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An inverse source time-fractional diffusion problem via an input-output mapping

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Abstract

In this paper, we investigate an inverse source problem involving a one-dimensional diffusion equation of a time-fractional Riemann-Liouville derivative with $0 < \alpha < 1$. *First, results on the existence and regularity of the weak solution of the direct problem are obtained. For the determination of the unknown time-dependent source term, we use a monotone and distinguishable input-output mapping defined by the additional over-determination integral data for the considered sub-diffusion problem. Finally, the uniqueness of the solution of the inverse problem is proved.*

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1. Introduction

Problems of determination of unknown source term in a diffusion equation play a crucial role in engineering, physics and applied mathematics. They are called inverse source diffusion problems and intensive investigations have been carried out concerning this kind of inverse problems by various theoretical and numerical methods, see for example [1, 3, 4, 6, 7, 9].

On the other hand, fractional integrals and derivatives which are generalizations of ordinary ones to an arbitrary fractional order, have a nonlocal character and are used to describe memory and hereditary properties of various phenomena and processes in science and engineering such as chemistry, mechanics, control and viscoelasticity. Hence, fractional partial differential equations become an important tool in modeling many real-life problems, in some cases when the standard diffusion equations have a disagreement with experimental data due to non Gaussian diffusion and there are some important applications of the anomalous diffusion processes; to know more about see [8, 27]. Note that, by a time-fractional diffusion equation we mean a parabolic-like partial differential equation with the partial time derivative of fractional order. It is called subdiffusion equation when $0 < \alpha < 1$. There are many works on the direct problems for subdiffusion equations such as an initial-boundary value problem [5, 15, 13, 14, 16, 17, 21, 24, 25]. Also, there has been growing interest in studying inverse problems of time-fractional partial differential equations by using different approaches [2, 10, 11, 18, 19, 20, 21, 22, 28]. The recovering of unknown source term is frequently investigated and various inverse problems were developed.

In this paper, we consider the inverse problem of finding a pair of functions $\{u(x,t), c(t)\}$ on $\Omega_T = \{(x,t) : 0 < x < 1, 0 < t \leq T\}$ which satisfies the time-fractional diffusion equation

$$\partial^{\alpha}_{0+,t} u(x,t) - u_{xx}(x,t) = c(t)f(x); \quad (x,t) \in \Omega_T \tag{1.1}$$

along with the fractional integral initial condition

$$\lim_{t \longrightarrow 0+} I^{1-\alpha}_{0^+,t} u(x,t) = \varphi(x); \quad x \in [0,1] \tag{1.2}$$

and the Dirichlet boundary conditions

$$u(0,t) = 0 = u(1,t); \quad t \in (0,T]. \tag{1.3}$$

$\varphi(x)$, $f(x)$ are given functions, $\partial_{0+,t}^{\alpha}$ and $I_{0+,t}^{1-\alpha}$ stand for Riemann-Liouville time fractional partial derivative of order $0 < \alpha < 1$ and integral of order $1-\alpha$, respectively.

The initial and boundary conditions are not sufficient to obtain the solution, an additional condition is required to determine the time-dependent source term $c\,(t)$. The additional data may be given in an interior point, on the boundary or on the whole domain. Here, we give the overdetermination integral condition

$$\int_0^1 xu(x,t)dx = g\,(t)\,; \qquad t \in (0,T] \tag{1.4}$$

where $g(t)$ is a given function.

Our main goal is to investigate theoretical aspects of the problem by analyzing the weak solution and uniquely recover the source term from the additional data (1.4). We introduce the input-output mapping $G\,(c) = g\,(t)$ on some admissible functions space $\mathcal{H}$ and determine analytically its series representation by using the additional integral overdetermination data. Hence, the inverse problem is reduced to the problem of the invertibility of the input-output mapping.

The new in this study that the problem is involving with Riemann-Liouville fractionnal derivative with $0 < \alpha < 1$ and the fractional integral initial condition.

The rest of the paper is structured as following. After some preliminaries about fractional calculus in the next section, we obtain, in section 3, existence and regularity results for the unique weak solution of the direct problem (1.1)-(1.3) using the Fourier method and Duhamel's principle (see [26] for the fractional case) to give the spectral representation of the solution. In section 4, we obtain the input-output mapping explicitly from the additional data $g\,(t)$ and discuss its monotonicity and distinguishability via the source term $c\,(t)$. This ensures the unique determination of $c\,(t)$. Then, we check the uniqueness of the pair $\{u(x,t), c(t)\}$ solution of the inverse problem.

2. Preliminaries

In this section, we present some useful definitions and results of fractional calculus which can be found in these books [12, 23].

Definition 1. *The left-sided Riemann-Liouville fractional integral of order $0<\alpha<1$ of $f\in L^1(0,T)$ is defined by*

$$I_{0+}^{\alpha}f(t):=\frac{1}{\Gamma(\alpha)}\int_0^t\frac{f(s)}{(t-s)^{1-\alpha}}ds,\quad t>0,$$

where $\Gamma(\alpha)$ is the Euler gamma function.

Proposition 2. *For the Euler gamma function $\Gamma(z)$ the following hold:*

$$\Gamma(z+1)=z\Gamma(z);\ \int_0^1 t^{z-1}(1-t)^{w-1}dt=\frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)};\ z,w\in\mathbf{R}^+.$$

Definition 3. *The left-sided Riemann-Liouville fractional derivative of order $0<\alpha<1$ is defined by*

$$D_{0+}^{\alpha}f(t):=\frac{d}{dt}I_{0^+}^{1-\alpha}f(t)=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t\frac{f(s)}{(t-s)^{\alpha}}ds,$$

for all $f\in L^1(0,T)$ such that $I_{0+}^{1-\alpha}f^{1,1}(0,T)$, a Sobolev space.

