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Solution of linear and non-linear partial differential equations of fractional order

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Abstract

We know that the solution of partial differential equations by analytical method is better than the solution by approximate or series solution method. In this paper, we discuss the solution of linear and non-linear fractional partial differential equations involving derivatives with respect to time or space variables by converting them into the partial differential equations of integer order. Also we develop an analytical formulation to solve such fractional partial differential equations. Moreover, we discuss the method to solve the fractional partial differential equations in space as well as time variables simultaneously with the help of some examples.

Mathematics Subject Classification: *26A33, 35R11, 34A08.*

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1. Introduction

The theory of fractional derivation has known great importance in mathematical research from last few decades. Fractional differential quations do not have any known method to get exact solution but there are methods which gives the approximate and numerical solutions of fractional order differential equations. Fractional derivative [7, 12, 13, 14, 15] was discovered in a discussion between L. Hospital and Leibniz through a letter. Many mathematicians like Hadmard, Erdelyi-Kobe, Fourier, Euler, Mittag-Leffler, Laplace, Riemann, Grunewald etc. tried to develop the definition of fractional derivative.The definition of fractional derivative don't have a standard form. But the most used definitions of fractional derivation are Caputo definition [15] and Riemann Liouville [15]. Here, these fractional derivatives do not provide some properties of algebra of derivative and Mean Value Theorems. To overcome these difficulties, R. Khalil, M. Al Horani, A. Yusuf and M. Sababhed. [9], came up with an idea that extends the limit definition of the derivative. He derived some results of fractional derivative by using his new definition of fractional derivative. R. Almeida, M. Guzowska and T. Odzijewicz [2] introduced different definition of the fractional derivative. He also discussed some important results by using his definition of fractional derivative. Katugampola [6], introduced the idea of fractional derivative by using his new definition.
Recently a new definition of conformable fractional derivative has been introduced by R. Khalil, M. Abu-Hammad [8]. As this definition satisfies some classical and fundamental properties [See [4, 11]] of a fractional derivative and hence many new researchers are taking deep interest to develop the theory of fractional derivative by using this definition of conformable fractional derivative. Many applications and properties of conformable derivative and generalized form of conformable fractional derivative can be found in [1, 3, 9, 10, 6]. The work on fractional order partial differential equations has been already done as series solution or Mittag-Leffler function but not analytically. In this paper, we try to present the solution of fractional order partial differential equations in the form of boundary value problems in which fractional derivative is involved in both time as well as space variables.

2. α-Fractional Derivative and Integral

In [16], we introduced the concept of α-fractional derivative and integral by doing some appropriate modification in the classical definition of an ordinary derivative, which is defined as follows.

Definition 2.1. **α-Fractional Derivative [16]** *Let Φ: $[0,\infty) \to \mathbf{R}$ and $\zeta > 0$ then α-fractional derivative of order α is given by*

$$T_\alpha(\Phi(\zeta)) = \lim_{\mu \to 0} \frac{\Phi(\zeta e^{\mu \zeta^{1-\alpha}}) - \Phi(\zeta)}{\mu},$$

for all $\zeta > 0$, $\alpha \in (0,1]$.

Definition 2.2. **α-Fractional Integral [16]** *Let $0 \leq a \leq \zeta$ and Φ be a function defined on $(a,\zeta]$, then the α-fractional integral is defined as*

$$I_\alpha^a \Phi(\zeta) = \int \frac{\Phi(\zeta)}{\zeta^{2-\alpha}} d\zeta,$$

provided the integral exists. It is interesting to note that, for $\alpha = 1$ the definition coincides with the classical definition of first order derivative.

Following properties can be obtained directly using basic definition of α-fractional derivative.

