Proyecciones (Antofagasta. On line) | vol. 40, n. 3 (2021) | pp. 681-696.

10.22199/issn.0717-6279-4386

PROYECCIONES
Journal of Mathematics

REVISTAS CIENTÍFICAS
de la Universidad Católica del Norte.
revistas.ucn.cl

ISSN 0717-6279 (On line)

Continuity in fuzzy bitopological ordered spaces

Runu Dhar[1] orcid.org/0000-0002-2445-3608

[1]Maharaja Bir Bikram Univesity, Dept. of Applied Mathematics, Agartala, TR, India.
runu.dhar@gmail.com

Received: 06 August 2020 | Accepted: 30 November 2020

Abstract:

The aim of the present paper is to introduce and study different forms of continuity in fuzzy bitopological ordered spaces. The concepts of different mappings such as pairwise fuzzy I -continuous mappings, pairwise fuzzy D -continuous mappings, pairwise fuzzy B -continuous mappings, pairwise fuzzy I -open mappings, pairwise fuzzy D -open mappings, pairwise fuzzy B -open mappings, pairwise fuzzy I -closed mappings, pairwise fuzzy D -closed mappings and pairwise fuzzy B -closed mappings have been introduced. Some of the basic properties and characterization theorems of these mappings have been investigated.

Keywords: Fuzzy topological ordered space; Fuzzy bitopological ordered space; Pairwise fuzzy x continuity (x = I, D, B).

MSC (2020): 54A40.

Cite this article as (IEEE citation style):

R. Dhar, "Continuity in fuzzy bitopological ordered spaces", *Proyecciones (Antofagasta, On line)*, vol. 40, no. 3, pp. 681-696, 2021, doi: 10.22199/issn.0717-6279-4386

1. Introduction and Preliminaries

The fundamental concept of fuzzy sets was first introduced by Zadeh [10] in 1965. In 1968, Chang [2] introduced the notion of fuzzy topological spaces as a generalization of topological spaces. Since then many topologist have contributed to the theory of fuzzy topological spaces. Kelly [5] first introduced the concept of bitopological spaces. The concept of fuzzy bitopological spaces was introduced by Kandil [4] in 1989 as a natural generalization of fuzzy topological spaces. After the introduction of the definition of a fuzzy bitopological space by Kandil, a large number of topologists have turned their attention to the generalization of different concepts of a single fuzzy topological spaces. Kumar [7, 8] introduced and studied different types mappings in fuzzy bitopological spaces. Also Dhar [3] introduced and studied some pairwise weakly fuzzy mappings. Singal and Singal [9] initiated the study of bitopological ordered spaces. Bakier and Sayed [1] introduced and studied the notion of continuity, openness and closedness in bitopological ordered spaces. The study of the relationship between fuzzy topology and order was initiated by Katsaras [6] in 1981. In this paper, pairwise fuzzy I-continuous mappings, pairwise fuzzy D-continuous mappings, pairwise fuzzy B-continuous mappings, pairwise fuzzy D-open mapping, pairwise fuzzy B-open mappings, pairwise fuzzy I-closed mappings, pairwise fuzzy D-closed mappings and pairwise fuzzy B-closed mappings for fuzzy bitopological ordered spaces have been introduced and investigated.

Definition 1.1. *[1] Let $(X, \leq)$ be a partially ordered set (i.e. a set X together with a reflexive, antisymmetric and transitive relation). For a subset $A \subseteq X$, we write: $L(A) = \{y \in X : y \leq x \text{ for some } x \in A\}$, $M(A) = \{y \in X : x \leq y \text{ for some } x \in A\}$. In particular, if A is a singleton set, say $\{x\}$, then we write $L(x)$ and $M(x)$ respectively. A subset A of X is said to be decreasing (resp. increasing) if $A = L(A)$ (resp. $A = M(A)$). The complement of a decreasing (respectively an increasing) set is an increasing (respectively a decreasing) set.*

Definition 1.2. *[6] Let X be a non-empty set. A preorder on X is a relation $\leq$ on X which is reflexive and transitive. A preorder on X which is also anti-symmetric is called a partial order or simply an order. By an ordered set we mean a set X with an order on it and we denote it by $(X, \leq)$. An ordered set on which there is given a fuzzy topology is called a fuzzy topological ordered space and we denote it by $(X, \tau, \leq)$.*

Definition 1.3. *[6] A fuzzy set μ in a pre- ordered set $(X, \leq)$, is called*
(i) increasing if $x \leq y\mu(x) \leq \mu(y)$,
(ii) decreasing if $x \leq y\mu(x) \geq \mu(y)$,
(iii) order-convex if $x \leq y \leq z\mu(y) \geq$ min $\{\mu(x), \mu(z)\}$.

The smallest increasing fuzzy set, in $(X, \leq)$, which contains μ is called the increasing hull denoted by $i(\mu)$. The decreasing (respectively order-convex) hull $d(\mu)$ (resp. $c(\mu)$) is defined analogously.

Proposition 1.4. *[6] Let μ be a fuzzy set in a preordered set $(X, \leq)$. Then, for each $x \in X$, we have*
(i) $i(\mu)(x) =$ sup $\{\mu(y) : y \leq x\}$;
(ii) $d(\mu)(x) =$ sup $\{\mu(y) : y \geq x\}$;
(iii) $c(\mu)(y) =$ min $\{\mu(x), \mu(z)\}$, $x \leq y \leq z$.

Definition 1.5. *[6] Let μ be a fuzzy set, in a preordered set $(X, \leq)$ with a fuzzy topology τ. Then we have*
(i) $D(\mu) =$ inf $\{\rho : \rho \geq \mu, \rho$ is closed and decreasing$\}$;
(ii) $I(\mu) =$ inf $\{\rho : \rho \geq \mu, \rho$ is closed and increasing $\}$.
Clearly, $D(\mu)$ (respectively $I(\mu)$) is the smallest closed decreasing (respectively increasing) fuzzy set in $(X, \geq)$ which contains μ.