Proposition 4. *For $0<\alpha<1$, $f\in L^1[0,T]$, $I_{0^+}^{1-\alpha}f\in W^{1,1}(0,T)$,*

$$I_{0+}^{\alpha}D_{0+}^{\alpha}f(t)=f(t)-\frac{t^{\alpha-1}}{\Gamma(\alpha)}I_{0^+}^{1-\alpha}f(0^+);\ D_{0+}^{\alpha}I_{0+}^{\alpha}f(t)=f(t).$$

We need to define an appropriate space: $C_\gamma[0,T]$, $0\le\gamma<1$ the weighted space of functions f defined on $(0,T]$ such that $t^\gamma f\in C[0,T]$ which is a Banach space with the norm $\|u\|_{C_\gamma[0,T]}=\|t^\gamma u\|_{C[0,T]}$. Also, we define the α-weighted space

$$C_{1-\alpha}^{\alpha}[0,T]=\{u\in C_{1-\alpha}[0,T]:D_{0+}^{\alpha}u\in C_{1-\alpha}[0,T];0<\alpha<1\}.$$

Lemma 5. *Let $0<\alpha<1$,the functions $f\in L^1[0,T]$ and $K(t)$ has a measurable derivative $K'(t)$ almost everywhere on $[0,T]$, then for any $t\in[0,T]$,*

$$D_{0+}^{\alpha}\int_0^t f(s)K(t-s)ds=\int_0^t f(t-s)D_{0+,s}^{\alpha}K(s)ds+f(t)\lim_{s\to0+}I_{0^+,s}^{1-\alpha}K(s).\tag{2.1}$$

Let us present the **Mittag-Leffler Function** which is an important tool in fractional calculus.

Definition 6. *A two-parameter Mittag-Leffler function is defined by the series expansion*

$$E_{\alpha,\beta}(z) = \sum_{n\geq 0} \frac{z^n}{\Gamma(\alpha n+\beta)};\ \alpha,\beta,z\in \mathbf{C} \text{ with } \operatorname{Re}\alpha > 0. \tag{2.2}$$

In particular, for $\beta = 1$, $E_{\alpha,1}(z) = E_\alpha(z)$, for $\alpha = \beta = 1$; $E_1(z) = e^z$ and for $\alpha = \beta$, we define a special function called α-Exponential function defined by

$$e_\alpha(\lambda, z) = z^{\alpha-1} E_{\alpha,\alpha}(\lambda z^\alpha), \alpha, \lambda \in \mathbf{C}, z \in \mathbf{C}\setminus\{0\} \text{ with } \operatorname{Re}\alpha > 0.$$

Corollary 7. *The following properties hold on $(0, T]$ for $\alpha > 0, \beta > 0$, $\lambda \in \mathbf{R}$:*

(1) $0 < E_{\alpha,\beta}(\lambda t^\alpha) < \infty$; $\lim\limits_{t\to 0} E_{\alpha,\beta}(\lambda t^\alpha) = \dfrac{1}{\Gamma(\beta)}$ *and*

$$I^\alpha_{0+}\left(t^{\beta-1} E_{\alpha,\beta}(\lambda t^\alpha)\right) = t^{\alpha+\beta-1} E_{\alpha,\alpha+\beta}(\lambda t^\alpha).$$

(2) $e_\alpha(\lambda, t)$ *is bounded positive completely monotonic function and satisfies*

$$\begin{aligned} \int_0^t e_\alpha(\lambda, s)\, ds &< \infty; \\ D^\alpha_{0+}\left(e_\alpha(\lambda, t)\right) &= \lambda\, e_\alpha(\lambda, t). \end{aligned}$$

Theorem 8. *For a sequence of functions $(f_i(t))_{i\geq 0}$ defined on $(0, T]$, suppose the following conditions are fulfilled:*

(i) For a given $\alpha > 0$ the α-derivatives $D^\alpha_{0+} f_i(t), i \geq 0$; $t \in (0, T]$ exists.

(ii) $\sum_{i=1}^\infty f_i(t)$ and $\sum_{i=1}^\infty D^\alpha_{0+} f_i(t)$ are uniformly convergent on the interval $[\varepsilon, T]$ for any $\varepsilon > 0$.

Then the function defined by the series $\sum_{i=1}^\infty f_i(t)$ is $\alpha-$ differentiable and satisfies

$$D^\alpha_{0+} \sum_{i=1}^\infty f_i(t) = \sum_{i=1}^\infty D^\alpha_{0+} f_i(t).$$

Theorem 9. *The Cauchy type fractional problem*

$$\left\{\begin{array}{c} D^{\alpha}_{0+}u(t) = \lambda u(t),\ 0<\alpha<1,\ t\in(0,T],\ \lambda\in\mathbf{R}^{*}, \\ \lim\limits_{t\longrightarrow 0+} I^{1-\alpha}_{0+}u(t) = c, \end{array}\right. \tag{2.3}$$

has a unique solution $u\in C^{\alpha}_{1-\alpha}[0,T]$ *given by* $u(t)=ce_{\alpha}(\lambda,t)$.

Remark 10. *For* $0<\alpha<1,\ t\in(0,T]$, $u\in C_{1-\alpha}[0,T]$; $\lim\limits_{t\longrightarrow 0+} I^{1-\alpha}_{0+}u(t) = c$ *is equivalent to* $\lim\limits_{t\longrightarrow 0+} t^{1-\alpha}u(t)=c/\Gamma(\alpha)$.

3. Direct Problem

In this section, we will establish the spectral representation, existence, uniqueness and some regularity results for the weak solution of (1.1)-(1.3). To this end, we state some assumptions on c, f and φ obviously with $H^2(0,1)$ and $H^1_0(0,1)$ denote the Sobolev spaces.

(A1) $c\in C_{1-\alpha}[0,T]\cap L^2(0,T)$ is positive and $c(t)\neq 0$ for each $t\in(0,T]$.

(A2) $f\in H^1_0(0,1)$ with $f'(0)=f'(1)$.

(A3) $\varphi\in H^1_0(0,1)$ with $\varphi'(0)=\varphi'(1)$.