Theorem 2.3. *[16] If $\alpha \in (0,1]$ and Φ, Ψ are α-differentiable at a point $r > 0$, then*

1. $T_\alpha(a\Phi + b\Psi) = aT_\alpha\Phi + bT_\alpha\Psi$, *for all $a, b \in \mathbf{R}$.*
2. $T_\alpha(r^p) = pr^{p+1-\alpha}$, *for all $p \in \mathbf{R}^+$.*
3. $T_\alpha(\Phi\Psi) = \Phi T_\alpha(\Psi) + \Psi T_\alpha(\Phi)$.
4. $T_\alpha\left(\frac{\Phi}{\Psi}\right) = \left(\frac{\Psi(r)T_\alpha\Phi(r) - \Phi(r)T_\alpha\Psi(r)}{\Psi(r)^2}\right)$.
5. $T_\alpha(\lambda) = 0$, *for all constant functions $\Phi(r) = \lambda$.*

Theorem 2.4. *[16] If Φ is a α-differentiable function at point $\zeta > 0$, then*

$$T_\alpha\Phi(\zeta) = \zeta^{2-\alpha}\frac{d\Phi(\zeta)}{d\zeta}.$$

Proof: Let Φ be the α-differentiable function at $\zeta > 0$ then, we have

$$\begin{aligned} T_\alpha\Phi(\zeta) &= \lim_{\mu\to 0}\left(\frac{\Phi\left(\zeta e^{\mu\zeta^{1-\alpha}}\right)-\Phi(\zeta)}{\mu}\right) \\ \Rightarrow T_\alpha\Phi(\zeta) &= \lim_{\mu\to 0}\left(\frac{\Phi\left(\zeta e^{\mu\zeta^{1-\alpha}}-\zeta+\zeta\right)-\Phi(\zeta)}{\mu}\right). \end{aligned}$$

If $\zeta e^{\mu\zeta^{1-\alpha}} - \zeta = k$ then $\mu = \frac{log\left(\frac{k+\zeta}{\zeta}\right)}{\zeta^{1-\alpha}}$ and $T_\alpha\Phi(\zeta) = \zeta^{2-\alpha}\frac{d\Phi(\zeta)}{d\zeta}$.

3. Linear α-Fractional Partial Differential Equations

In this section, we use the method of separation of variables to solve the partial differential equations of fractional order.

3.1. Linear Fractional Wave Equation of Finite Length

Consider the fractional wave equation

$$\frac{\partial^{1+\alpha}\vartheta}{\partial\xi^{1+\alpha}} - c^2\frac{\partial^{1+\alpha}\vartheta}{\partial\nu^{1+\alpha}} = 0, \tag{3.1}$$

subject to the conditions

$$\begin{aligned} \vartheta(\nu,0) = \Phi(\nu), &\quad 0<\nu\le l, \\ \vartheta_\xi(\nu,0) = \Psi(\nu), &\quad 0<\nu\le land \\ \vartheta(0,\xi) = \vartheta(l,\xi) &= 0, 0<\xi. \end{aligned}$$

If $\vartheta = \chi(\nu)\gamma(\xi)$, then by using the method of separation of variables, above equation (3.1) can be written as

$$\gamma^{(1+\alpha)}\chi = c^2\chi^{(1+\alpha)}\gamma,$$

where

$$\chi^{(1+\alpha)} = \frac{\partial^{1+\alpha}\chi}{\partial\nu^{1+\alpha}}, \gamma^{(1+\alpha)} = \frac{\partial^{1+\alpha}\gamma}{\partial\xi^{1+\alpha}}.$$

On choosing a constant k, such that

$$\frac{\chi^{(1+\alpha)}}{\chi} = \frac{\gamma^{(1+\alpha)}}{c^2\gamma} = k,$$

we obtain the following differential equations

$$\chi^{(1+\alpha)} - k\chi = 0 \tag{3.2}$$

$$\gamma^{(1+\alpha)} - c^2 k\gamma = 0. \tag{3.3}$$

Using Theorem [2.4], in equation (3.2), we have

$$\nu^{1-\alpha}\frac{d\chi}{d\nu} = k\chi$$

$$\Rightarrow \chi = Ae^{(k\nu^{\alpha})/\alpha}. \tag{3.4}$$

Similarly, from equation(3.3), we obtain

$$\gamma = Be^{c^2(k\xi^{\alpha})/\alpha}. \tag{3.5}$$

By equations (3.4) and (3.5), we have

$$\vartheta = \chi\gamma = Ae^{(k\nu^{\alpha})/\alpha}Be^{c^2(k\xi^{\alpha})/\alpha}.$$

Applying given boundary condition, we have

$$\vartheta(0,\xi) = \chi(0)\gamma(\xi) = 0, \forall\xi > 0.$$

Since $\gamma(\xi) \neq 0$, hence we get $\chi(0) = 0$. Similarly as $\vartheta(l,\xi) = 0$, then we have $\chi(l) = 0$. After solving this equation we can find eigenvalues of given problem.

