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Unit groups of group algebras of abelian groups of order 32

Suchi Bhatt
M. M. M. University of Technology, India
and
Harish Chandra
M. M. M. University of Technology, India
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Abstract

Let F be a finite field of characteristic $p > 0$ with $q = p^n$ elements. In this paper, a complete characterization of the unit groups $U(FG)$ of group algebras FG for the abelian groups of order 32, over finite field of characteristic $p > 0$ has been obtained.

Key words: *Group algebras, Unit groups, Jacobson radical.*

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1. Introduction

Let FG be the group algebra of a group G over a field F. Suppose $U(FG)$ be the group of all invertible elements of the group algebra FG, called unit group of FG. In this paper, we study the unit groups of group algebra for abelian groups of order 32. Suppose $V(FG)$ be the normalized unit group, $\omega(G)$ be the augmentation ideal of G, $J(FG)$ is the Jacobson radical of the group algebra and $V = 1 + J(FG)$. It is known fact that $U(FG) \cong V(FG) \times F^*$. An element $g \in G$ is called p-regular if $(p, o(g)) = 1$, where $\mathrm{Char}F = p > 0$. Notation used in this paper are same as in [2]. Our problem is based on the Witt-Berman theorem [6, Ch.17, Theorem 5.3], which states that the number of non-isomorphic simple FG-modules is equal to the number of F-conjugacy classes of p-regular elements of G. Problem of finding unit groups of group algebras generated a considerable interest in recent decade and can be easily seen in [5, 7, 8, 10, 13-15]. Recently in [1, 12], Sahai and Ansari have characterized the unit groups of group algebras of groups of orders 16 and 20. Let G be a group of order 32, we have seven non-isomorphic abelian groups C_{32}, $C_{16} \times C_2$, $C_8 \times C_4$, $C_8 \times C_2^2$, $C_4^2 \times C_2$, $C_4 \times C_2^3$ and C_2^5. Here, we have obtained the structure of the unit groups of the group algebras for all these groups over any finite field of characteristics $p > 0$. We denote $GL(n, F)$ the general linear group of degree n over F, $M(n, F)$ the algebra of all $n \times n$ matrices over F, $CharF$ the characteristic of F, C_n is the cyclic group of order n and $F^* = F \setminus \{0\}$.

2. Preliminaries

Following are the important results which we have used frequently.

Lemma 1. [4, *Proposition* 1.2] *The number of simple components of $FG/J(FG)$ is equal to the number of cyclotomic F-classes in G.*

Lemma 2. [3, *Lemma* 2.1] *Let F be a finite field of characteristic p with $|F| = q = p^n$. Then $U(FC_p^k) = C_p^{n(p^k-1)} \times C_{p^n-1}$.*

Lemma 3. [9, *Lemma* 2.3] *Let F be a finite field of characteristic p with $|F| = q = p^n$. Then*

$$U(FC_{p^k}) \cong \begin{cases} C_p^{n(p-1)} \times C_{p^n-1} & \text{if } k = 1; \\ \prod_{s=1}^{k} C_{p^s}^{h_s} \times C_{p^n-1}, & \text{otherwise}, \end{cases}$$

where $h_k = n(p-1)$ and $h_s = np^{k-s-1}(p-1)^2$ for all s, $1 \leq s < k$.

Lemma 4. [11] *Let G be a group and R be a commutative ring. Then the set of all finite class sums forms an R-basis of $\zeta(RG)$, the center of RG.*

Lemma 5. [11] *Let FG be a semisimple group algebra. If G' denotes the commutator subgroup of G, then*

$$FG = FG_{e_{G'}} \oplus \Delta(G, G')$$

where $FG_{e_{G'}} \cong F(G/G')$ is the sum of all commutative simple components of FG and $\Delta(G, G')$ is the sum of all the others.

3. Main Results

Theorem 1. *Let F be a finite field of characteristic $p > 0$, having $q = p^n$ elements and $G \cong C_{32}$.*

1. *If $p = 2$. Then,*

$$U(FC_{32}) \cong C_{32}^{n} \times C_{16}^{n} \times C_{8}^{2n} \times C_{4}^{4n} \times C_{2}^{8n} \times C_{2^n-1}.$$

2. *If $p \neq 2$. Then,*

$$U(FC_{32}) \cong \begin{cases} C_{p^n-1}^{32}, & \text{if } q \equiv 1 \, mod\, 32; \\ C_{p^n-1}^{2} \times C_{p^{2n}-1}^{15}, & \text{if } q \equiv -1 \, mod\, 32; \\ C_{p^{8n}-1}^{2} \times C_{p^{4n}-1}^{2} \times C_{p^{2n}-1}^{3} \times C_{p^n-1}^{2}, & \text{if } q \equiv 3,\, -5,\, 11,\, -13 \, mod\, 32; \\ C_{p^{8n}-1}^{2} \times C_{p^{4n}-1}^{2} \times C_{p^{2n}-1}^{2} \times C_{p^n-1}^{4}, & \text{if } q \equiv -3,\, 5,\, -11,\, 13 \, mod\, 32; \\ C_{p^n-1}^{2} \times C_{p^{4n}-1}^{4} \times C_{p^{2n}-1}^{7}, & \text{if } q \equiv 7 \, mod\, 32; \\ C_{p^n-1}^{8} \times C_{p^{2n}-1}^{4} \times C_{p^{4n}-1}^{4}, & \text{if } q \equiv -7 \, mod\, 32; \\ C_{p^n-1}^{2} \times C_{p^{2n}-1}^{15}, & \text{if } q \equiv 15 \, mod\, 32; \\ C_{p^n-1}^{16} \times C_{p^{2n}-1}^{8}, & \text{if } q \equiv -15 \, mod\, 32. \end{cases}$$