The initial boundary value problem (1.1)-(1.3) has a formal solution $u(x,t)$ defined in the domain $\overline{\Omega}_T$ as a Fourier series of the form

$$u(x,t)=\sum_{n\geq 0} u_n(t)\,X_n(x); \tag{3.1}$$

where the eigenfunctions $X_n(x)=\sin n\pi x; n\geq 1$, corresponding to the eigenvalues $\lambda_n=(n\pi)^2, n\geq 1$ of the spectral problem

$$\left\{\begin{array}{c} -X"(x)=\lambda X(x);\quad x\in(0,1) \\ X(0)=0=X(1), \end{array}\right. \tag{3.2}$$

form an orthogonal basis for the space $L^2(0,1)$.

Let us define what we mean by a weak solution.

Definition 1. *We call* $u(x,t)\in C_{1-\alpha}\left([0,T],H^1_0(0,1)\right)$ *is a weak solution to the subdiffusion problem (1.1)-(1.3) if it satisfies the equation*

$$\int_0^1 \left(\partial^{\alpha}_{0+,t}u(x,t)-u_{xx}(x,t)-c(t)f(x)\right)\phi(x)\,dx=0,\quad \text{for any } t\in(0,T];$$

and

$$\int_0^1 \left(\lim_{t \longrightarrow 0+} I_{0+,t}^{1-\alpha} u(x,t) - \varphi(x) \right) \phi(x)\, dx = 0;$$

for any $\phi \in H_0^1(0,1)$.

Theorem 2. *Under assumptions (A1)-(A3), there exists a unique weak solution defined by (3.1) of the time-fractional diffusion problem (1.1)-(1.3) where the coefficients* $u_n(t), n \geq 1$ *are given by*

$$u_n(t) = \varphi_n e_\alpha(-\lambda_n, t) + f_n \int_0^t c(s) e_\alpha(-\lambda_n, t-s) ds; \tag{3.3}$$

where $\varphi_n = 2\int_0^1 \varphi(x) X_n(x)\, dx, n \geq 1$ *and* $f_n = 2\int_0^1 f(x) X_n(x)\, dx, n \geq 1.$

Furthermore,for each $n \geq 1$, u_n *is in* $C_{1-\alpha}^{\alpha}[0,T]$.

Proof. In order to simplify the initial-boundary values problem (1.1)-(1.3), we put $u(x,t) = v(x,t) + w(x,t)$ where $w(x,t)$ is the solution of

$$\begin{cases} \partial_{0+,t}^{\alpha} w(x,t) - w_{xx} = 0; & (x,t) \in (0,1) \times (0,T] \\ \lim\limits_{t \longrightarrow 0+} I_{0+,t}^{1-\alpha} w(x,t) = \varphi(x); & x \in [0,1] \\ w(0,t) = 0 = w(1,t); & t \in (0,T] \end{cases} \tag{3.4}$$

and $v(x,t)$ is the solution of

$$\begin{cases} \partial_{0+,t}^{\alpha} v(x,t) - v_{xx} = c(t) f(x); & (x,t) \in (0,1) \times (0,T] \\ \lim\limits_{t \longrightarrow 0+} I_{0+,t}^{1-\alpha} v(x,t) = 0; & x \in [0,1] \\ v(0,t) = 0 = v(1,t); & t \in (0,T]. \end{cases} \tag{3.5}$$

The formal solution of (3.4) is given by $w(x,t) = \sum_{n=1}^{\infty} w_n(t) X_n(x)$, where $w_n(t) = 2\int_0^1 w(x,t) X_n(x)\, dx, n \geq 1$ are solutions of the fractional problem

$$\begin{cases} D_{0+}^{\alpha} w_n(t) = -\lambda_n w_n(t), \quad t \in (0,T]; n \geq 1 \\ \lim\limits_{t \longrightarrow 0+} I_{0+}^{1-\alpha} w_n(t) = \varphi_n, \end{cases} \tag{3.6}$$

given in $C_{1-\alpha}^{\alpha}[0,T]$ by

$$w_n(t) = \varphi_n e_\alpha(-\lambda_n, t)\,;\ t \in (0, T]\,;\ n \geq 1. \tag{3.7}$$

Therefore, the solution of (3.4) can be written as the series:

$$w(x,t) = \sum_{n\geq 1} \varphi_n e_\alpha(-\lambda_n, t)\, X_n(x)\,. \tag{3.8}$$

To determine $v(x,t) = \sum_{n\geq 1} v_n(t)\, X_n(x)$ solution of (3.5), we use Duhamel's principle. So, we put $v(x,t) = \int\limits_0^t V(x,t,s)\, ds$ and by Lemma 5 we get $V(x,t,s)$ is the solution of

$$\begin{cases} \partial^\alpha_{s+,t} V(x,t,s) - V_{xx}(x,t,s) = 0; & x \in (0,1);\ 0 < s < t \leq T \\ \lim\limits_{t\to s^+} I^{1-\alpha}_{0+,s} V(x,t,s) = f(x)c(s); & x \in [0,1] \\ V(0,t,s) = 0 = V(1,t,s); & 0 < s < t \leq T, \end{cases} \tag{3.9}$$

satisfying, in view of (3.8),

$$V(x,t,s) = \sum_{n\geq 1} V_n(t,s)\, X_n(x) = \sum_{n\geq 1} f_n c(s) e_\alpha(-\lambda_n, t-s)\, X_n(x)\,, \tag{3.10}$$

where $V_n(.,s) \in C^\alpha_{1-\alpha}[0,T]$ for each $0 < s < t$. Thus,

$$v(x,t) = \sum_{n\geq 1} f_n \int\limits_0^t c(s)\, e_\alpha(-\lambda_n, t-s)\, ds X_n(x)\,. \tag{3.11}$$

Finally, to sum up (3.8) and (3.11), the spectral representation of the solution to the problem (1.1)- (1.3) is of the form (3.1)-(3.3).