Case-i. If $k = \mu^2$, then a solution of the above problem is given by

$$\chi(\nu) = Ae^{(-\mu\nu^{\alpha})/\alpha} + Be^{(\mu\nu^{\alpha})/\alpha}, \text{ where } A \text{ and } B \text{ are arbitrary constants.} \tag{3.6}$$

By using the given boundary conditions, we have

$$A + B = Ae^{-\alpha\mu l^{\alpha})/\alpha} + Be^{\alpha\mu l^{\alpha})/\alpha}.$$

This is only possible, if $A = B = 0$. Hence there is no eigenvalues in this case.

Case-ii. If $k = 0$, the solution of the problem takes the form

$$\chi(\nu) = A + B\nu.$$

The boundary conditions implies that $A = 0$ and $A + Bl = 0$. Therefore $A = B = 0$. Hence there is no eigenvalues in this case also.

Case-iii. If $k = -\mu^2$, then the solution of the problem is of the form

$$\chi(\nu) = \left(A\cos(\mu\nu^\alpha/\alpha) + B\sin(\mu\nu^\alpha/\alpha)\right).$$

The conditions $\chi(0) = 0 \Rightarrow A = 0$ and $\chi(l) = 0 \Rightarrow B\sin(\mu l^\alpha/\alpha) = 0$. As $B = 0$ gives only a trivial solution and hence we must have, $\sin(\mu l^\alpha/\alpha) = 0$, for a non trivial solution. Therefore $(\mu l^\alpha/\alpha) = n\pi, \quad n = 1, 2, ...$, and hence $\mu_n = \frac{\alpha n\pi}{l^\alpha}$. These are eigenvalues and the corresponding eigen functions are $\sin n\pi$. Therefore $\chi_n = B_n \sin n\pi, \quad n = 1, 2, ...$
For each μ_n, we have

$$\gamma_n(\xi) = C_n\cos(n\pi c\xi) + D_n\sin(n\pi c\xi),$$

where C_n and D_n are arbitrary constants. Hence

$$\vartheta_n(\nu, \xi) = (a_n\cos(n\pi c\xi) + b_n\sin(n\pi c\xi))\sin n\pi, \quad n = 1, 2, ...$$

is a solution of given fractional wave equation of finite length.

3.2. Linear Fractional Boundary Value Problem of Finite Length (Type-I)

Consider the fractional boundary value problem

$$\frac{\partial^2\vartheta}{\partial\nu^2} = \frac{\partial^{3/2}\vartheta}{\partial\xi^{3/2}}, \tag{3.7}$$

with the boundary conditions
$\vartheta(0, \xi) = 0, \quad \vartheta(\pi, \xi) = 0$ *and* $\vartheta(\nu, 0) = \zeta\sin 4\nu$.

If $\vartheta = \chi(\nu)\gamma(\xi)$, then by using the method of separation of variables, above equation (3.7) can be written as

$$\chi''\gamma - \chi\gamma^{(3/2)} = 0.$$

By choosing a constant $k \in R$ such that $\frac{\chi^{''}}{\chi} = \frac{\gamma^{(3/2)}}{\gamma} = k$, we obtain differential equations

$$\chi^{''} - k\chi = 0 \tag{3.8}$$

$$\gamma^{(3/2)} - k\gamma = 0. \tag{3.9}$$

Using Theorem [2.4] in equation (3.9), we have

$$\begin{aligned} & \xi^{2-3/2}\, \frac{d\gamma}{d\xi} = k\gamma \\ \Rightarrow \quad & \frac{d\gamma}{\gamma} = \frac{kd\xi}{\xi^{1/2}} \\ \Rightarrow \quad & \gamma = Ce^{2k\sqrt{\xi}} \\ \Rightarrow \quad & \vartheta = \chi\gamma = \left(Ae^{\nu\sqrt{k}} + Be^{-\nu\sqrt{k}}\right) Ce^{2k\sqrt{\xi}} \end{aligned}$$

$$\Rightarrow \vartheta = \left(\Psi_1 e^{\nu\sqrt{k}} + \Psi_2 e^{-\nu\sqrt{k}}\right) e^{2k\sqrt{\xi}}, \tag{3.10}$$

where $\Psi_1 = AC, \Psi_2 = BC$ are some constants.

