\|\frac{R^{(n)}(\lambda, A)}{n!}\right\| &= \|\sum_{k=n}^{\infty}\binom{k}{n}\lambda^{k-n}A^k\|, && (2.10)\\
&\leq \sup_{k\geq n}|\binom{k}{n}|_p|\lambda|_p^{k-n}\|A^k\|, && (2.11)\\
&\leq \sup_{k\geq n}|\lambda|_p^{k-n}\|A^k\|, && (2.12)\\
&\leq 1. && (2.13)
\end{aligned}$$

Thus, for all $n \in \mathbf{N}$ and $t \in \Omega_{|\alpha|}$,

$$\left\|\frac{R^{(n)}(\lambda, A)}{n!}\right\| \leq 1. \tag{2.14}$$

From 2.9 and 2.14, we have for all $n \in \mathbf{N}$, $\lambda \in \Omega_{|\alpha|}$,

$$\|(R(\lambda, A) - I)^n R(\lambda, A)\| \leq |\lambda|_p^n. \tag{2.15}$$

Conversely, let $A \in B(X)$ be a spectral operator, we assume that 2.8, for all $\lambda \in \Omega_{|\alpha|}$, $R(\lambda, A) = \sum_{n=0}^{\infty} \lambda^n A^n$. Set for all $\lambda \in \Omega_{|\alpha|}$, $k \in \mathbf{N}$, $S_k(\lambda) = \lambda^{-k}(R(\lambda, A) - I)^k R(\lambda, A)$, then for all $\lambda \in \Omega_{|\alpha|}$, $k \in \mathbf{N}$, $\|S_k(\lambda)\| \leq 1$. Since A and $R(\lambda, A)$ commute, we have:

$$\begin{aligned}
S_k(\lambda) &= \lambda^{-k}\Big(\Big(I - (I - \lambda A)\Big)R(\lambda, A)\Big)^k R(\lambda, A), && (2.16)\\
&= \lambda^{-k}(\lambda A R(\lambda, A))^k R(\lambda, A), && (2.17)\\
&= A^k R(\lambda, A)^{k+1}. && (2.18)
\end{aligned}$$

Then for all $\lambda \in \Omega_{|\alpha|}$, $k \in \mathbf{N}$,

$$\begin{aligned}
(2.19) \qquad \|A^k\| &= \|(I-\lambda A)^{k+1}S_k(\lambda)\|, \\
(2.20) \qquad &\leq \|(I-\lambda A)^{k+1}\|\|S_k(\lambda)\|, \\
(2.21) \qquad &\leq \|\sum_{j=0}^{k+1}\binom{k+1}{j}(-\lambda A)^j\|, \\
(2.22) \qquad &\leq \max\{1, \|\lambda A\|, \|\lambda^2 A^2\|, \cdots, \|\lambda^{k+1}A^{k+1}\|\},
\end{aligned}$$

for $\lambda \to 0$, we have for all $k \in \mathbf{N}$, $\|A^k\| \leq 1$. □

For A densely closed linear operator on X, the resolvant set $\rho(A)$ is the set of all $\lambda \in \mathbf{K}$ such that the range $Im(\lambda - A)$ is dense in X and that $\lambda I - A$ has the continous inverse $(\lambda I - A)^{-1}$ on $D((\lambda I - A)^{-1}) = Im(\lambda I - A)$, $\Big($where $Im(\lambda I - A)$ denote the range of $(\lambda I - A)\Big)$. In the next statements, we assume that $\mathbf{K}$ is a non-archimedean non trivially complete valued field with valuation $|\cdot|$.

Theorem 2.14. *Let X be a non-archimedeab Banach space of countable type over $\mathbf{K}$, let $(A^n)_{n\in\mathbf{N}}$ be a discrete semigroup of genrator A be densely defined closed linear operator on X such that $Im(A) \subset D(A)$. Suppose that $\rho(A) \neq \emptyset$, then A is bounded.*

Proof. Let $(A^n)_{n\in\mathbf{N}}$ be a discrete semigroup, suppose that $\rho(A) \neq \emptyset$, let $\lambda \in \rho(A)$, hence $(\lambda I - A)^{-1} \in B(X)$, then there exists $M > 0$ such that

$$(2.23) \qquad for\ all\ x \in D(\lambda I - A) = D(A),\ \|(\lambda I - A)x\| \geq M\|x\|.$$

Or A be densely closed linear operator, then $Im(\lambda I - A)$ is closed. In fact, $x_n \in D(A)$ and $z \in X$, $(\lambda I - A)x_n \to z$. By inequality 2.23, (x_n) is a Chauchy sequence in X. Or X is complete, then $x_n \to x$, for some $x \in X$. Thus, $x_n \to x$ and $Ax_n \to \lambda x - z$. By the closedness of A, we have $x \in D(A)$ and $Ax = \lambda x - z$. Since $\lambda \in \rho(A)$, $Im(\lambda I - A)$ is dense in X, then $Im(\lambda I - A) = X$. Consequently, $X = Im(\lambda I - A) \subseteq D(A)$, hence $D(A) = X$, then A is bounded. □

Proposition 2.15. *Let X be a non-archimedean Banach space over $\mathbf{K}$, let $A \in B(X)$ such that $\|A\| < 1$. Let $q(\lambda)$ be an arbitrary polynomial and set $p(\lambda) = 1 - (1 - \lambda)q(\lambda)$. Then we have*

$\|p(A)\| = \|(I - A)^{-1} - q(A)\|$.

Proof. Let $A \in B(X)$ such that $\|A\| < 1$. Let $q(\lambda)$ be an arbitrary polynomial and set $p(\lambda) = 1 - (1-\lambda)q(\lambda)$, then

$$\begin{aligned} (I-A)^{-1} - q(A) &= (I-A)^{-1}\Big(I - (I-A)q(A)\Big) && (2.24)\\ &= (I-A)^{-1}p(A). && (2.25)\end{aligned}$$

Thus,

$$\begin{aligned} \|(I-A)^{-1} - q(A)\| &= \|(I-A)^{-1}p(A)\|, && (2.26)\\ &\leq \|p(A)\|. && (2.27)\end{aligned}$$

On the other hand,

$$p(A) = (I-A)\Big((I-A)^{-1} - q(A)\Big). \tag{2.28}$$

Hence,

$$\begin{aligned} \|p(A)\| &= \|(I-A)\Big((I-A)^{-1} - q(A)\Big)\|, && (2.29)\\ &\leq \|(I-A)\|\|\Big((I-A)^{-1} - q(A)\Big)\|, && (2.30)\\ &\leq \|\Big((I-A)^{-1} - q(A)\Big)\|. && (2.31)\end{aligned}$$

Then,

$$\|p(A)\| = \|(I-A)^{-1} - q(A)\|.$$

$\square$

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Abdelkhalek El amrani
Department of mathematics and computer science,
Sidi Mohamed Ben Abdellah University,
Faculty of Sciences Dhar El Mahraz,
Fès,
Morocco
e-mail: abdelkhalek.elamrani@usmba.ac.ma

Jawad Ettayb
Department of mathematics and computer science,
Sidi Mohamed Ben Abdellah University,
Faculty of Sciences Dhar El Mahraz,
Fès,
Morocco
e-mail: jawad.ettayb@usmba.ac.ma
Corresponding author

and

Aziz Blali
Department of Mathematics,
Sidi Mohamed Ben Abdellah University,
ENS
B. P. 5206 Bensouda-Fès,
Morocco
e-mail: aziz.blali@usmba.ac.ma