In view of Theorem 9 and (A1)-(A3), for each fixed $n \geq 1$, the solutions of (3.5) and (3.6) are unique. Hence, for any $n \geq 1$, $w_n(t) + v_n(t) = u_n(t)$ is unique in $C^\alpha_{1-\alpha}[0,T]$. This leads to the uniqueness of the weak solution $u(x,t)$ with the spectral representation (3.1)- (3.3). This completes the proof. □

Now, we deal with the regularity properties of $u(x,t)$ for a fixed term-source $c(t)$ in $C_{1-\alpha}[0,T]$. Let us denote,

$$M := \sup_{n>0} \sup_{0\leq s<t\leq T} E_{\alpha,\alpha}(-\lambda_n (t-s)^\alpha); \tag{3.12}$$

$$M_1 := \sup_{n>0} \sup_{0\le s<t\le T} \lambda_n e_\alpha\left(-\lambda_n, t-s\right); \tag{3.13}$$

$$Q_1 := T^{\alpha} \frac{\Gamma^2(\alpha)}{\Gamma(2\alpha)} M. \tag{3.14}$$

$$Q_2 := T^{1-\alpha} M_1; \;\; Q_3 := \frac{T}{\alpha} M_1. \tag{3.15}$$

Theorem 3. *The unique weak solution of (1.1)-(1.3) is in $C_{1-\alpha}([0,T]; L^2(0,1))$ and satisfies the following approximation*

$$\|u\|_{C_{1-\alpha}([0,T];L^2(0,1))} \le M \|\varphi\|_{L^2(0,1)} + Q_1 \|c\|_{C_{1-\alpha}[0,T]} \|f\|_{L^2(0,1)}, \tag{3.16}$$

for some positive constants M and Q_1 given by (3.14).

Proof. The $L^2(0,1)$ norm of $w(x,t)$ with respect to x, satisfies for each $t \in (0,T]$

$$\|w(.,t)\|^2_{L^2(0,1)} = \frac{1}{2} \sum_{n\ge 1} |w_n(t)|^2.$$

This yields, for $t \in (0,T]$,

$$\begin{aligned}
\|w(.,t)\|^2_{L^2(0,1)} &\le \frac{1}{2} \sum_{n\ge 1} \varphi_n^2 \left[e_\alpha(-\lambda_n, t)\right]^2 \\
&\le \frac{1}{2} \left[M t^{\alpha-1}\right]^2 \sum_{n\ge 1} \varphi_n^2 \le t^{2(\alpha-1)} M^2 \|\varphi\|^2_{L^2 0,1)};
\end{aligned}$$

where M is given by (3.12). By similar approximations and in view of (A2) we get from (3.11) for each $t \in (0,T]$

$$\begin{aligned}
\|v(.,t)\|^2_{L^2(0,1)} &= \frac{1}{2} \sum_{n\ge 1} |v_n(t)|^2 \\
&\le \frac{M^2}{2} \|c\|^2_{C_{1-\alpha}[0,T]} \sum_{n\ge 1} |f_n|^2 \left[\int_0^t s^{\alpha-1}(t-s)^{\alpha-1} ds\right]^2 \\
&\le \left(t^{2\alpha-1} \frac{\Gamma^2(\alpha)}{\Gamma(2\alpha)}\right)^2 \|c\|^2_{C_{1-\alpha}[0,T]} M^2 \|f\|^2_{L^2(0,1)}.
\end{aligned}$$

Hence, for each $t \in (0,T]$ we get

$$\begin{aligned}\|u(.,t)\|_{L^2\ [0,1]} &\le \|w(.,t)\|_{L^2(0,1)} + \|v(.,t)\|_{L^2(0,1)} \\ &\le t^{\alpha-1} M \|\varphi\|_{L^2(0,1)} + t^{2\alpha-1}\frac{\Gamma^2(\alpha)}{\Gamma(2\alpha)} \|c\|_{C_{1-\alpha}[0,T]} M \|f\|_{L^2\ (0,1)}.\end{aligned}$$

This leads to the result (3.16) with (3.14) which completes the proof.
□

Theorem 4. *The unique weak solution of (1.1)-(1.3) $u \in C_{1-\alpha}([0,T];$ $H_0^1(0,1) \cap H^2(0,1))$ such that $\partial_{0+,t}^{\alpha} u \in L^2((0,T);L^2(0,1))$ and satisfies the following approximations*

$$\begin{aligned}\|u\|_{C_{1-\alpha}([0,T];H_0^1(0,1))} + \left\|\partial_{0+,t}^{\alpha} u\right\|_{C_{1-\alpha}([0,T];L^2(0,1))} &\le A_1 \|\varphi\|_{L^2(0,1)} \\ &+ A_2 \|c\|_{C_{1-\alpha}[0,T]} \|f\|_{L^2(0,1)},\end{aligned} \tag{3.17}$$

for some positive constants A_i, $i = 1,2$ given by

$$A_1 := Q_2\left(\frac{1}{\pi}+1\right)\ ;\ A_2 := Q_3\left(\frac{1}{\pi}+1\right)+1; \tag{3.18}$$

where Q_2, Q_3 are defined by (3.15).

Proof. From (3.1)-(3.3) the spectral form of $u(x,t)$, we deduce $u_x(x,t) = w_x(x,t) + v_x(x,t)$. Then, under (A3), we get

$$\begin{aligned}\|w_x(.,t)\|^2_{L^2\ (0,1)} &= \frac{1}{2}\sum_{n\ge1} \lambda_n w_n^2(t) \\ &\le \frac{1}{2}\sum_{n\ge1} |\varphi_n|^2 \lambda_n [e_\alpha(-\lambda_n,t)]^2 \\ &\le \frac{M_1^2}{2}\sum_{n\ge1}\frac{(\varphi_n)^2}{\lambda_n} \le \frac{M_1^2}{\pi^2}\|\varphi\|^2_{L^2(0,1)};\end{aligned}$$

where M_1 is given by (3.13). Also, we get

$$\begin{aligned}\|v_x(.,t)\|^2_{L^2\ (0,1)} &\leq \frac{1}{2}\sum_{n\geq 1}\frac{|f_n|^2}{\lambda_n}\|c\|^2_{C_{1-\alpha}[0,T]} \\ &\quad\times\left[\int_0^t s^{\alpha-1}\lambda_n e_\alpha(-\lambda_n, t-s)\,ds\right]^2 \\ &\leq \frac{M_1^2}{\pi^2}\|f\|^2_{L^2(0,1)}\|c\|^2_{C_{1-\alpha}[0,T]}\left[\frac{t^\alpha}{\alpha}\right]^2.\end{aligned}$$

As the $H_0^1(0,1)$ norm of $u(x,t)$ with respect to x is

$$\|u(.,t)\|_{H_0^1(0,1)} = \|u_x(.,t)\|_{L^2(0,1)};$$

we obtain

$$\|u\|_{C_{1-\alpha}([0,T];H_0^1(0,1))} \leq \frac{Q_2}{\pi}\|\varphi\|_{L^2(0,1)} + \frac{Q_3}{\pi}\|c\|_{C_{1-\alpha}[0,T]}\|f\|_{L^2(0,1)}, \tag{3.19}$$

with (3.15). Thus, $u \in C_{1-\alpha}([0,T]; H_0^1(0,1))$.