For different values of k, we get different solutions as follows:

Case-i. If $k = 0$, then $\vartheta = (\Psi_1 + \Psi_2) = constant$.

Case-ii. If $k = \mu^2$, then $\vartheta = (\Psi_1 e^{\nu\mu} + \Psi_2 e^{-\nu\mu})e^2k\sqrt{\xi}$.

Case-iii. If $k = -\mu^2$, then $\vartheta = (\Psi_1 \sin\mu\nu + \Psi_2 \cos\mu\nu)e^-2\mu^2\sqrt{\xi}$.
From the Case-iii, we get most consistent and bounded solution of given problem. By using boundary condition $\vartheta(0, \xi) = 0$, then equation (3.10) becomes

$$0 = \Psi_2 e^{-2\mu^2\sqrt{\xi}} \Rightarrow 0 = \Psi_2$$

and

$$\vartheta(\pi, \xi) = 0 = (\Psi_1 \sin\mu\nu)e^{-2\mu^2\sqrt{\xi}}$$

$$\Rightarrow 0 = (\Psi_1 \sin\mu\pi)e^{-2\mu^2\sqrt{\xi}} \text{ and hence } \mu = n.$$

Thus, we have

$$\vartheta(\nu, \xi) = (\Psi_1 \sin n\pi)e^{-2n^2\sqrt{\xi}}.$$

Now, as $\vartheta(\nu, 0) = \zeta \sin 4\nu = (\Psi_1 \sin n\pi)$, we have $\Psi_1 = \zeta, n = 4$.

Hence

$$\vartheta(\nu, \xi) = \zeta \sin 4\nu e^{-32\sqrt{\xi}}$$

is a solution of given linear fractional boundary value problem of finite length.

3.3. Linear Fractional Boundary Value Problem of Finite Length (Type-II)

Consider the fractional boundary value problem

$$\frac{\partial^\alpha \vartheta}{\partial \nu^\alpha} - 3\vartheta = 5\frac{\partial^\beta \vartheta}{\partial \xi^\beta}, \alpha \geq 1, \beta \geq 1 \tag{3.11}$$

subject to the condition

$$\vartheta(\nu, 0) = \sigma e^{\frac{5(\nu^{\alpha-1})}{\alpha-1}}.$$

If $\vartheta = \chi(\nu)\gamma(\xi)$, then given equation becomes

$$\chi^{(\alpha)}\gamma - 3\chi\gamma = 5\chi\gamma^{(\beta)}.$$

Using method of separation of variables, we get

$$\frac{\chi^{(\alpha)}}{\chi} - 3 = 5\frac{\gamma^{(\beta)}}{\gamma} = k,$$

where $k$is a constant.

Consider the fractional differential equation

$$\frac{\chi^{(\alpha)}}{\chi} - 3 = k \text{ i.e. } \chi^{(\alpha)} = (k+3)\chi. \tag{3.12}$$

Using Theorem 2.4 in equation (3.12), we have

$$\nu^{2-\alpha}\frac{d\chi}{d\nu} = (k+3)\chi$$

$$\Rightarrow \chi = Ae^{(k+3)\frac{\nu^{\alpha-1}}{\alpha-1}}. \tag{3.13}$$

Similarly,

$$5\frac{\gamma^{(\beta)}}{\gamma} = k \Rightarrow 5\xi^{2-\beta}d\gamma/d\xi = k\gamma$$

$$\Rightarrow \frac{d\gamma}{\gamma} = k/5 \frac{d\xi}{\xi^{2-\beta}}$$

$$\Rightarrow \gamma = Be^{k/5 \frac{\xi^{\beta-1}}{\beta-1}}. \tag{3.14}$$

Thus, from equations (3.13) and (3.14) we have

$$\vartheta = \chi(\nu)Y(\xi)$$

$$\Rightarrow \vartheta = Ae^{(k+3)\frac{\nu^{\alpha-1}}{\alpha-1}} Be^{k/5\frac{\xi^{\beta-1}}{\beta-1}}$$