2. Pairwise fuzzy I- continuous, pairwise fuzzy D-continuous and pairwise fuzzy B- continuous mappings

Definition 2.1. *A fuzzy bitopological ordered space $(X, \tau_1, \tau_2, \leq)$ is a fuzzy bitopological space (X, τ_1, τ_2) equipped with a partial order $\leq$ (that is, reflexive, transitive and antisymetric).*

Definition 2.2. *A fuzzy subset μ of $(X, \tau_1, \tau_2, \leq)$ is said to be increasing if $x \leq y\mu(x) \leq \mu(y)$, for $x, y \in X$.*

Definition 2.3. *A fuzzy subset μ of $(X, \tau_1, \tau_2, \leq)$ is said to be decreasing if $x \leq y\mu(x) \geq \mu(y)$, for $x, y \in X$.*

Definition 2.4. *For a fuzzy subset A of a fuzzy bitopological ordered space $(X, \tau_1, \tau_2, \leq)$,*
$H_i^l(A) = \wedge\{F : F$ is τ_i-decreasing fuzzy closed subset of X containing $A\}$,
$H_i^m(A) = \wedge\{F : F$ is τ_i-increasing fuzzy closed subset of X containing $A\}$,
$H_i^b(A) = \wedge\{F : F$ is a fuzzy closed subset of X containing A with $F = L(F) = M(F)\}$,

$O_i^l(A) = \vee\{G : G$ *is* τ_i*-decreasing fuzzy open subset of* X *contained in* $A\}$,
$O_i^m(A) = \vee\{G : G$ *is* τ_i*-increasing fuzzy open subset of* X *contained in* $A\}$,
$O_i^b(A) = \vee\{G : G$ *is both* τ_i*-increasing and* τ_i*-decreasing fuzzy open subset of* X *contained in* $A\}$.

Clearly, $H_i^m(A)$ (respectively $H_i^l(A)$, $H_i^b(A)$) is the smallest τ_i-increasing (respectively τ_i-decreasing, both τ_i-increasing and τ_i-decreasing) fuzzy closed set containing A. Moreover $\overline{A_i} \leq H_i^m(A) \leq H_i^b(A)$ and where $\overline{A_i}$ stands for the τ_i-closure of A in $(X, \tau_1, \tau_2, \leq), i = 1, 2$. Further A is τ_i-decreasing (respectively τ_i-increasing) fuzzy closed if and only if $A = H_i^m(A) = H_i^l(A)$. Clearly, $O_i^m(A)$ (respectively $O_i^l(A)$, $O_i^b(A)$) is the largest τ_i-increasing (respectively τ_i-decreasing, both τ_i-increasing and τ_i-decreasing) fuzzy open set contained in A. Moreover $O_i^b(A) \leq O_i^m(A) \leq A_i^o$ and $O_i^b(A) \leq O_i^l(A)$, where A_i^o denotes the τ_i-interior of A in $(X, \tau_1, \tau_2, \leq), i \neq j$. If A and B are two τ_1 fuzzy subsets of a fuzzy bitopological ordered space $(X, \tau_1, \tau_2, \leq), i \neq j$ such that $A \leq B$, then $O_i^m(A) \leq O_i^l(B) \leq B_i^o$. $\Omega(O_i^m(X))$ (respectively $\Omega(O_i^l(X))$, $\Omega(O_i^b(X))$) denotes the collection of all τ_i-increasing (respectively τ_i-decreasing, both τ_i-increasing and τ_i-decreasing) fuzzy open subsets of a fuzzy bitopological ordered space $(X, \tau_1, \tau_2, \leq)$.

Definition 2.5. *A mapping* $f : (X, \tau_1, \tau_2, \leq) \to (X^*, \tau_1^*, \tau_2^*, \leq^*)$ *from a fuzzy bitopological ordered space* $(X, \tau_1, \tau_2, \leq)$ *to another fuzzy bitopological ordered space* $(X^*, \tau_1^*, \tau_2^*, \leq^*)$ *is called a pairwise fuzzy* I*-continuous (respectively a pairwise fuzzy* D*-continuous, a pairwise fuzzy* B*-continuous) mapping if* $f^{-1}(G) \in \Omega(O_i^m(X))$ *(respectively* $f^{-1}(G) \in \Omega(O_i^l(X))$, $f^{-1}(G) \in \Omega(O_i^b(X))$*), whenever* G *is a* τ_i^**-fuzzy open subset of* $(X^*, \tau_1^*, \tau_2^*, \leq^*), i = 1, 2$.

It is evident that every pairwise fuzzy x-continuous mapping is pairwise fuzzy continuous for $x = I, D, B$ and that every pairwise fuzzy B-continuous mapping is both pairwise fuzzy I-continuous and pairwise fuzzy D- continuous.

Example 2.6. *Let* $X = \{a, b, c\}$ *and* λ *and* μ *are fuzzy sets defined as follows:*

$$\lambda(a) = 0.3, \qquad \lambda(b) = 0.4, \qquad \lambda(c) = 0.5$$

$$\mu(a) = 0.7, \qquad \mu(b) = 0.8, \qquad \mu(c) = 0.9.$$

Let $\tau_1 = \{0, \lambda, 1\}, \tau_2 = \{0, \mu, 1\}$ *and* $\leq = \{(a, a), (b, b), (c, c), (a, b), (b, c), (a, c)\}$. *Clearly* $(X, \tau_1, \tau_2, \leq)$ *is a fuzzy bitopological ordered space. Let* f *be the identity mapping from* $(X, \tau_1, \tau_2, \leq)$ *onto itself.* λ *is* τ_1*- fuzzy open and* μ *is* τ_2*-fuzzy open.*

But$f^{-1}(\lambda)(a) = \lambda(f(a)) = \lambda(a) = 0.3$ *which is neither a* τ_1*- increasing nor a* τ_1*- decreasing fuzzy open set.*