By the above similar arguments, we get

$$\begin{aligned}\|w_{xx}(.,t)\|^2_{L^2\ (0,1)} &= \frac{1}{2}\sum_{n\geq 1}\lambda_n^2 w_n^2(t) \leq M_1^2\|\varphi\|^2_{L^2(0,1)}; \\ \|v_{xx}(.,t)\|^2_{L^2\ (0,1)} &= \frac{1}{2}\sum_{n\geq 1}\lambda_n^2 v_n^2(t) \leq \left(\frac{t^\alpha}{\alpha}\right)^2\|c\|^2_{C_{1-\alpha}[0,T]}M_1^2\|f\|^2_{L^2(0,1)}.\end{aligned}$$

Hence,

$$\|u_{xx}(.,t)\|_{L^2\ (0,1)} \leq M_1\left(\|\varphi\|_{L^2(0,1)} + \frac{t^\alpha}{\alpha}\|c\|_{C_{1-\alpha}[0,T]}\|f\|_{L^2(0,1)}\right).$$

This implies that $u(.,t) \in H^2(0,1)$ for $t \in (0,T]$. Next, note that $D^\alpha_{0+}u_n(t)$, $n \geq 1$ exist for $t \in (0,T]$ and satisfies by fractional calculus

$$\begin{aligned}D^\alpha_{0+}u_n(t) &= \varphi_n D^\alpha_{0+,t}\ e_\alpha(-\lambda_n, t) + \int_0^t f_n c(t-s)\,D^\alpha_{0+,s}\ e_\alpha(\lambda_n, s)\,ds \\ &\quad + f_n c(t)\lim_{s\to 0+} I^{1-\alpha}_{0+,s} e_\alpha(-\lambda_n, s) \\ &= -\varphi_n\lambda_n\ e_\alpha(-\lambda_n, t) - \int_0^t f_n c(t-s)\,\lambda_n\ e_\alpha(\lambda_n, s)\,ds \\ &\quad + f_n c(t)\lim_{s\to 0+} E_{\alpha,1}(-\lambda_n s^\alpha).\end{aligned}$$

Thus, by hypotheses, we get

$$\begin{aligned}\left|\sum_{n\geq 1} D_{0+}^{\alpha} u_n(t) X_n(x)\right| &\leq M_1 |\varphi(x)| + |f(x)| \left(M_1 \int_0^t c(t-s)\, ds + c(t)\right) \\ &\leq M_1 |\varphi(x)| \\ &\quad + |f(x)| \left(M_1 \frac{t^{\alpha}}{\alpha} \|c\|_{C_{1-\alpha}[0,T]} + t^{\alpha-1} \|c\|_{C_{1-\alpha}[0,T]}\right).\end{aligned}$$

In consequent, the series $\sum_{n\geq 1} D_{0+}^{\alpha} u_n(t) X_n(x)$ is uniformly convergent for $t \in [\epsilon, T], \epsilon > 0$ and in view of Theorem 8

$$\sum_{n\geq 1} D_{0+}^{\alpha} u_n(t) X_n(x) = D_{0+}^{\alpha} \sum_{n\geq 1} u_n(t) X_n(x).$$

By (1.1), we have for $t \in (0, T]$

$$\left\|\partial_{0+,t}^{\alpha} u(.,t)\right\|_{L^2\ (0,1)} \leq \|u_{xx}(.,t)\|_{L^2(0,1)} + |c(t)| \, \|f\|_{L^2(0,1)}.$$

In view of the approximation of $u_{xx}(x,t)$ we get

$$\begin{aligned}\left\|\partial_{0+,t}^{\alpha} u(.,t)\right\|_{L^2\ (0,1)} &\leq M_1 \left(\|\varphi\|_{L^2(0,1)} + \|c\|_{C_{1-\alpha}[0,T]} \frac{T^{\alpha}}{\alpha} \|f\|_{L^2(0,1)}\right) \\ &\quad + |c(t)| \, \|f\|_{L^2(0,1)},\end{aligned}$$

which implies in view of $A1$ that the time-fractional derivative of u is in $L^2((0,T); L^2(0,1))$ and yields

$$\left\|\partial_{0+,t}^{\alpha} u\right\|_{C_{1-\alpha}([0,T];L^2(0,1))} \leq Q_2 \|\varphi\|_{L^2(0,1)} + (1+Q_3) \|c\|_{C_{1-\alpha}[0,T]} \|f\|_{L^2(0,1)}, \tag{3.20}$$

with Q_2 and Q_3 are defined by (3.15). Then, between (3.20) and (3.20) we conclude (3.17). In view of (3.16) and (3.20) we deduce that $u(x,.) \in C_{1-\alpha}^{\alpha}[0,T]$ almost everywhere on $[0,1]$ in $L^2(0,1)$. □

4. Inverse source Problem

Now, to prove the existence and uniqueness of the time dependent source term $c(t)$ we study the monotonicity and distinguishability of the input-output mapping that can be determined as well.

Therefore, multiplying (3.1) by x and integrating over $[0,1]$, $g(t)$ can be determined analytically by a series representation, for $t \in (0,T]$,

$$g(t) = \sum_{n\geq 1} \frac{(-1)^{n+1}}{n\pi} \varphi_n t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n t^\alpha) \tag{4.1}$$

$$+ \sum_{n\geq 1} \frac{(-1)^{n+1}}{n\pi} f_n \int_0^t c(s)(t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n (t-s)^\alpha) ds,$$

which is well defined in view of (A1)-(A3).