Proof. The presentation of C_{32} is given by

$$C_{32} =< a \mid a^{32} = 1 > .$$

1. If $p = 2$, then $|F| = q = 2^n$. Since $G \cong C_{32} \cong C_{2^5}$, therefore using Lemma 3, we have

$$U(FC_{32}) \cong C_{32}^{n} \times C_{16}^{n} \times C_{8}^{2n} \times C_{4}^{4n} \times C_{2}^{8n} \times C_{2^n-1}.$$

2. If $p \neq 2$, then p does not divides $|C_{32}|$, therefore by Maschke's theorem, FC_{32} is semisimple over F. Hence by Wedderburn decomposition theorem and by Lemma 5, we have

$$FC_{32} \cong (\bigoplus_{i=1}^{r} M(n_i, D_i))$$

where for each i, $n_i \geq 1$ and Di's are finite field extensions of F. Since group is abelian, therefore dimension constraint gives $n_i = 1$, for every i. It is clear that C_{32} has 32 conjugacy classes. Now for any $k \in N$, $x^{q^k} = x, \forall x \in \zeta(FC_{32})$ if and only $\widehat{C_i^{q^t}} = \widehat{C_i}$, for all $1 \leq i \leq 32$. It exists if and only if $32|q^k - 1$ or $32|q^k + 1$. If $D_i^* =< y_i >$ for all i, $1 \leq i \leq r$, then $x^{q^k} = x, \forall x \in \zeta(FC_{32})$ if and only if $y_i^{q^k} = 1$, which holds if and only if $[D_i : F]|k$, for all $1 \leq i \leq r$. Hence the least number t such that $32|q^k - 1$ or $32|q^k + 1$,

$$t = l.c.m.\{[D_i : F]|1 \leq i \leq r\}.$$

Therefore all conjugacy classes of C_{32} are p-regular and m=32. By observation we have following possibilities for q:

(a) If $q \equiv 1\, mod\, 32$, then $t = 1$;
(b) If $q \equiv -1\, mod\, 32$, then $t = 2$;
(c) If $q \equiv 3, -5, 11, -13\ mod\, 32$, then $t = 8$;
(d) If $q \equiv -3, 5, -11, 13\ mod\, 32$, then $t = 8$;
(e) If $q \equiv 7\ mod\ 32$, then $t = 4$;
(f) If $q \equiv -7\ mod\ 32$, then $t = 4$;
(g) If $q \equiv 15\ mod\ 32$, then $t = 2$;
(h) If $q \equiv -15\ mod\ 32$, then $t = 2$.

Now we will find T and the number of p-regular F-conjugacy classes, denoted by c. By Lemma 4, $dim_F(\zeta(FC_{32})) = 32$, therefore $\sum_{i=1}^{r}[D_i : F] = 32$. We have the following cases:

1. If $q \equiv 1\, mod\, 32$, then $T = \{1\}\, mod\ 32$. Thus p-regular F-conjugacy classes are the conjugacy classes of C_{32} and c=32. Hence $FC_{32} \cong F^{32}$.

2. If $q \equiv -1\, mod\, 32$, then $T = \{1, -1\}\, mod\ 32$. Thus p-regular F-conjugacy classes are $\{1\}$, $\{a^{16}\}$, $\{a^{\pm i}\}$, $1 \leq i \leq 15$ and c=17. Hence $FC_{32} \cong F^2 \oplus F_2^{15}$.

3. If $q \equiv 3, -5, 11, -13 \, mod\, 32$, then $T = \{1, 3, 9, 11, 17, 19, 25, 27\}\, mod$ 32. Thus p-regular F-conjugacy classes are $\{1\}$, $\{a, a^3, a^9, a^{11}, a^{17}, a^{19}, a^{25}, a^{27}\}$, $\{a^2, a^6, a^{18}, a^{22}\}$, $\{a^4, a^{12}\}$, $\{a^5, a^7, a^{13}, a^{15}, a^{21}, a^{23}, a^{29}, a^{31}\}$, $\{a^8, a^{24}\}$, $\{a^{10}, a^{14}, a^{26}, a^{30}\}$, $\{a^{16}\}$, $\{a^{20}, a^{28}\}$ and c=9. Hence $FC_{32} \cong F_8^2 \oplus F_4^2 \oplus F_2^3 \oplus F^2$.

4. If $q \equiv -3, 5, -11, 13 \, mod\, 32$, then $T = \{1, 5, 9, 13, 17, 21, 25, 29\}\, mod$ 32. Thus p-regular F-conjugacy classes are $\{1\}$, $\{a, a^5, a^9, a^{13}, a^{17}, a^{21}, a^{25}, a^{29}\}$, $\{a^2, a^{10}, a^{18}, a^{26}\}$, $\{a^4, a^{20}\}$, $\{a^3, a^7, a^{11}, a^{15}, a^{19}, a^{23}, a^{27}, a^{31}\}$, $\{a^8\}$, $\{a^6, a^{14}, a^{22}, a^{30}\}$, $\{a^{16}\}$, $\{a^{24}\}$, $\{a^{12}, a^{28}\}$ and c=10. Hence $FC_{32} \cong F_8^2 \oplus F_4^2 \oplus F_2^2 \oplus F^4$.