Also $f^{-1}(\mu)(a) = \mu(f(a)) = \mu(a) = 0.7$ *which is neither a* τ_2*- increasing nor a* τ_2*-decreasing fuzzy open set. Thus f is not pairwise fuzzy x continuous for x = I, D, B. However f is fuzzy continuous.*

Example 2.7. *Let* $X = \{a, b, c\} = X^*$ *and* λ *and* μ *are fuzzy sets defined as follows:*

$$\lambda(a) = 0.3, \qquad \lambda(b) = 0.4, \qquad \lambda(c) = 0.5$$

$$\mu(a) = 0.7, \qquad \mu(b) = 0.8, \qquad \mu(c) = 0.9$$

Let $\tau_1 = \{0, \lambda, 1\} = \tau_1^*, \tau_2 = \{0, \mu, 1\} = \tau_2^*$ *and* $\leq = \{(a, a), (b, b), (c, c), (a, c)\}$. $\leq^* = \{(a, a), (b, b), (c, c), (a, b), (b, c), (a, c)\}$. *Clearly* $(X, \tau_1, \tau_2, \leq)$ *and* $(X^*, \tau_1^*, \tau_2^*, \leq^*)$ *are fuzzy bitopological ordered spaces. Let g be the identity mapping from* $(X, \tau_1, \tau_2, \leq)$ *onto* $(X^*, \tau_1^*, \tau_2^*, \leq^*)$. *g is not pairwise fuzzy B-continuous. However g is a pairwise fuzzy D-continuous mapping. The following example supports that a pairwise fuzzy I-continuous mapping need not be a pairwise fuzzy B-continuous mapping.*

Example 2.8. *Let* $X = \{a, b, c\} = X^*$. *Let* $\tau_1 = \{0, (0.3, 0.4, 0.5), 1\}, \tau_1^* = \{0, (0, 0,\ 0.5), 1\},\ \tau_2 = \{0, (0.7, 0.8, 0.9), 1\}, \tau_2^* = \{0, (0, 0.2, 0.9), 1\}$ *and* $\leq = \{(a, a), (b, b), (c, c),\ (a, b), (b, c), (a, c)\} = \leq^*$. *Clearly* $(X, \tau_1, \tau_2, \leq)$ *and* $(X^*, \tau_1^*, {}_2^*, \leq^*)$ *are fuzzy bitopological ordered spaces. Define* $h : (X, \tau_1, \tau_2, \leq) \to (X^*, \tau_1^*, {}_2^*, \leq^*)$ *by* $h(a) = b, h(b) = a$ *and* $h(c) = c$. *h is pairwise fuzzy I-continuous but not a pairwise fuzzy B-continuous mapping.*

Thus we have the following diagram:
For a mapping $f : (X, \tau_1, \tau_2, \leq) \to (X^*, \tau_1^*, \tau_2^*, \leq^*)$, where $P \to Q$ (respectively $P \not\leftrightarrow Q$) represents P implies Q but Q need not imply P (respectively P and Q are independent of each other).

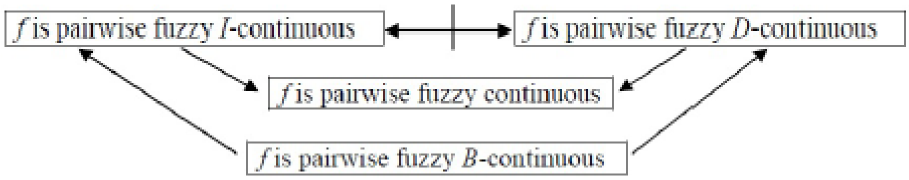

FIGURE 1

The following Theorem characterizes pairwise fuzzy I-continuous mapping.

Theorem 2.9. *For a mapping $f : (X, \tau_1, \tau_2, \le) \to (X^*, \tau_1^*, \tau_2^*, \le)$, the following statements are equivalent :*
(1) f is pairwise fuzzy I-continuous.
(2) $f(H_i^l(A)) \le \overline{(f(A)_i)}$ for any fuzzy subset $A \le X$, $i = 1, 2$.
(3) $H_i^l(f^{-1}(B)) \le f^{-1}\overline{(B)_i}$ for any fuzzy subset $B \le X^*$, $i = 1, 2$.
(4) For every τ_i^* fuzzy closed subset K of $(X^*, \tau_1^*, \tau_2^*, \le)$, $f^{-1}(K)$ is a τ_i decreasing fuzzy closed subset of $(X, \tau_1, \tau_2, \le)$, $i = 1, 2$.

Proof. (1) (2). Since $X^* \setminus \overline{(f(A))_i}$ is τ_i-fuzzy open in X^* and f is pairwise fuzzy I-continuous, then $f^{-1}(X \setminus \overline{(f(A))_i})$ is a τ_i- increasing fuzzy open set in X. Then $X \setminus f^{-1}(X \setminus \overline{((f(A))_i})$ is a τ_i-decreasing fuzzy closed subset of X. Since $X \setminus f^{-1}(X \setminus \overline{(f(A))_i}) = f^{-1}\overline{(f(A))_i}$, so $f^{-1}\overline{(f(A))_i}$ is a τ_i-decreasing fuzzy closed subset of X. Since $A \le f^{-1}\overline{(f(A))_i}$ and is the smallest τ_i-decreasing fuzzy closed set containing A, then $H_i^l(A) \le f^{-1}\overline{(f(A))_i}$, $f(H_i^l(A)) \le f(f^{-1}\overline{(f(A))_i}) \le \overline{(f(A))_i}$. Thus $f(H_i^l(A)) \le \overline{(f(A))_i}$.