Moreover (A1)-(A3), we assume the following assumptions:

(A4) $g \in C^{\alpha}_{1-\alpha}[0,T]$ and $\lim\limits_{t\to 0^+} I^{1-\alpha}_{0^+} g(t) = \int_0^1 x\varphi(x)dx$.

(A5) $(-1)^n f_n \geq 0; n \geq 1$ and $\displaystyle\int_0^1 xf(x)dx \neq 0$.

(A6) $\varphi_n \geq 0; n \geq 1$ and $\displaystyle\int_0^1 x\varphi(x)dx \neq 0$.

Let us denote the set of admissible time-dependant source terms $c(t)$ by

$$\mathcal{H} = \{c \in C_{1-\alpha}[0,T] : \ 0 < C_0 \leq c(t) \leq C_1, t \in (0,T] \ \} \subset C_{1-\alpha}[0,T]$$

and by $\mathcal{G} \subset C^{\alpha}_{1-\alpha}[0,T]$ the set of measured (free noisy) output data $g(t)$. Then, define the input-output mapping $G(.) : \mathcal{H} \to \mathcal{G}$ in view of the right-hand side of (4.1) as follows:

$$G(c) = W_\varphi + V_c = g, \quad g \in \mathcal{G}, \tag{4.2}$$

where W_φ and V_c are the part of (4.1) depending of φ and c respectively given by

$$W_\varphi(t) = \int_0^1 xw(x,t)\,dx = \sum_{n\geq 1} \frac{(-1)^{n+1}}{n\pi} \varphi_n t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n t^\alpha); \tag{4.3}$$

and

$$V_c(t) = \int_0^1 xv(x,t)\,dx = \sum_{n\geq 1} \frac{(-1)^{n+1}}{n\pi} f_n \int_0^t c(s)(t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n (t-s)^\alpha) ds. \tag{4.4}$$

Hence, the inverse problem of determination of the time-source term $c(t)$ in (1.1)-(1.4) is reduced to the problem of invertibility of G.

Remark 1. *The existence of $c(t)\, in \mathcal{H}$ derive from the relation $g(t) - W_{\varphi}(t) = V_c(t)$.*

Let $u(x,t)$, $\widetilde{u}(x,t)$ be two solutions of the direct problem (1.1)-(1.3) and $g(t)$, $\widetilde{g}(t)$ be the overdetermination data corresponding to the admissible time-source terms c, $\widetilde{c} \in \mathcal{H}$ respectively. Now, to ensure the existence of the solution of the inverse problem let's show the monotonicity of G.

Theorem 2. *Let assumptions (A1)-(A6) hold. Then, the operator G is monotone on $\mathcal{H}$.*

Proof. Given $c, \widetilde{c} \in \mathcal{H}$ such that $0 < c(t) \leq \widetilde{c}(t)\,; t \in (0,T]$, without problems we can suppose that $f(x)$ is positive and have

$$\begin{cases} \partial^{\alpha}_{0+,t} u(x,t) - u_{xx}(x,t) \leq \partial^{\alpha}_{0+,t}\, \widetilde{u}(x,t) - \widetilde{u}_{xx}(x,t)\,; \quad (x,t) \in \Omega \\ \lim\limits_{t \longrightarrow 0+} I^{1-\alpha}_{0+,t} u(x,t) = \lim\limits_{t \longrightarrow 0+} I^{1-\alpha}_{0+,t}\, \widetilde{u}(x,t); \quad x \in [0,1]. \end{cases}$$

Multiply by x and integrate over $[0,1]$ to obtain

$$\text{(4.5)} \quad \begin{cases} D^{\alpha}_{0+}\left(\widetilde{g}(t) - g(t)\right) \geq \int\limits_0^1 x\left(\widetilde{u}_{xx}(x,t) - u_{xx}(x,t)\right) dx, \qquad t \in (0,T] \\ \lim\limits_{t \longrightarrow 0+} I^{1-\alpha}_{0+}\left(\widetilde{g}(t) - g(t)\right) = 0. \end{cases}$$

We have for $t \in (0,T]$

$$\begin{aligned} \int\limits_0^1 x\left(\widetilde{u}_{xx}(x,t) - u_{xx}(x,t)\right) dx &= \widetilde{u}_x(1,t) - u_x(1,t) \\ &= \sum_{n \geq 1} (-1)^n f_n \left(\widetilde{h}_n(t) - h_n(t)\right); \end{aligned}$$

where, for $n \geq 1$,

$$\widetilde{h}_n(t) - h_n(t) = n\pi \int\limits_0^t \left(\widetilde{c}(s) - c(s)\right)(t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n (t-s)^{\alpha}) ds \geq 0.$$

Apply I^{α}_{0+} to the first equation of *(4.5), we get*

$$\widetilde{g}(t) - g(t) \geq \sum_{n\geq 1} (-1)^n f_n I^{\alpha}_{0+} \left(\widetilde{h}_n(t) - h_n(t)\right) \geq 0;$$

which implies the monotonicity of G. The proof is complete. □

Although $w(x,t)$ the solution of (3.4) does not depend on the source term $c(t)$ but it influences its uniqueness compared to the initial data. So, we must present a uniqueness result concerning W_φ the part of G related on the input data φ.

Lemma 3. *Let $w_1(x,t)$, $w_2(x,t)$ be the solutions of the direct problem (3.4) related to the initial conditions $\varphi(x)$, $\psi(x)$ and W_φ, W_ψ be the part of G related to the input data $\varphi(x)$, $\psi(x)$ respectively given by (4.3) where $\varphi(x)$, $\psi(x)$ satisfy assumptions (A3) and (A6). If*

$$W_\varphi(t) = W_\psi(t), \ 0 < t \leq T,$$

then,

$$\varphi(x) = \psi(x) \tag{4.6}$$

almost everywhere on $[0,1]$.