$$\Rightarrow \vartheta = \varsigma e^{(k+3)\frac{\nu^{\alpha-1}}{\alpha-1}} e^{k/5\frac{\xi^{\beta-1}}{\beta-1}}$$

$$\Rightarrow \vartheta = \varsigma e^{(k+3)\frac{\nu^{\alpha-1}}{\alpha-1}+k/5\frac{\xi^{\beta-1}}{\beta-1}},$$

where $AB = \varsigma$.
As $\vartheta(\nu, 0) = \sigma e^{5\frac{\nu^{\alpha-1})}{\alpha-1}} \Rightarrow \varsigma = \sigma$ and $k = 2$. Hence $\vartheta = \varsigma e^{5\frac{\nu^{\alpha-1}}{\alpha-1}} e^{2/5\frac{\xi^{\beta-1}}{\beta-1}}$ is a solution of given linear fractional boundary value problem of finite length.

3.4. Linear Time Fractional Partial Differential Equation

Consider the time fractional partial differential equation

$$D_\xi^\alpha \vartheta = \nu^2 \vartheta_{\nu\nu} + \nu \vartheta_\nu \tag{3.15}$$

with the conditions $\vartheta(\nu, \xi) = \nu, \vartheta(0, \xi) = 0, \vartheta(\nu, 1) = e^t$ *and* ϑ *is bounded.*

If $\vartheta = \chi(\nu)\gamma(\xi)$, so that given equation becomes

$$\chi\gamma^{(\alpha)} = \nu^2\chi''\gamma + \nu\chi'\gamma$$

$$\Rightarrow \gamma^{(\alpha)}/\gamma = \nu^2\chi''/\chi + \nu\chi'/\chi. \tag{3.16}$$

Using Theorem [2.4] in equation (3.16) we have

$$\xi^{2-\alpha}\gamma'/\gamma = \nu^2\chi''/\chi + \nu\chi'/\chi = k,$$

for some constant$k$.
Now for

$$\xi^{2-\alpha}\gamma'/\gamma = k$$

$$i.e. \xi^{2-\alpha}\gamma'/\gamma = k \Rightarrow \xi^{2-\alpha}\frac{d\gamma}{d\xi} = \gamma k$$

$$\Rightarrow \gamma = Ce^{k\frac{\xi^{\alpha-1}}{\alpha-1}}.$$

Similarly, for

$$\nu^2\chi''/\chi + \nu\chi'/\chi = k,$$

we get Cauchy-Euler equation

$$\nu^2\chi'' + x\chi' - k\chi = 0. \tag{3.17}$$

Substituting $\nu = e^z$, then equation (3.17) becomes $(D_1^2 - k)\chi = 0$, where $D_1 \equiv d/dz$, and solution of this differential equation is given by

$$\chi = Ae^{-z\sqrt{k}} + Be^{z\sqrt{k}}$$

$$\Rightarrow \chi = A\nu^{-\sqrt{k}} + B\nu^{\sqrt{k}}$$

$$\Rightarrow \vartheta = (A\nu^{-\sqrt{k}} + B\nu^{\sqrt{k}})Ce^{k\frac{\xi^{\alpha-1}}{\alpha-1}}$$

$$\Rightarrow \vartheta = (\Psi_1\nu^{-\sqrt{k}} + \Psi_2\nu^{\sqrt{k}})e^{k\frac{\xi^{\alpha-1}}{\alpha-1}}.$$

Using given condition $\vartheta(\nu, 1) = e^{\xi}$, we obtain

$$\alpha = 1, k = 1 \& \Psi_1 + \Psi_2 = 1$$

$$\Rightarrow \nu = (\Psi_1\nu^{-1} + \Psi_2\nu)e^{\xi}.$$

Since

$$\vartheta(0, \xi) = 0 \Rightarrow \frac{\nu}{e^{\xi}} = (\Psi_1\nu^{-1} + \Psi_2\nu)$$

$$\Rightarrow \Psi_1 = 0 \& \Psi_2 = 1.$$

Thus $\vartheta(\nu, \xi) = \nu e^{\xi}$ is a required solution of linear time fractional partial differential equation.