5. If $q \equiv 7\, mod\, 32$, then $T = \{1, 7, 17, 23\}\, mod$ 32. Thus, p-regular F-conjugacy classes are $\{1\}$, $\{a, a^7, a^{17}, a^{23}\}$, $\{a^2, a^{14}\}$, $\{a^3, a^5, a^{19}, a^{21}\}$, $\{a^4, a^{28}\}$, $\{a^6, a^{10}\}$, $\{a^8, a^{24}\}$, $\{a^9, a^{15}, a^{25}, a^{31}\}$, $\{a^{11}, a^{13}, a^{27}, a^{29}\}$, $\{a^{12}, a^{20}\}$, $\{a^{16}\}$, $\{a^{18}, a^{30}\}$, $\{a^{22}, a^{26}\}$ and $c = 13$. Hence $FC_{32} \cong F^2 \oplus F_4^4 \oplus F_2^7$.

6. If $q \equiv -7\, mod\, 32$, then $T = \{1, 9, 17, 25\}\, mod$ 32. Thus, p-regular F-conjugacy classes are $\{1\}$, $\{a, a^9, a^{17}, a^{25}\}$, $\{a^2, a^{18}\}$, $\{a^3, a^{11}, a^{19}, a^{27}\}$, $\{a^4\}$, $\{a^6, a^{22}\}$, $\{a^5, a^{13}, a^{21}, a^{29}\}$, $\{a^7, a^{15}, a^{23}, a^{31}\}$, $\{a^8\}$, $\{a^{10}, a^{26}\}$, $\{a^{12}\}$, $\{a^{16}\}$, $\{a^{14}, a^{30}\}$, $\{a^{20}\}$, $\{a^{24}\}$, $\{a^{28}\}$ and $c = 16$. Hence $FC_{32} \cong F^8 \oplus F_2^4 \oplus F_4^4$.

7. If $q \equiv 15\, mod\, 32$, then $T = \{1, 15\}\, mod$ 32. Thus, p-regular F-conjugacy classes are $\{1\}$, $\{a, a^{15}\}$, $\{a^2, a^{30}\}$, $\{a^3, a^{13}\}$, $\{a^4, a^{28}\}$, $\{a^5, a^{11}\}$, $\{a^6, a^{26}\}$, $\{a^7, a^9\}$, $\{a^8, a^{24}\}$, $\{a^{10}, a^{22}\}$, $\{a^{12}, a^{20}\}$, $\{a^{14}, a^{18}\}$, $\{a^{17}, a^{31}\}$, $\{a^{19}, a^{29}\}$, $\{a^{21}, a^{27}\}$, $\{a^{23}, a^{25}\}$, $\{a^{16}\}$ and c=17. Hence, $FC_{32} \cong F^2 \oplus F_2^{15}$.

8. If $q \equiv -15\, mod\, 32$, then $T = \{1, 17\}\, mod$ 32. Thus, p-regular F-conjugacy classes are $\{1\}$, $\{a, a^{17}\}$, $\{a^2\}$, $\{a^{30}\}$, $\{a^3, a^{19}\}$, $\{a^4\}$, $\{a^{28}\}$, $\{a^5, a^{21}\}$, $\{a^6\}$, $\{a^{26}\}$, $\{a^7, a^{23}\}$, $\{a^8\}$, $\{a^{24}\}$, $\{a^9, a^{25}\}$, $\{a^{10}\}$, $\{a^{22}\}$, $\{a^{11}, a^{27}\}$, $\{a^{13}, a^{29}\}$, $\{a^{15}, a^{31}\}$, $\{a^{12}\}$, $\{a^{20}\}$, $\{a^{16}\}$, $\{a^{14}\}$, $\{a^{18}\}$ and c=24. Hence, $FC_{32} \cong F^{16} \oplus F_2^8$. Thus our result follows.

□

Theorem 2. *Let F be a finite field of characteristic $p > 0$ having $q = p^n$ elements and $G \cong C_{16} \times C_2$.*

1. *If $p = 2$. Then, $U(F[C_{16} \times C_2]) \cong C_{16}^n \times C_8^n \times C_4^{2n} \times C_2^{20n} \times C_{2^n-1}$.*

2. If $p \neq 2$. Then,

$$U(F[C_{16} \times C_2]) \cong \begin{cases} C_{p^n-1}^{32}, & \text{if} \quad q \equiv 1\, mod\ 16; \\ C_{p^n-1}^{4} \times C_{p^{2n}-1}^{14}, & \text{if} \quad q \equiv -1\, mod\ 16; \\ C_{p^n-1}^{4} \times C_{p^{2n}-1}^{6} \times C_{p^{4n}-1}^{4}, & \text{if} \quad q \equiv 3, -5\, mod\ 16; \\ C_{p^n-1}^{8} \times C_{p^{2n}-1}^{4} \times C_{p^{4n}-1}^{4}, & \text{if} \quad q \equiv -3, 5\, mod\ 16; \\ C_{p^n-1}^{4} \times C_{p^{2n}-1}^{14}, & \text{if} \quad q \equiv 7\, mod\ 16; \\ C_{p^n-1}^{16} \times C_{p^{2n}-1}^{8}, & \text{if} \quad q \equiv -7\, mod\ 16. \end{cases}$$