Proof. $w_\varphi(x,t)$, $w_\psi(x,t) \in C_{1-\alpha}([0,T]; L^2[0,1])$ *the solutions of the direct problem (3.4) related to the initial conditions* $\varphi(x)$ and $\psi(x)$ are *given by (3.8)*, respectively. It is clear that $W_\varphi(t), W_\psi(t)$ *are in* $C_{1-\alpha}[0,T]$. Let us suppose that

$$\begin{aligned} \varphi(x) &> \psi(x); \quad x \in (x_i, x_{i+1}), i = 1, ..., m \\ \varphi(x) &= \psi(x); \ \text{otherwise}, \end{aligned}$$

for some $x_i \in [0,1], i = 1, ..., m+1$. Then, *we get*

$$\begin{aligned} 0 &< \sum_{i=1}^{m} \int_{x_i}^{x_{i+1}} x[\varphi(x) - \psi(x)]\,dx = \int_0^1 x[\varphi(x) - \psi(x)]\,dx \\ &\leq \int_0^1 x \sum_{n\geq 1} [\varphi_n - \psi_n] X_n(x)\,dx. \end{aligned}$$

Thus, by the fact that

$$\lim_{t\to 0^+} t^{1-\alpha}t^{\alpha-1}\Gamma(\alpha)\, E_{\alpha,\alpha}(-\lambda_n t^\alpha) = 1, \tag{4.7}$$

we get

$$\begin{aligned} 0 &< \int_0^1 x \sum_{n\geq 1} |\varphi_n - \psi_n| \lim_{t\to 0^+} t^{1-\alpha}\Gamma(\alpha) t^{\alpha-1} E_{\alpha,\alpha}(-(n\pi)^2 t^\alpha) X_n(x)\, dx \\ &\leq \sup_{t\in[0,T]} t^{1-\alpha}\Gamma(\alpha) \int_0^1 x \ \ |w_\varphi(x,t) - w_\psi(x,t)|\, dx \\ &= \Gamma(\alpha)\, \|W_\varphi - W_\psi\|_{C_{1-\alpha}[0,T]} = 0. \end{aligned}$$

This gives us a contradiction. Then, $\varphi(x) = \psi(x)$ almost everywhere on $[0,1]$. □

To can identify $c(t)$ uniquely we will study the distinguishability of the unknown function $c(t)$ via the input-output mapping G in the sense that $G(c) \neq G(\widetilde{c})$ implies $c \neq \widetilde{c}$ and this means the injectivity of its inverse G^{-1}. So, we give the following result.

Theorem 4. *Assume that assumptions (A1)-(A6) hold. Then the input-output mapping $G(c)$ corresponding to the additional data (1.4), is distinguishable in the class of admissible source parameters $\mathcal{H}$.*

Proof. We will prove that $G(c)$ is Lipschitz continuous on $\mathcal{H}$. Let c, $\widetilde{c} \in \mathcal{H}$ such that $c(t) \neq \widetilde{c}(t)$; $t \in (0,T]$ then $z(x,t) = u(x,t) - \widetilde{u}(x,t)$ is a solution of the problem

$$\begin{cases} \partial^\alpha_{0+,t} z(x,t) - z_{xx} = (c(t) - \widetilde{c}(t))\, f(x); \quad (x,t) \in (0,1)\times(0,T] \\ \lim\limits_{t\longrightarrow 0+} I^{1-\alpha}_{0^+,t} z(x,t) = 0; \quad x \in [0,1] \\ z(0,t) = 0 = z(1,t); \qquad t \in (0,T]. \end{cases} \tag{4.8}$$

In view of (3.1)-(3.3), z is given by

$$z(x,t) = \sum_{n\geq 1} \int_0^t (c(s) - \widetilde{c}(s))\, (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n (t-s)^\alpha) ds f_n X_n(x).$$

Then, we get

$$\int_0^1 xz(x,t)\,dx = \int_0^1 x \sum_{n\geq 1} \int_0^t (c(s) - \widetilde{c}(s))$$
$$\times (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n (t-s)^{\alpha}) ds \langle f; X_n \rangle X_n(x)\,dx,$$

which implies that

$$|g(t) - \widetilde{g}(t)| \leq \int_0^1 x \left| \sum_{n\geq 1} f_n X_n(x) \right| dx \, \|c - \widetilde{c}\|_{C_{1-\alpha}[0,T]} M \int_0^t s^{\alpha-1} (t-s)^{\alpha-1} ds.$$

Therefore, in view of (A2) and (A5) we obtain

$$\|G(c) - G(\widetilde{c})\|_{C_{1-\alpha}[0,T]} = \|g - \widetilde{g}\|_{C_{1-\alpha}[0,T]} \leq H \|c - \widetilde{c}\|_{C_{1-\alpha}[0,T]},$$

where $H := M \dfrac{T^{\alpha}\Gamma^2(\alpha)}{\Gamma(2\alpha)} \displaystyle\int_0^1 x |f(x)|\,dx$. In consequent, if $G(c) \neq G(\widetilde{c})$ for each c, $\widetilde{c} \in \mathcal{H}$ then $c(t) \neq \widetilde{c}(t)$ on $(0,T]$ which is the wanted property. □

Theorem 5. *The source term $c(t)$ can be determined uniquely in the problem (1.1)-(1.3) by the additional data (1.4).*

Proof. The uniqueness of $c(t)$ depends on $v(x,t)$ the solution of (3.5). Then for c, $\widetilde{c} \in \mathcal{H}$

$$g(t) - \widetilde{g}(t) = \int_0^1 x (v(x,t) - \widetilde{v}(x,t))\,dx.$$

Recall that $v(x,t) = \displaystyle\int_0^t V(x,t,s)\,ds$ where $V(x,t,s)$ given by (3.10) is the solution of the problem (3.9). Then, in view of Lemma 3, the initial condition $f(x)c(s)$ of this problem can be determined uniquely in $L^2(0,1)$ by the additional data $\displaystyle\int_0^1 xV(x,t,s)\,dx$. Accordingly, under (A1)-(A5) *and by setting* $c(t) > \widetilde{c}(t) > 0$ on $(0,T]$ *we have*

$$0 < (c(s) - \widetilde{c}(s)).$$

Then,

$$0 < \left| \int_0^1 xf\left(x\right)dx \right| \left(c\left(s\right) - \widetilde{c}\left(s\right)\right) = \left| \int_0^1 x \sum_{n\geq 1} f_n X_n\left(x\right)dx \right| \left(c\left(s\right) - \widetilde{c}\left(s\right)\right).$$