4. Non-Linear α-Fractional Partial Differential Equations

4.1. Non-Linear Fractional Initial Value Partial Differential Equation (Type-I)

Consider the fractional initial value problem

$$\frac{\partial\vartheta}{\partial\nu} + \vartheta\frac{\partial^{\alpha}\vartheta}{\partial\xi^{\alpha}} = 0, \tag{4.1}$$

subject to the condition $\vartheta(\nu, 0) = l$.

If $\vartheta = \chi(\nu)\gamma(\xi)$, then by using method of separation of variables, then equation (4.1) becomes

$$\chi'\gamma + (\chi\gamma)\chi\gamma^{(\alpha)} = 0$$

$$\Rightarrow \frac{\chi'}{\chi^2} = -\gamma^{(\alpha)} = k,$$

where $k \in R.Now \frac{\chi'}{\chi^2} = k \Rightarrow \frac{d\chi}{d\nu} = \chi^2 k$

$$\Rightarrow \chi = \frac{-1}{k\nu + c}. \tag{4.2}$$

Also

$$-\gamma^{(\alpha)} = k \Rightarrow -\xi^{2-\alpha}\frac{d\gamma}{d\xi} = k$$

$$\Rightarrow \gamma = \frac{-k\xi^{\alpha-1}}{\alpha-1} + w,$$

where $w \in R.$(4.3)

By equations (4.2) and (4.3), we have

$$\vartheta(\nu,\xi) = \frac{-1}{k\nu + c}\left(\frac{-k\xi^{\alpha-1}}{\alpha-1} + w\right).$$

Using given condition $\vartheta(\nu, 0) = l$ we obtain $w = -l(k\nu + c)$. Hence the solution of given non-linear fractional initial value problem is

$$\vartheta(\nu,\xi) = \left(\frac{k\xi^{\alpha-1}}{(k\nu + c)(\alpha-1)} + l\right).$$

Example 4.1. *Consider the fractional initial value problem*

$$\frac{\partial\vartheta}{\partial\nu} + \vartheta\frac{\partial^{3/2}\vartheta}{\partial\xi^{3/2}} = 0, \tag{4.4}$$

subject to the condition $\vartheta(\nu, 0) = l$.

If $\vartheta = \chi(\nu)\gamma(\xi)$ then by using method of separation of variables, then equation (4.4) can be written as

$$\chi'\gamma + (\chi\gamma)\chi\gamma^{(3/2)} = 0$$

$$\Rightarrow \frac{\chi'}{\chi^2} = -\gamma^{(3/2)} = k,$$

where $k \in R, \Rightarrow \frac{\chi'}{\chi^2} = k$

$$\Rightarrow \chi = \frac{-1}{k\nu + c}. \tag{4.5}$$

Similarly,

$$-\gamma^{(3/2)} = k \Rightarrow -\xi^{\frac{1}{2}} \frac{d\gamma}{d\xi} = k$$

$$\Rightarrow \gamma = -2k\xi^{\frac{1}{2}} + w. \tag{4.6}$$

By equations (4.5) and (4.6), we have

$$\vartheta(\nu, \xi) = \frac{-1}{k\nu^{1/2} + c}(-2k\xi^{\frac{1}{2}} + w).$$

As $\vartheta(\nu, 0) = l$ then $w = -l(k\nu^{1/2} + c)$ and hence the solution of given non-linear fractional initial value problem is

$$\vartheta(\nu, \xi) = \left(\frac{2k\xi^{\frac{1}{2}}}{(k\nu^{1/2} + c)} + l \right).$$

4.2. Non-Linear Fractional Initial Value Partial Differential Equation (Type-II)

Consider the fractional initial value problem

$$\frac{\partial^\alpha \vartheta}{\partial \nu^\alpha} + \vartheta \frac{\partial^\beta \vartheta}{\partial \xi^\beta} = 0, \tag{4.7}$$

subject to the condition $\vartheta(0, \xi) = \nu$.