(2) $\Rightarrow$ (3). Let $A = f^{-1}(B)$. Then $f(A) = f(f^{-1}(B)) \le B$. This implies $\overline{(f(A))_i} \le \overline{(B)_i}$. Now $H_i^l(f^{-1}(B)) \le H_i^l(A) \le f^{-1}(f(H_i^l(A))) \le f^{-1}\overline{(f(A))_i}$ [by (2)]. But $f^{-1}\overline{(f(A))_i} \le f^{-1}\overline{(B)_i}$. Thus $H_i^l(f^{-1}(B)) \le f^{-1}\overline{(B)_i}$.

(3)$\Rightarrow$ (4). $H_i^l(f^{-1}(K)) \le f^{-1}\overline{(K)_i}$ for any τ_i^*-fuzzy closed set K of $(X^*, \tau_1^*, \tau_2^*, \le)$. Thus $f^{-1}(K)$ is a τ_i-decreasing fuzzy closed set in $(X, \tau_1, \tau_2, \le)$ whenever K is a τ_i^*-fuzzy closed set in $(X^*, \tau_1^*, \tau_2^*, \le)$.

(4)⇒(1). Let G be a τ_i^*-fuzzy open set in $(X^*, \tau_1^*, \tau_2^*, \leq)$. Then $f^{-1}(X \setminus (G))$ is a τ_i-decreasing fuzzy closed set in $(X, \tau_1, \tau_2, \leq)$, since $X^* \setminus (G)$ is a fuzzy closed set in $(X^*, \tau_1^*, \tau_2^*, \leq)$. But $X \setminus (f^{-1}(G)) = f^{-1}(X \setminus (G))$. Thus $X \setminus (f^{-1}(G))$ is a τ_i-decreasing fuzzy closed set in $(X, \tau_1, \tau_2, \leq)$. So $f^{-1}(G)$ is a τ_i-increasing fuzzy open set in $(X, \tau_1, \tau_2, \leq)$. Thus f is pairwise fuzzy I-continuous. □

The following two theorems characterize pairwise fuzzy D-continuous mapping and pairwise fuzzy B-continuous mapping, whose proofs are similar to as that of the above Theorem 2.9.

Theorem 2.10. *For a mapping $f : (X, \tau_1, \tau_2, \leq) \to (X^*, \tau_1^*, \tau_2^*, \leq^*)$, the following statements are equivalent :*
(1) f is pairwise fuzzy D-continuous.
(2) $f(H_i^m(A)) \leq \overline{(f(A))}_i$ for any fuzzy subset $A \leq X$, $i = 1, 2$.
(3) $H_i^m(f^{-1}(B)) \leq f^{-1}\overline{(B)}_i$ for any fuzzy subset $B \leq X^*$, $i = 1, 2$.
(4) For every τ_i^*-fuzzy closed subset K of $(X^*, \tau_1^*, \tau_2^*, \leq)$, $f^{-1}(K)$ is a τ_i-increasing fuzzy closed subset of $(X, \tau_1, \tau_2, \leq)$, $i = 1, 2$.

Theorem 2.11. *For a mapping $f : (X, \tau_1, \tau_2, \leq) \to (X^*, \tau_1^*, \tau_2^*, \leq)$, the following statements are equivalent :*
(1) f is pairwise fuzzy B-continuous.
(2) $f(H_i^b(A)) \leq \overline{(f(A))}_i$ for any $A \leq X$, $i = 1, 2$.
(3) $H_i^b(f^{-1}(B)) \leq f^{-1}\overline{(B)}_i$ for any $B \leq X^*$, $i = 1, 2$.
(4) For every τ_i^*-fuzzy closed subset K of $(X^*, \tau_1^*, \tau_2^*, \leq)$, $f^{-1}(K)$ is both τ_i-increasing and τ_i-decreasing fuzzy closed subset of $(X, \tau_1, \tau_2, \leq)$, $i = 1, 2$.

Theorem 2.12. *Let $f : (X, \tau_1, \tau_2, \leq_1) \to (Y, \nu_1, \nu_2, \leq_2)$ and $g : (Y, \nu_1, \nu_2, \leq_2) \to (Z, \eta_1, \eta_2, \leq_3)$ be any two mappings. Then*
(1) $g \circ f : (X, \tau_1, \tau_2, \leq_1) \to (Z, \eta_1, \eta_2, \leq_3)$ is pairwise fuzzy x-continuous for $x = I, D, B$.
(2) $g \circ f : (X, \tau_1, \tau_2, \leq_1) \to (Z, \eta_1, \eta_2, \leq_3)$ is pairwise fuzzy x-continuous and g is pairwise fuzzy continuous for $x = I, D, B$.
(3) $g \circ f : (X, \tau_1, \tau_2, \leq_1) \to (Z, \eta_1, \eta_2, \leq_3)$ is pairwise fuzzy x-continuous and g is pairwise fuzzy y-continuous for $x, y \in \{I, D, B\}$.

3. Pairwise fuzzy I-open, pairwise fuzzy D-open and pairwise fuzzy B-open mappings

Definition 3.1. *A mapping $f : (X, \tau_1, \tau_2, \leq) \to (X^*, \tau_1^*, \tau_2^*, \leq^*)$ from a fuzzy bitopological ordered space $(X, \tau_1, \tau_2, \leq)$ to another fuzzy bitopo-*

logical ordered space $(X, \tau_1^*, \tau_2^*, \leq^*)$ *is called a pairwise fuzzy* I*-open (respectively a pairwise fuzzy* D*-open, a pairwise fuzzy* B*-open) mapping if* $f(G) \in \Omega(O_i^m(X^*))$ *(respectively* $f(G) \in \Omega(O_i^l(X^*))$, $f(G) \in \Omega(O_i^b(X^*)))$, *whenever* G *is a* τ_i*-fuzzy open subset of* $(X, \tau_1, \tau_2, \leq)$, $i = 1, 2$.