Proof. The presentation of $G \cong C_{16} \times C_2$ is given by

$$C_{16} \times C_2 =< a, b \mid a^{16} = b^2 = 1,\ ab = ba > .$$

1. If $p = 2$, then FG is non-semisimple and $|F| = q = 2^n$. It is well known that $U(FG) \cong V(FG) \times F^*$ and $|V(FG)| = 2^{31n}$ as $dim_F J(FG) = 31$. Obviously exponent of $V(FG)$ is 16. Suppose $V(FG) \cong C_{16}^{l_1} \times C_8^{l_2} \times C_4^{l_3} \times C_2^{l_4}$ such that $2^{31n} = 16^{l_1} \times 8^{l_2} \times 4^{l_3} \times 2^{l_4}$. Now we will compute l_1, l_2, l_3 and l_4. Set $W_1 = \Big\{\gamma_1 \in \omega(G) : \gamma_1^2 = 0 \,\text{and there exists}\, \beta \in \omega(G),\ \text{such that}\, \gamma_1 = \beta^8\Big\}$, $W_2 = \Big\{\gamma_2 \in \omega(G) : \gamma_2^2 = 0 \,\text{and there exists}\, \beta \in \omega(G),\ \text{such that}\, \gamma_2 = \beta^4\Big\}$ and $W_3 = \Big\{\gamma_3 \in \omega(G) : \gamma_3^2 = 0 \,\text{and there exists}\, \beta \in \omega(G),\ \text{such that}\, \gamma_3 = \beta^2\Big\}$. Now if $\gamma = \sum_{j=0}^{1}\sum_{i=0}^{15} \alpha_{16j+i}a^ib^j \in \omega(G)$, then $\sum_{i=0}^{15} \alpha_{2i+j} = 0$, for $j = 0, 1$. Also $\gamma^2 = \sum_{j=0}^{7}\sum_{i=0}^{3} \alpha_{8i+j}^2 a^{2j}$, $\gamma^4 = \sum_{j=0}^{3}\sum_{i=0}^{7} \alpha_{4i+j}^4 a^{4j}$ and $\gamma^8 = \sum_{j=0}^{1}\sum_{i=0}^{15} \alpha_{2i+j}^8 a^{8j}$. Let $\beta = \sum_{j=0}^{1}\sum_{i=0}^{15} \beta_{16j+i}a^ib^j$, such that $\gamma_1 = \beta^8$. Now applying condition $\gamma_1^2 = 0$ and by direct computation we have $\alpha_i = 0$, for all $i \neq 0, 8$ and $\alpha_0 = \alpha_8$. Thus $W_1 = \Big\{\alpha_0(1 + a^8), \alpha_0 \in F\Big\}$, $|W_1| = 2^n$ and $l_1 = n$. Similarly, applying the conditions $\gamma_2 = \beta^4$ and $\gamma_2^2 = 0$, we have $\alpha_i = 0$, for all $i \neq 0, 4$ and $\alpha_0 = \alpha_4$. Thus $W_2 = \Big\{\alpha_0(1 + a^4), \alpha_0 \in F\Big\}$, $|W_2| = 2^n$ and $l_2 = n$. Again, applying the conditions $\gamma_3 = \beta^8$ and $\gamma_3^2 = 0$. We have $\alpha_i = 0$, for all $i \neq 0, 2, 8, 10$ and $\alpha_0 = \alpha_8$, $\alpha_2 = \alpha_{10}$. Thus $W_3 = \Big\{(\alpha_0 + \alpha_2 a^2)(1 + a^8), \alpha_0, \alpha_2 \in F\Big\}$, $l_3 = 2n$ and $l_4 = 20n$. Hence $V(FG) \cong C_{16}^n \times C_8^n \times C_4^{2n} \times C_2^{20n}$ and hence the result.

2. If $p \neq 2$, then $|F| = p^n$. Using the similar arguments as in Theorem 1, $F[C_{16} \times C_2]$ is semisimple and we have m=16, $\sum_{i=1}^{r}[D_i : F] = 32$. By observation we have following possibilities for q:

(a) If $q \equiv 1$ mod 16, then $t = 1$;
(b) If $q \equiv -1$ mod 16, then $t = 2$;
(c) If $q \equiv 3, -5\,mod$ 16, then $t = 4$;
(d) If $q \equiv -3, 5mod$ 16, then $t = 4$;
(e) If $q \equiv 7$ mod 16, then $t = 2$;
(f) If $q \equiv -7$ mod 16, then $t = 2$.

Hence we have the following cases:

1. If $q \equiv 1\,mod\,16$, then $T = \{1\}\,mod$ 16. Thus, p-regular F-conjugacy classes are the conjugacy classes of $C_{16} \times C_2$ and c=32. Hence $F[C_{16} \times C_2] \cong F^{32}$.

2. If $q \equiv -1\,mod\,16$, then $T = \{1, -1\}\,mod$ 16. Thus, p-regular F-conjugacy classes are $\{1\}$, $\{b\}$, $\{a^8\}$, $\{a^{\pm i}\}$, where $1 \leq i \leq 7$, $\{a^8b\}$, $\{a^jb, a^{-j}b\}$, where $1 \leq j \leq 7$ and c=18. Hence $F[C_{16} \times C_2] \cong F^4 \oplus F_2^{14}$.

3. If $q \equiv 3, -5\,mod$ 16, then $T = \{1, 3, 9, 11\}\,mod$ 16. Thus, p-regular F-conjugacy classes are $\{1\}$, $\{b\}$, $\{a, a^3, a^{-7}, a^{-5}\}$, $\{a^{-1}, a^{-3}, a^5, a^7\}$, $\{a^2, a^6\}$, $\{a^{-2}, a^{-6}\}$, $\{a^{\pm 4}\}$, $\{a^8\}$, $\{ab, a^3b, a^{-7}b, a^{-5}b\}$, $\{a^{-1}b, a^{-3}b, a^5b, a^7b\}$, $\{a^2b, a^6b\}$, $\{a^{-2}b, a^{-6}b\}$, $\{a^{\pm 4}b\}$, $\{a^8b\}$ and c=14. Hence $F[C_{16} \times C_2] \cong F_2^6 \oplus F_4^4 \oplus F^4$.