If $\vartheta = \chi(\nu)\gamma(\xi)$, then with the help of method of separation of variables we may write above equation (4.7) as

$$\chi^{(\alpha)}\gamma + (\chi\gamma)\chi\gamma^{(\beta)} = 0 \Rightarrow \frac{\chi^{(\alpha)}}{\chi^2} + \gamma^{(\beta)} = 0$$

$$\Rightarrow \frac{\chi^{(\alpha)}}{\chi^2} = -\gamma^{(\beta)} = k, \text{ where } k \in R.$$

On separating variables, we have

$$\frac{\chi^{(\alpha)}}{\chi^2} = k \Rightarrow \nu^{2-\alpha} \frac{d\chi}{\chi^2} = kd\nu$$

$$\Rightarrow \frac{-1}{\chi} = \frac{k\nu^{\alpha-1}}{\alpha-1} + c$$

$$\Rightarrow \chi = \frac{-1}{\frac{k\nu^{\alpha-1}}{\alpha-1} + c}. \tag{4.8}$$

Similarly, as $\gamma^{(\beta)} = -k$ then $\xi^{2-\beta}\frac{d\gamma}{d\xi} = -k$

$$\Rightarrow \gamma = -\frac{k\xi^{\beta-1}}{\beta-1} + A. \tag{4.9}$$

By equations (4.8) and (4.9), we have

$$\vartheta(\nu, \xi) = \frac{-1}{\frac{k\nu^{\alpha-1}}{\alpha-1} + c}\left(-\frac{k\xi^{\beta-1}}{\beta-1} + A\right).$$

Since $\vartheta(0, \xi) = \nu$, therefore $-\nu c = \left(-\frac{k\xi^{\beta-1}}{\beta-1} + A\right) \Rightarrow -c\nu + \frac{k\xi^{\beta-1}}{\beta-1} = A$. Hence the solution of given non-linear fractional initial value problem is

$$\vartheta(\nu, \xi) = \frac{c\nu}{k\frac{\nu^{\alpha-1}}{\alpha-1} + c}.$$

Example 4.2. *Consider the fractional initial value problem*

$$\frac{\partial^{3/2}\vartheta}{\partial\nu^{3/2}} + \vartheta\frac{\partial^{3/2}\vartheta}{\partial\xi^{3/2}} = 0, \tag{4.10}$$

subject to the condition $\vartheta(\nu, 0) = l$.

If $\vartheta = \chi(\nu)\gamma(\xi)$, then by using method of separation of variables, above equation can be written as

$$\chi^{(3/2)}\gamma + (\chi\gamma)\chi\gamma^{(3/2)} = 0$$

$$\Rightarrow \frac{\chi^{(3/2)}}{\chi^2} = -\gamma^{(3/2)} = k,$$

where $k \in R$.

On separating variables, we have

$$\frac{\chi^{(3/2)}}{\chi^2} = k \Rightarrow \nu^{1/2}\frac{d\chi}{d\nu} = \chi^2 k$$

$$\Rightarrow \chi = \frac{-1}{k\nu^{1/2} + c}. \tag{4.11}$$

Similarly,

$$-\gamma^{(3/2)} = k \Rightarrow -\xi^{\frac{1}{2}} \frac{d\gamma}{d\xi} = k$$

$$\Rightarrow \gamma = -2k\xi^{\frac{1}{2}} + w. \tag{4.12}$$

By equations (4.11) and (4.12), we have

$$\Rightarrow \vartheta(\nu, \xi) = \frac{-1}{k\nu^{1/2} + c}(-2k\xi^{\frac{1}{2}} + w).$$

As $\vartheta(\nu, 0) = l$ then $w = -l(k\nu^{1/2} + c)$ and hence the solution of given non-linear fractional initial value problem is

$$\vartheta(\nu, \xi) = \left(\frac{2k\xi^{\frac{1}{2}}}{(k\nu^{1/2} + c)} + l \right).$$

5. Conclusion

In this paper, we have discussed the solution of linear and non-linear fractional order partial differential equations involving derivatives with respect to time or space variables by converting it into integer order. We also discussed the analytical method to solve the simultaneous fractional derivative, in space and time variables.

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