It is evident that every pairwise fuzzy x-open mapping is a pairwise fuzzy open mapping for $x = I, D, B$ and that every pairwise fuzzy B-open mapping is both pairwise fuzzy I-open and pairwise fuzzy D-open.
The following example shows that a pairwise fuzzy D-open mapping need not be a pairwise fuzzy B-open mapping.

Example 3.2. *Let* $(X, \tau_1, \tau_2, \leq)$ *be a fuzzy bitopological ordered space and* f *be a mapping as in Example 2.6. Then* f *is a pairwise fuzzy open mapping but* f *is not pairwise fuzzy* x*-open for* $x = I, D, B$.

Example 3.3. *Let* $X, X^*, \tau_1, \tau_2, \tau_1^*, \tau_2^*, \leq$ *and* $\leq^*$ *be as in Example 2.7. Let* θ *be the identity mapping from* $(X, \tau_1, \tau_2, \leq)$ *onto* $(X^*, \tau_1^*, \tau_2^*, \leq^*)$. θ *is pairwise fuzzy* D*-open but not a pairwise fuzzy* B*-open mapping.*

The following example shows that a pairwise fuzzy I-open mapping need not be a pairwise fuzzy B-open mapping.

Example 3.4. *Let* $X, X^*, \tau_1, \tau_2, \tau_1^*, \tau_2^*, \leq$ *and* $\leq^*$ *be as in Example 2.8. Let* $\pi : (X, \tau_1, \tau_2, \leq) \to (X^*, \tau_1^*, \tau_2^*, \leq^*)$ *by* $\pi(a) = b, \pi(b) = a$ *and* $\pi(c) = c$. π *is a pairwise fuzzy* I*-open mapping but not a pairwise fuzzy* B*-open mapping.*

Thus we have the following diagram:

For a mapping $f : (X, \tau_1, \tau_2, \leq) \to (X^*, \tau_1^*, \tau_2^*, \leq^*)$, where $P \to Q$ (respectively $P \not\leftrightarrow Q$) represents P implies Q but Q need not imply P (respectively P and Q are independent of each other).

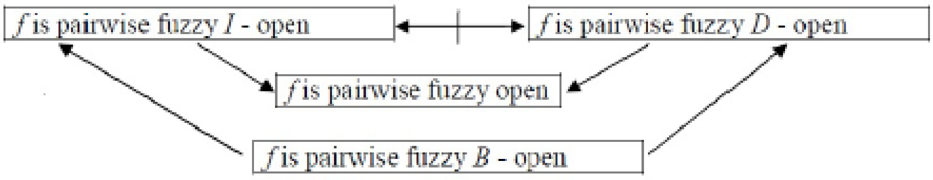

FIGURE 2

Before characterizing pairwise fuzzy I-open (respectively pairwise fuzzy D-open, pairwise fuzzy B-open) mapping, we establish the following useful Lemma.

Lemma 3.5. *Let A be any fuzzy subset of a fuzzy bitopological ordered space $(X, \tau_1, \tau_2, \leq)$. Then*
(1) $X \setminus H_i^l(A) = O_i^m(X \setminus A), i = 1, 2$.
(2) $X \setminus H_i^m(A) = O_i^l(X \setminus A), i = 1, 2$.
(3) $X \setminus H_i^b(A) = O_i^b(X \setminus A), i = 1, 2$.

Proof. (1) $X \setminus H_i^l(A) = X \setminus (\wedge\{F : F$ is a τ_i-decreasing fuzzy closed subset of X containing $A\}$
$= \vee(X \setminus F : F$ is a τ_i-decreasing fuzzy closed subset of X containing $A\}$
$= \vee(G : G$ is a τ_i- increasing fuzzy open subset of X contained in $X \setminus A\}$
$= O_i^m(X \setminus A)$.
The proofs for (2) and (3) are analogous to that of (1) and so omitted. □

The following Theorem characterizes pairwise fuzzy I-open mappings.

Theorem 3.6. *For any mapping $f : (X, \tau_1, \tau_2, \leq) \to (X^*, \tau_1^*, \tau_2^*, \leq^*)$, the following statements are equivalent :*
(1) f is pairwise fuzzy I-open mapping.
(2) $f(A_i^o)) \leq O_i^m(f(A))$ for any fuzzy subset $A \leq X, i = 1, 2$.
(3) $(f^{-1}(B))_i^o \leq f^{-1}(O_i^m(B))$ for any fuzzy subset $B \leq X^*, i = 1, 2$.
(4) $f^{-1}(H_i^l(B)) \leq H_i^l(f^{-1}(B))$ for any fuzzy subset $B \leq X^*, i = 1, 2$.

Proof. (1)(3) . Since $(f^{-1}(B))_i^o$ is τ_i fuzzy open in X and f is pairwise fuzzy I- open, then $f((f^{-1}(B))_i^o)$ is a τ_i-increasing fuzzy open set in X^*. Also $f((f^{-1}(B))_i^o) \leq f((f^{-1}(B)) \leq B$. Then $f((f^{-1}(B))_i^o \leq O_i^m(B)$ since $O_i^m(B)$ is the largest τ_i-increasing fuzzy open set contained in B. Therefore