4. If $q \equiv -3, 5\,mod$ 16, then $T = \{1, 5, 9, 13\}\,mod$ 16. Thus, p-regular F-conjugacy classes are $\{1\}$, $\{b\}$, $\{a, a^5, a^{-3}, a^{-7}\}$, $\{a^{-1}, a^{-5}, a^3, a^7\}$, $\{a^2, a^{-6}\}$, $\{a^{-2}, a^6\}$, $\{a^4\}$, $\{a^{-4}\}$, $\{a^8\}$, $\{ab, a^5b, a^{-3}b, a^{-7}b\}$, $\{a^{-1}b, a^{-5}b, a^3b, a^7b\}$, $\{a^2b, a^{-6}b\}$, $\{a^{-2}b, a^6b\}$, $\{a^4b\}$, $\{a^{-4}b\}$, $\{a^8b\}$ and c=16. Hence $F[C_{16} \times C_2] \cong F_2^4 \oplus F_4^4 \oplus F^8$.

5. If $q \equiv 7\,mod$ 16, then $T = \{1, 7\}\,mod$ 16. Thus, p-regular F-conjugacy classes are $\{1\}$, $\{b\}$, $\{a, a^7\}$, $\{a^3, a^5\}$, $\{a^{-1}, a^{-7}\}$, $\{a^{-3}, a^{-5}\}$, $\{a^{\pm 2}\}$, $\{a^{\pm 6}\}$, $\{a^{\pm 4}\}$, $\{a^8\}$, $\{ab, a^7b\}$, $\{a^3b, a^5b\}$, $\{a^{-1}b, a^{-7}b\}$, $\{a^{-3}b, a^{-5}b\}$, $\{a^{\pm 2}b\}$, $\{a^{\pm 6}b\}$, $\{a^{\pm 4}b\}$, $\{a^8b\}$ and c=18. Hence $F[C_{16} \times C_2] \cong F_2^{14} \oplus F^4$.

6. If $q \equiv -7\,mod$ 16, then $T = \{1, 9\}\,mod$ 16. Thus, p-regular F-conjugacy classes are $\{1\}$, $\{b\}$, $\{a, a^{-7}\}$, $\{a^3, a^{-5}\}$, $\{a^{-1}, a^7\}$, $\{a^{-3}, a^5\}$, $\{a^2\}$, $\{a^{-2}\}$, $\{a^6\}$, $\{a^{-6}\}$, $\{a^4\}$, $\{a^{-4}\}$, $\{a^8\}$,- $\{ab, a^{-7}b\}$, $\{a^3b, a^{-5}b\}$, $\{a^{-1}b, a^7b\}$, $\{a^{-3}b, a^5b\}$, $\{a^2b\}$, $\{a^{-2}b\}$, $\{a^6b\}$, $\{a^{-6}b\}$, $\{a^4b\}$, $\{a^{-4}b\}$, $\{a^8b\}$ and c=24. Hence $F[C_{16} \times C_2] \cong F_2^8 \oplus F^{16}$. Thus we have the result.

□

Theorem 3. *Let F be a finite field of characteristic $p > 0$ having $q = p^n$ elements and $G \cong C_8 \times C_4$.*

1. *If $p = 2$. Then,*

$$U(F[C_8 \times C_4]) \cong C_8^n \times C_4^{5n} \times C_2^{18n} \times C_{2^n-1}.$$

2. *If $p \neq 2$. Then,*

$$U(F[C_8 \times C_4]) \cong \begin{cases} C_{p^n-1}^{32}, & if \quad q \equiv 1 \, mod\ 8; \\ C_{p^n-1}^{4} \times C_{p^{2n}-1}^{14}, & if \quad q \equiv -1 \, mod\ 8; \\ C_{p^n-1}^{4} \times C_{p^{2n}-1}^{14}, & if \quad q \equiv 3 \, mod\ 8; \\ C_{p^n-1}^{16} \times C_{p^{2n}-1}^{8}, & if \quad q \equiv -3 \, mod\ 8. \end{cases}$$