Hence, by (4.7) and as $V\left(x,t,s\right)$ is given by (3.10), we have,

$$\begin{aligned} & \left| \int_0^1 x\left(c\left(s\right) - \widetilde{c}\left(s\right)\right) \sum_{n\geq 1} f_n X_n\left(x\right) \right. \\ & \left. \times \lim_{t-s\to 0^+} \left(t-s\right)^{1-\alpha} \left(t-s\right)^{\alpha-1} \Gamma\left(\alpha\right) E_{\alpha,\alpha}(-\lambda_n\left(t-s\right)^{\alpha})dx \right| \\ \leq \; & \Gamma\left(\alpha\right) \sup_{0\leq s\leq t} \left(t-s\right)^{1-\alpha} \left| \int_0^1 x\left[V\left(x,t,s\right) - \widetilde{V}\left(x,t,s\right)\right]dx \right|. \end{aligned}$$

From the fact that $V_c\left(t\right) = \int_0^1 x \int_0^t V\left(x,t,s\right)dsdx$ and the monotonicity of $G\left(c\right)$, we conclude

$$\begin{aligned} & \left| \int_0^1 xf\left(x\right)dx \right| \sup_{t\in[0,T]} t^{1-\alpha}\left(c\left(t\right) - \widetilde{c}\left(t\right)\right) \int_0^t s^{\alpha-1}ds \\ \leq \; & \Gamma\left(\alpha\right) \sup_{t\in[0,T]} t^{1-\alpha} \int_0^t \int_0^1 x\left[V\left(x,t,s\right) - \widetilde{V}\left(x,t,s\right)\right]dxds \end{aligned}$$

then,

$$\begin{aligned} \left\|c - \widetilde{c}\right\|_{C_{1-\alpha}[0,T]} & \leq C_2 \sup_{t\in[0,T]} t^{1-\alpha}\left|V_c\left(t\right) - V_{\widetilde{c}}(t)\right| \\ & \leq C_2 \left\|V_c - V_{\widetilde{c}}\right\|_{C_{1-\alpha}[0,T]}; \end{aligned}$$

where $C_2 := \dfrac{\alpha\Gamma\left(\alpha\right)}{T^{\alpha}} \left[\left| \int_0^1 xf(x)dx \right|\right]^{-1}$. Thus,

$$\left\|c - \widetilde{c}\right\|_{C_{1-\alpha}[0,T]} \leq C_2 \left\|g - \widetilde{g}\right\|_{C_{1-\alpha}[0,T]}. \tag{4.9}$$

So, $c(t)$ can be obtained uniquely on $(0,T]$ by the output data $g(t)$ and the proof is completed. $\square$

Now, let us prove the uniqueness of the solution of the inverse problem (1.1)-(1.4).

Theorem 6. *Under the assumptions (A1)-(A5), $\{u(x,t);c(t)\}$ the solution of the inverse problem (1.1)-(1.4) is unique.*

Proof. Let $\{u(x,t);c(t)\}$ and $\{\widetilde{u}(x,t);\widetilde{c}(t)\}$ be two solutions of the inverse problem (1.1)-(1.4) then for $(x,t)\in\Omega$

$$\begin{aligned}|u(x,t)-\widetilde{u}(x,t)| &\leq \sum_{n\geq 1}|f_n|\int_0^t |c(s)-\widetilde{c}(s)|(t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(t-s)^{\alpha})ds\\ &\leq M\sum_{n\geq 1}|f_n|\,t^{2\alpha-1}\frac{\Gamma^2(\alpha)}{\Gamma(2\alpha)}\|c-\widetilde{c}\|_{C_{1-\alpha}[0,T]}.\end{aligned}$$

Thus, for $x\in(0,1)$

$$\|u(x,.)-\widetilde{u}(x,.)\|_{C_{1-\alpha}[0,T]}\leq K\|c-\widetilde{c}\|_{C_{1-\alpha}[0,T]}, \tag{4.10}$$

where $K := MT^{\alpha}\dfrac{\Gamma^2(\alpha)}{\Gamma(2\alpha)}\sum_{n\geq 1}|f_n|$ is a constant by the fact that $f\in H_0^1(0,1)$ which implies that $\sum_{n\geq 1}|f_n|$ is convergent in view of Bessel's inequality. In consequent,

$$\|u-\widetilde{u}\|_{C_{1-\alpha}([0,T],L^2(0,1))}\leq K\|c-\widetilde{c}\|_{C_{1-\alpha}[0,T]} \tag{4.11}$$

Also, from (4.9) and by virtue of the Cauchy-Schawrtz inequality, we have

$$\begin{aligned}\|c-\widetilde{c}\|_{C_{1-\alpha}[0,T]} &\leq C_2\left\|\int_0^1 x\,(u(x,.)-\widetilde{u}(x,.))\,dx\right\|_{C_{1-\alpha}[0,T]}\\ &\leq C_2\|u-\widetilde{u}\|_{C_{1-\alpha}([0,T],L^2(0,1))}.\end{aligned} \tag{4.12}$$

The result is obtained between (4.11) and (4.12). $\square$

5. Conclusion

In this paper, we investigate an inverse source problem for the time-fractional of Riemann-Liouville type diffusion problem with an integral over-detemination data (1.4). First, we obtain the analytical solution of the direct problem (1.1)-(1.3) using Fourier's method, then, we study its regularity.

For the inverse problem, the obtained series representation of the measured output data $g(t)$ leads to the explicit form of the input-output mapping $G(c)$. Hence, we etablish the properties of monotonicity and distinguishability of this input-output mapping which implies the existence and injectivity of the inverse mapping and permit us the determination of the unknown time dependent term source $c(t)$ by the additional data $g(t)$. Finally uniqueness result is given.

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