$(f^{-1}(B))_i^o \leq f^{-1}(O_i^m(B))$.
(3)(4). Replacing B by $X \setminus B$ in (3), we get, $(f^{-1}(X \setminus B))_i^o \leq f^{-1}(O_i^m(X \setminus B))$. Since $f^{-1}(X \setminus B) = X \setminus (f^{-1}(B))$, then $(X \setminus (f^{-1}(B)))_i^o \leq f^{-1}(O_i^m(X \setminus B))$. Now $X \setminus (H_i^l(f^{-1}(B))) = O_i^m(X \setminus f^{-1}(B)) \leq (X \setminus (f^{-1}(B)))_i^o \leq f^{-1}(O_i^m(X \setminus B)) = f^{-1}(X \setminus (H_i^l(B))) = X \setminus (f^{-1}(H_i^l(B)))$ using the above Lemma 3.5. Therefore $f^{-1}(H_i^l(B)) \leq H_i^l(f^{-1}(B))$.
(4)(3). All the steps in (3)(4) are reversible.
(3)(2). Replacing B by $f(A)$ in (3), we get $(f^{-1}(f(A)))_i^o \leq f^{-1}(O_i^m(f(A)))$. Since $A_i^o \leq (f^{-1}(f(A)))_i^o$, then we have $A_i^o \leq f^{-1}(O_i^m(f(A)))$. This implies that $f(A_i^o) \leq f(f^{-1}(O_i^m(f(A)))) \leq O_i^m(f(A))$. Hence $f(A_i^o)) \leq O_i^m(f(A))$.
(2)(1) Let G be any τ_i fuzzy open subset of X. Then $f(G) = f(G_i^o) \leq O_i^m(f(G))$. So $f(G)$ is a τ_i^* increasing fuzzy open set in X^*. Therefore f is a pairwise fuzzy I- open mapping. □

The following two Theorems give characterizations for pairwise fuzzy D-open mapping and pairwise fuzzy B-open mapping, whose proofs are similar to as that of the above Theorem 3.6.

Theorem 3.7. *For any mapping $f : (X, \tau_1, \tau_2, \leq) \to (X^*, \tau_1^*, \tau_2^*, \leq^*)$, the following statements are equivalent :*
(1) f is pairwise fuzzy D-open.
(2) $f((A_i^o)) \leq O_i^l(f(A))$ for any fuzzy subset $A \leq X$, $i = 1, 2$.
(3) $(f^{-1}(B))_i^o \leq f^{-1}(O_i^l(B))$ for any fuzzy subset $B \leq X^*$, $i = 1, 2$.
(4) $f^{-1}(H_i^m(B)) \leq H_i^m(f^{-1}(B))$ for any fuzzy subset $B \leq X^*$, $i = 1, 2$.

Theorem 3.8. *For any mapping $f : (X, \tau_1, \tau_2, \leq) \to (X^*, \tau_1^*, \tau_2^*, \leq^*)$, the following statements are equivalent :*
(1) f is pairwise fuzzy B-open.
(2) $f((A_i^o)) \leq O_i^b(f(A))$ for any fuzzy subset $A \leq X$, $i = 1, 2$.
(3) $(f^{-1}(B))_i^o \leq f^{-1}(O_i^b(B))$ for any fuzzy subset $B \leq X^*$, $i = 1, 2$.
(4) $f^{-1}(H_i^b(B)) \leq H_i^b(f^{-1}(B))$ for any fuzzy subset $B \leq X^*$, $i = 1, 2$.

Theorem 3.9. *Let $f : (X, \tau_1, \tau_2, \leq_1) \to (Y, \nu_1, \nu_2, \leq_2)$ and $g : (Y, \nu_1, \nu_2, \leq_2) \to (Z, \eta_1, \eta_2, \leq_3)$ be any two mappings. Then*
(i) $g \circ f : (X, \tau_1, \tau_2, \leq_1) \to (Z, \eta_1, \eta_2, \leq_3)$ is pairwise fuzzy x-open if f is pairwise fuzzy open and g is pairwise fuzzy x-open for $x = I, D, B$.
(ii) $g \circ f : (X, \tau_1, \tau_2, \leq_1) \to (Z, \eta_1, \eta_2, \leq_3)$ is pairwise fuzzy x-open if both f and g are pairwise fuzzy x-open for $x = I, D, B$.
(iii) $g \circ f : (X, \tau_1, \tau_2, \leq_1) \to (Z, \eta_1, \eta_2, \leq_3)$ is pairwise fuzzy x-open if f is pairwise fuzzy y- open and g is pairwise fuzzy x- open for $x, y \in \{I, D, B\}$.

Proof. *Omitted.* □

4. Pairwise fuzzy I- closed, pairwise fuzzy D- closed and pairwise fuzzy B- closed mappings

Definition 4.1. *A mapping $f : (X, \tau_1, \tau_2, \leq) \to (X^*, \tau_1^*, \tau_2^*, \leq^*)$ from a fuzzy bitopological ordered space $(X, \tau_1, \tau_2, \leq)$ to another fuzzy bitopological ordered space $(X^*, \tau_1^*, \tau_2^*, \leq^*)$ is called a pairwise fuzzy I-closed (respectively a pairwise fuzzy D-closed, a pairwise fuzzy B-closed) mapping if $f(G) \in \Omega(H_i^m(X^*))$ (respectively $f(G) \in \Omega(H_i^l(X^*)), f(G) \in \Omega(H_i^b(X^*))$, whenever G is a τ_i-fuzzy open subset of $(X, \tau_1, \tau_2, \tau)$, where $\Omega(H_i^m(X^*))$ (respectively $\Omega(H_i^l(X^*))$, $\Omega(H_i^b(X^*))$ is the collection of all τ_i-increasing (respectively τ_i-decreasing, both τ_i- increasing and τ_i-decreasing) closed fuzzy subsets of $(X^*, \tau_1^*, \tau_2^*, \leq^*), i = 1, 2$. It is evident that every pairwise fuzzy x- closed mapping is a pairwise fuzzy closed mapping for $x = I, D, B$ and every pairwise fuzzy B-closed mapping is both pairwise fuzzy I-closed and pairwise fuzzy D-closed. The following example shows that a pairwise fuzzy closed mapping need not be pairwise fuzzy x- closed mapping for $x = I, D, B$.*

Example 4.2. *Let $(X, \tau_1, \tau_2, \leq)$ be a fuzzy bitopological ordered space and f be a mapping as in Example 2.6. Then f is a pairwise fuzzy closed mapping, f is not pairwise x- closed for $x = I, D, B$.*
The following example shows that a pairwise fuzzy I-closed mapping need not be a pairwise fuzzy B- closed mapping.