Proof. The presentation of $G \cong C_8 \times C_4$ is given by

$$C_8 \times C_4 =< a, b \mid a^8 = b^4 = 1, \, ab = ba > .$$

1. If $p = 2$, then FG is non-semisimple and $|F| = q = 2^n$. It is well known that $U(FG) \cong V(FG) \times F^*$ and $|V(FG)| = 2^{31n}$ as $dim_F J(FG) = 31$. Obviously exponent of $V(FG)$ is 8. Suppose $V(FG) \cong C_8^{l_1} \times C_4^{l_2} \times C_2^{l_3}$ such that $2^{31n} = 8^{l_1} \times 4^{l_2} \times 2^{l_3}$. Now we will compute l_1, l_2 and l_3. Set $W_1 = \Big\{\alpha \in \omega(G) : \alpha^2 = 0 \,\text{and there exists}\, \beta \in \omega(G), \text{ such that}\, \alpha = \beta^4\Big\}$, $W_2 = \Big\{\gamma \in \omega(G) : \gamma^2 = 0 \,\text{and there exists}\, \beta \in \omega(G), \text{ such that}\, \gamma = \beta^2\Big\}$.
   If $\alpha = \sum_{j=0}^{3}\sum_{i=0}^{7} \alpha_{8j+i}a^i b^j \in \omega(G)$, then $\sum_{i=0}^{7} \alpha_{4i+j} = 0$, for $j = 0, 1, 2, 3$. Let $\beta = \sum_{j=0}^{3}\sum_{i=0}^{7} \beta_{8j+i}a^i b^j$ such that $\alpha = \beta^4$. Now applying condition $\alpha^2 = 0$, $\alpha = \beta^4$ and by direct computation, we have $\alpha_i = 0$, for all $i \neq 0, 4$ and $\alpha_0 = \alpha_4$. Thus $W_1 = \Big\{\alpha_0(1+a^4), \alpha_0 \in F\Big\}$. Therefore $|W_1| = 2^n$ and $l_1 = n$. Similarly, applying the conditions $\gamma = \beta^2$, $\gamma^2 = 0$ and by direct computation, we have $|W_2| = 2^{5n}$, $l_2 = 5n$ and $l_3 = 18n$. Hence $V(FG) \cong C_8^n \times C_4^{5n} \times C_2^{18n}$ and hence the result.

2. If $p \neq 2$, then $|F| = p^n$. Using the similar arguments as in Theorem 1, $F[C_8 \times C_4]$ is semisimple and we have m=8, $\sum_{i=1}^{r}[D_i : F] = 32$. By observation we have following possibilities for q:

(a) If $q \equiv 1 \ mod \ 8$, then $t = 1$;
(b) If $q \equiv -1 \ mod \ 8$, then $t = 2$;
(c) If $q \equiv 3 \ mod \ 8$, then $t = 2$;
(d) If $q \equiv -3 \ mod \ 8$, then $t = 2$.

Hence we have the following cases: -

1. If $q \equiv 1 \ mod \ 8$, then $T = \{1\} \ mod \ 8$. Thus, p-regular F-conjugacy classes are the conjugacy classes of $C_8 \times C_4$ and c=32. Hence $F[C_8 \times C_4] \cong F^{32}$.

2. If $q \equiv -1 \ mod \ 8$, then $T = \{1, -1\} \ mod \ 8$. Thus, p-regular F-conjugacy classes are $\{1\}$, $\{b^2\}$, $\{b, b^3\}$, $\{a^{\pm 1}\}$, $\{a^{\pm 2}\}$, $\{a^{\pm 3}\}$, $\{a^4\}$, $\{ab, a^{-1}b^3\}$, $\{a^2b, a^{-2}b^3\}$, $\{a^3b, a^{-3}b^3\}$, $\{a^4b, a^4b^3\}$, $\{a^{-3}b, a^3b^3\}$, $\{a^{-2}b, a^2b^3\}$, $\{a^{-1}b, ab^3\}$, $\{ab^2, a^{-1}b^2\}$, $\{a^{-2}b^2, a^2b^2\}$, $\{a^3b^2, a^{-3}b^2\}$, $\{a^4b^2\}$ and c=18. Hence $F[C_8 \times C_4] \cong F^4 \oplus F_2^{14}$.

3. If $q \equiv 3 \ mod \ 8$, then $T = \{1, 3\} \ mod \ 8$. Thus, p-regular F-conjugacy classes are $\{1\}$, $\{b^2\}$, $\{b, b^3\}$, $\{a, a^3\}$, $\{a^2, a^{-2}\}$, $\{a^{-1}, a^{-3}\}$, $\{a^4\}$, $\{ab, a^3b^3\}$, $\{a^2b, a^{-2}b^3\}$, $\{a^{-1}b, a^{-3}b^3\}$, $\{a^4b, a^4b^3\}$, $\{ab^3, a^3b\}$, $\{a^2b^3, a^{-2}b\}$, $\{a^{-1}b^3, a^{-3}b\}$, $\{ab^2, a^3b^2\}$, $\{a^2b^2, a^{-2}b^2\}$, $\{a^{-1}b^2, a^{-3}b^2\}$, $\{a^4b^2\}$ and c=18. Hence $F[C_8 \times C_4] \cong F^4 \oplus F_2^{14}$.

4. If $q \equiv -3 \ mod \ 8$, then $T = \{1, 5\} \ mod \ 8$. Thus, p-regular F-conjugacy classes are $\{1\}$, $\{b\}$, $\{b^2\}$, $\{b^3\}$, $\{a, a^{-3}\}$, $\{a^2\}$, $\{a^{-2}\}$, $\{a^{-1}, a^3\}$, $\{a^4\}$, $\{ab, a^{-3}b\}$, $\{a^2b\}$, $\{a^{-2}b\}$, $\{a^{-1}b, a^3b\}$, $\{a^4b\}$, $\{ab^2, a^{-3}b^2\}$, $\{a^2b^2\}$, $\{a^{-2}b^2\}$, $\{a^{-1}b^2, a^3b^2\}$, $\{a^4b^2\}$, $\{ab^3, a^{-3}b^3\}$, $\{a^2b^3\}$, $\{a^{-2}b^3\}$, $\{a^{-1}b^3, a^3b^3\}$, $\{a^4b^3\}$ and c=24. Hence $F[C_8 \times C_4] \cong F^{16} \oplus F_2^8$.
   Thus we have the result.