Example 4.3. *Let $X, X^*, \tau_1, \tau_2, \tau_1^*, \tau_2^*, \leq$ and $\leq^*$ be as in Example 3.3. Let θ be the identity mapping from $(X, \tau_1, \tau_2, \leq)$ onto $(X^*, \tau_1^*, \tau_2^*, \leq^*)$. θ is pairwise fuzzy I- closed but not a pairwise fuzzy B-closed mapping.*

The following example shows that a pairwise fuzzy I-closed mapping need not be a pairwise fuzzy B- closed mapping.

Example 4.4. *Let $X, X^*, \tau_1, \tau_2, \tau_1^*, \tau_2^*, \leq, \leq^*$ and ϕ be as in Example 3.4. ϕ is a pairwise fuzzy D-closed mapping but not a pairwise fuzzy B-closed mapping.*

Thus we have the following diagram:
For a mapping $f : (X, \tau_1, \tau_2, \leq) \to (X^*, \tau_1^*, \tau_2^*, \leq^*)$, where $P \to Q$ (respectively $P \not\leftrightarrow Q$) represents P implies Q but Q need not imply P (respectively P and Q are independent of each other).

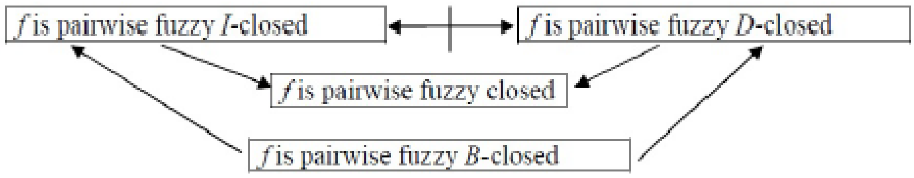

FIGURE 3

Theorem 4.5. *Let $f : (X, \tau_1, \tau_2, \leq) \to (X^*, \tau_1^*, \tau_2^*, \leq^*)$ be any mapping. Then f is pairwise fuzzy I-closed if and only if $H_i^m(f(A)) \leq f(\overline{(A)_i})$ for every $A \leq X, i = 1, 2$.*

Proof.

Necessity. Since f is pairwise fuzzy I-closed, then $f\overline{(A)}_i$ is a τ_i-increasing fuzzy closed subset of X and $f(A) \leq f\overline{(A)}_i$. Therefore $H_i^m(f(A)) \leq f\overline{(A)}_i$ since $H_i^m(f(A))$ is the smallest τ_i-increasing fuzzy closed set in X^* containing $f(A)$.

Sufficency. Let F be any fuzzy τ_i-closed subset of X^*. Then $f(F) \leq H_i^m(f(F)) \leq f\overline{(F)}_i = f(F)$. Thus $f(F) = H_i^m(f(F))$. So $f(F)$ is a τ_i-increasing fuzzy closed subset of X^*. Therefore f is a pairwise fuzzy I-closed mapping. □

The following two Theorems characterize pairwise fuzzy D-closed mapping and pairwise fuzzy B-closed mapping.

Theorem 4.6. *Let $f : (X, \tau_1, \tau_2, \leq) \to (X^*, \tau_1^*, \tau_2^*, \leq^*)$ be any mapping. Then f is pairwise fuzzy I closed if and only if $H_i^l(f(A)) \leq f\overline{(A)}_i$ for every $A \leq X, i = 1, 2$.*

Proof. Omitted. □

Theorem 4.7. *Let $f : (X, \tau_1, \tau_2, \leq) \to (X^*, \tau_1^*, \tau_2^*, \leq^*)$ be any mapping. Then f is pairwise fuzzy B-closed if and only if $H_i^b(f(A)) \leq f\overline{(A)}_i$ for every $A \leq X, i = 1, 2$.*

Proof. Omitted. □

Theorem 4.8. . *Let $f : (X, \tau_1, \tau_2, \leq) \to (X^*, \tau_1^*, \tau_2^*, \leq^*)$ be a pairwise fuzzy bijection mapping. Then*
(1) f is pairwise fuzzy I-open if and only if f is pairwise fuzzy D- closed.
(2) f is pairwise fuzzy I-closed if and only if f is pairwise fuzzy D-open.
(3) f is pairwise fuzzy B-open if and only if f is pairwise fuzzy B-closed.

Proof.

Necessity. Let F be any τ_i-fuzzy closed subset of X. Then $f(X \setminus F)$ is a τ_i^*-increasing fuzzy open subset of X^* since f is a pairwise fuzzy I-open mapping and $X \setminus F$ is a τ_i fuzzy open subset of X. Since f is a pairwise fuzzy bijection, then we have $f(X \setminus F) = X^* \setminus (f(F))$. So $f(F)$ is a τ_i^*-decreasing fuzzy closed subset of X^*. Therefore f is pairwise fuzzy D-closed.

Sufficiency. Let G be any τ_i-fuzzy open subset of X. Then $f(X \setminus G)$ is a τ_i-decreasing fuzzy closed subset of X^* since f is a pairwise fuzzy D-closed mapping and $X \setminus G$ is a τ_i-fuzzy closed subset of X. Since f is a pairwise fuzzy bijection, then we have $f(X \setminus G) = X \setminus (f(G))$. So $f(G)$ is a τ_i-decreasing fuzzy open subset of X^*. Therefore f is a pairwise fuzzy I-open mapping.
The proofs for (2) and (3) are similar to that of (1). □

Theorem 4.9. *Let $f : (X, \tau_1, \tau_2, \leq_1) \to (Y, \nu_1, \nu_2, \leq_2)$ and $g : (Y, \nu_1, \nu_2, \leq_2) \to (Z, \eta_1, \eta_2, \leq_3)$ be any two mappings. Then*
(1) $gof : (X, \tau_1, \tau_2, \leq_1) \to (Z, \eta_1, \eta_2, \leq_3)$ is pairwise fuzzy x- closed if f is pairwise fuzzy closed and g is pairwise fuzzy x- closed for $x = I, D, B$.
(2) $gof : (X, \tau_1, \tau_2, \leq_1) \to (Z, \eta_1, \eta_2, \leq_3)$ is pairwise fuzzy x- closed if both f and g are pairwise fuzzy x- closed for $x = I, D, B$.
(3) $gof : (X, \tau_1, \tau_2, \leq_1) \to (Z, \eta_1, \eta_2, \leq_3)$ is pairwise fuzzy x- closed if f is pairwise fuzzy y-closed and g is pairwise fuzzy x- closed for $x, y \in \{I, D, B\}$.