□

Theorem 4. *Let F be a finite field of characteristic $p > 0$ having $q = p^n$ elements and $G \cong C_8 \times C_2 \times C_2$.*

1. *If $p = 2$. Then,*

$$U(F[C_8 \times C_2 \times C_2]) \cong C_8^n \times C_4^n \times C_2^{26n} \times C_{2^n-1}.$$

2. *If $p \neq 2$. Then,*

$$U(F[C_8 \times C_2 \times C_2]) \cong \begin{cases} C_{p^n-1}^{32}, & if \quad q \equiv 1\, mod\ 8; \\ C_{p^n-1}^{8} \times C_{p^{2n}-1}^{12}, & if \quad q \equiv -1\, mod\ 8; \\ C_{p^n-1}^{8} \times C_{p^{2n}-1}^{12}, & if \quad q \equiv 3\, mod\ 8; \\ C_{p^n-1}^{16} \times C_{p^{2n}-1}^{8}, & if \quad q \equiv -3\, mod\ 8. \end{cases}$$

Proof. The presentation of $G \cong C_8 \times C_2 \times C_2$ is given by

$$C_8 \times C_2 \times C_2 =< a, b, c \mid a^8 = b^2 = c^2 = 1,\ ab = ba, bc = cb, ac = ca > .$$

1. If $p = 2$, then FG is non-semisimple and $|F| = q = 2^n$. It is well known that $U(FG) \cong V(FG) \times F^*$ and $|V(FG)| = 2^{31n}$ as $dim_F J(FG) = 31$. Obviously exponent of $V(FG)$ is 8. Suppose $V(FG) \cong C_8^{l_1} \times C_4^{l_2} \times C_2^{l_3}$ such that $2^{31n} = 8^{l_1} \times 4^{l_2} \times 2^{l_3}$. Now we will compute l_1, l_2 and l_3. Set $W_1 = \Big\{\alpha \in \omega(G) : \alpha^2 = 0 \text{ and there exists } \beta \in \omega(G),\ \text{such that } \alpha = \beta^4\Big\}$, $W_2 = \Big\{\gamma \in \omega(G) : \gamma^2 = 0 \text{ and there exists } \beta \in \omega(G),\ \text{such that } \gamma = \beta^2\Big\}$.
   Let $\alpha = \sum_{k=0}^{1}\sum_{j=0}^{1}\sum_{i=0}^{7} \alpha_{8(j+2k)+i} a^i b^j c^k \in \omega(G)$ and $\beta = \sum_{k=0}^{1}\sum_{j=0}^{1}\sum_{i=0}^{7} \beta_{8(j+2k)+i} a^i b^j c^k$ such that $\alpha = \beta^4$. Now applying the conditions $\alpha^2 = 0$, $\alpha = \beta^4$ and by direct computation, we have $\alpha_i = 0$, for all $i \neq 0, 4$ and $\alpha_0 = \alpha_4$. Thus $W_1 = \Big\{\alpha_0(1+a^4), \alpha_0 \in F\Big\}$. Therefore $|W_1| = 2^n$ and $l_1 = n$. Similarly, applying the conditions $\gamma = \beta^2$, $\gamma^2 = 0$ and by direct computation, we have $\alpha_i = 0$, for all $i \neq 0, 2$ and $\alpha_0 = \alpha_2$. Thus $W_2 = \Big\{\alpha_0(1 + a^2), \alpha_0 \in F\Big\}$. Therefore $|W_2| = 2^n$, $l_2 = n$ and $l_3 = 26n$. Hence $V(FG) \cong C_8^n \times C_4^n \times C_2^{26n}$ and hence the result follows.

2. If $p \neq 2$, then $|F| = p^n$. Using the similar arguments as in Theorem 1, $F[C_8 \times C_2 \times C_2]$ is semisimple and m=8, $\sum_{i=1}^{r}[D_i : F] = 32$. Here the number of p-regular F-conjugacy classes, denoted by w. By observation we have following possibilities for q:

(a) If $q \equiv 1\ mod\ 8$, then $t = 1$;
(b) If $q \equiv -1\ mod\ 8$, then $t = 2$;
(c) If $q \equiv 3\, mod\ 8$, then $t = 2$;
(d) If $q \equiv -3\, mod\ 8$, then $t = 2$.
Now we have the cases:

1. If $q \equiv 1 \, mod \, 8$, then $T = \{1\} \, mod \; 8$. Thus, p-regular F-conjugacy classes are the conjugacy classes of $C_8 \times C_2 \times C_2$ and w=32. Hence $F[C_8 \times C_2 \times C_2] \cong F^{32}$.

2. If $q \equiv -1 \, mod \, 8$, then $T = \{1, 7\} \, mod \; 8$. Thus, p-regular F-conjugacy classes are $\{1\}$, $\{a, a^7\}$, $\{a^2, a^6\}$, $\{a^3, a^5\}$, $\{a^4\}$, $\{b\}$, $\{c\}$, $\{ab, a^7b\}$, $\{a^2b, a^6b\}$, $\{a^3b, a^5b\}$, $\{a^4b\}$, $\{ac, a^7c\}$, $\{a^2c, a^6c\}$, $\{a^3c, a^5c\}$, $\{a^4c\}$, $\{bc\}$, $\{abc, a^7bc\}$, $\{a^2bc, a^6bc\}$, $\{a^3bc, a^5bc\}$, $\{a^4bc\}$ and w=20. Hence $F[C_8 \times C_2 \times C_2] \cong F^8 \oplus F_2^{12}$.