Theorem 4.10. *Let $f : (X, \tau_1, \tau_2, \leq) \to (X^*, \tau_1^*, \tau_2^*, \leq^*)$ be a pairwise fuzzy bijection mapping. Then the following statements are equivalent :*
(1) f is a pairwise fuzzy I- open mapping.
(2) f is a pairwise fuzzy D- closed mapping.
(3) f^{-1} is a pairwise fuzzy I- continuous.

Theorem 4.11. *Let* $f : (X, \tau_1, \tau_2, \leq) \to (X^*, \tau_1^*, \tau_2^*, \leq^*)$ *be a pairwise fuzzy bijection mapping. Then the following statements are equivalent :*
(1) f *is a pairwise fuzzy* D*- open mapping.*
(2) f *is a pairwise fuzzy* I*- closed mapping.*
(3) f^{-1} *is a pairwise fuzzy* D*- continuous.*

Theorem 4.12. *Let* $f : (X, \tau_1, \tau_2, \leq) \to (X^*, \tau_1^*, \tau_2^*, \leq^*)$ *be a pairwise fuzzy bijection mapping. Then the following statements are equivalent :*
(1) f *is a pairwise fuzzy* B*- open mapping.*
(2) f *is a pairwise fuzzy* B*- closed mapping.*
(3) f^{-1} *is a pairwise fuzzy* B*- continuous.*

Theorem 4.13. *Let* $f : (X, \tau_1, \tau_2, \leq) \to (X^*, \tau_1^*, \tau_2^*, \leq^*)$ *be a pairwise fuzzy* I*-closed mapping and* B, C *are two fuzzy subsets of* X^**. Then*
(1) if U *is a* τ_i*-fuzzy open neighbourhood of* $f^{-1}(B)$*, then there exists a* τ_i^**-decreasing fuzzy neighbourhood* V *of* B *such that* $f^{-1}(B) \leq f^{-1}(V) \leq U, i = 1, 2,$
(2) if $f^{-1}(B)$ *and* $f^{-1}(C)$ *have disjoint* τ_i*-fuzzy neighbourhoods, then* $f^{-1}(B)$ *and* $f^{-1}(C)$ *have disjoint* τ_i*-decreasing fuzzy open neighbourhoods,* $i = 1, 2.$

Proof. (1) Let U be a τ_i-fuzzy open neighbourhood of $f^{-1}(B)$. Take $X^* \setminus V = f(X \setminus U)$. Since f is a pairwise fuzzy I- closed mapping and $X \setminus U$ is a τ_i-fuzzy closed set, then $X^* \setminus V = f(X \setminus U)$ is a τ_i^*-increasing fuzzy closed subset of X^*. Thus V is a τ_i^*-decreasing fuzzy open subset of X^*. Since $f^{-1}(B) \leq U$, then $X^* \setminus V = f(X \setminus U) \leq f(f^{-1}(X * \setminus B)) \leq X^* \setminus B$. So $B \leq V$. Thus V is a τ_i^*-decreasing fuzzy open neighbourhood of B. Further $X \setminus U \leq f^{-1}(f(X \setminus U)) = f^{-1}(X * \setminus V) = X \setminus (f^{-1}(V))$. Thus $f^{-1}(B) \leq f^{-1}(V) \leq U$.
(2) Omitted. □

Theorem 4.14. *Let* $f : (X, \tau_1, \tau_2, \leq) \to (X^*, \tau_1^*, \tau_2^*, \leq^*)$ *be a pairwise fuzzy* D*-closed mapping and* B, C *are two fuzzy subsets of* X^**. Then*
(1) if U *is a* τ_i*-fuzzy open neighbourhood of* $f^{-1}(B)$*, then there exists a* τ_i^**- decreasing fuzzy neighbourhood* V *of* B *such that* $f^{-1}(B) \leq f^{-1}(V) \leq U, i = 1, 2,$
(2) if $f^{-1}(B)$ *and* $f^{-1}(C)$ *have disjoint* τ_i*-fuzzy neighbourhoods, then* $f^{-1}(B)$ *and* $f^{-1}(C)$ *have disjoint* τ_i*- increasing fuzzy open neighbourhoods,* $i = 1, 2.$

Theorem 4.15. *Let* $f : (X, \tau_1, \tau_2, \leq) \to (X^*, \tau_1^*, \tau_2^*, \leq^*)$ *be a pairwise fuzzy* B*-closed mapping and* B, C *are two fuzzy subsets of* X^**. Then*
(1) if U *is a* τ_i*-fuzzy open neighbourhood of* $f^{-1}(B)$*, then there exists a* τ_i^**- fuzzy open neighbourhood* V *of* B *which is both* τ_i^**-increasing and* τ_i^**-decreasing such that* $f^{-1}(B) \leq f^{-1}(V) \leq U, i = 1, 2$*.*
(2) if $f^{-1}(B)$ *and* $f^{-1}(C)$ *have disjoint* τ_i*-fuzzy neighbourhoods, then* $f^{-1}(B)$ *and* $f^{-1}(C)$ *have disjoint* τ_i*-fuzzy open neighbourhoods which is both* τ_i^**-increasing and* τ_i^**-decreasing* $i = 1, 2$*.*

Acknowledgment.

The author thanks the reviewer for the comments those improved the presentation the article.

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