3. If $q \equiv 3 \, mod \, 8$, then $T = \{1, 3\} \, mod \; 8$. Thus, p-regular F-conjugacy classes are $\{1\}$, $\{a, a^3\}$, $\{a^2, a^6\}$, $\{a^5, a^7\}$, $\{a^4\}$, $\{b\}$, $\{c\}$, $\{ab, a^3b\}$, $\{a^2b, a^6b\}$, $\{a^5b, a^7b\}$, $\{a^4b\}$, $\{ac, a^3c\}$, $\{a^2c, a^6c\}$, $\{a^5c, a^7c\}$, $\{a^4c\}$, $\{bc\}$, $\{abc, a^3bc\}$, $\{a^2bc, a^6bc\}$, $\{a^5bc, a^7bc\}$, $\{a^4bc\}$ and w=20. Hence $F[C_8 \times C_2 \times C_2] \cong F^8 \oplus F_2^{12}$.

4. If $q \equiv -3 \, mod \, 8$, then $T = \{1, 5\} \, mod \; 8$. Thus, p-regular F-conjugacy classes are $\{1\}$, $\{a, a^5\}$, $\{a^2\}$, $\{a^6\}$, $\{a^3, a^7\}$, $\{a^4\}$, $\{b\}$, $\{c\}$, $\{ab, a^5b\}$, $\{a^2b\}$, $\{a^6b\}$, $\{a^3b, a^7b\}$, $\{a^4b\}$, $\{ac, a^5c\}$, $\{a^2c\}$, $\{a^6c\}$, $\{a^3c, a^7c\}$, $\{a^4c\}$, $\{bc\}$, $\{abc, a^5bc\}$, $\{a^2bc\}$, $\{a^6bc\}$, $\{a^3bc, a^7bc\}$, $\{a^4bc\}$ and w=24. Hence $F[C_8 \times C_2 \times C_2] \cong F^{16} \oplus F_2^8$.
   Thus we have the result.

□

Theorem 5. *Let F be a finite field of characteristic $p > 0$ having $q = p^n$ elements and $G \cong C_4^2 \times C_2$.*

1. If $p = 2$. Then,

$$U(F[C_4^2 \times C_2]) \cong C_4^{3n} \times C_2^{25n} \times C_{2^n - 1}.$$

2. If $p \neq 2$. Then,

$$U(F[C_4^2 \times C_2]) \cong \begin{cases} C_{p^n-1}^{32}, & if \quad q \equiv 1 \, mod \; 4; \\ C_{p^n-1}^{8} \times C_{p^{2n}-1}^{12}, & if \quad q \equiv -1 \, mod \; 4. \end{cases}$$

Proof. The presentation of $G \cong C_4^2 \times C_2$ is given by

$$C_4^2 \times C_2 =< a, b, c \mid a^4 = b^4 = c^2 = 1,\ ab = ba, bc = cb, ac = ca > .$$

1. If $p = 2$, then FG is non-semisimple and $|F| = q = 2^n$. It is well known that $U(FG) \cong V(FG) \times F^*$ and $|V(FG)| = 2^{31n}$ as $dim_F J(FG) = 31$. Obviously exponent of $V(FG)$ is 4. Suppose $V(FG) \cong C_4^{l_1} \times C_2^{l_2}$ such that $2^{31n} = 4^{l_1} \times 2^{l_2}$. Now we will compute l_1 and l_2. Set $W = \Big\{\alpha \in \omega(G) : \alpha^2 = 0$ and there exists $\beta \in \omega(G)$, such that $\alpha = \beta^2\Big\}$. If $\alpha = \sum_{k=0}^{1}\sum_{j=0}^{3}\sum_{i=0}^{3} \alpha_{4(j+4k)+i} a^i b^j c^k \in \omega(G)$, then $\sum_{i=0}^{3} \alpha_{2(j+2k)+i} = 0$, for $j = 0, 1, 2, 3$ and $k = 0, 1$. Let $\beta = \sum_{k=0}^{1}\sum_{j=0}^{3}\sum_{i=0}^{3} \beta_{4(j+4k)+i} a^i b^j c^k$ such that $\alpha = \beta^2$. Now applying the conditions $\alpha^2 = 0$, $\alpha = \beta^2$ and by direct computation, we have $\alpha_i = 0$, for all $i \neq 0, 2, 8, 10$ and $\alpha_0 = \alpha_2$. Thus $W = \Big\{\alpha_0(1 + a^2) + (\alpha_8 + \alpha_{10}a^2)b^2, \alpha_0, \alpha_8, \alpha_{10} \in F\Big\}$. Therefore $|W| = 2^{3n}$, $l_1 = 3n$ and $l_2 = 25n$. Hence $V(FG) \cong C_4^{3n} \times C_2^{25n}$ and the result follows.

2. If $p \neq 2$, then $|F| = p^n$. Using the similar arguments as in Theorem 1, $F[C_4^2 \times C_2]$ is semisimple and m=4, $\sum_{i=1}^{r}[D_i : F] = 32$. By observation we have following possibilities for q:

(a) If $q \equiv 1\ mod\ 4$, then $t = 1$;
(b) If $q \equiv -1\ mod\ 4$, then $t = 2$.

Now we have the cases:

1. If $q \equiv 1\, mod\, 4$, then $T = \{1\}\, mod\ 4$. Thus, p-regular F-conjugacy classes are the conjugacy classes of $C_4^2 \times C_2$ and w=32. Hence $F[C_4^2 \times C_2] \cong F^{32}$.

2. If $q \equiv -1\, mod\, 4$, then $T = \{1, 3\}\, mod\ 4$. Thus, p-regular F-conjugacy classes are $\{1\}$, $\{a, a^3\}$, $\{a^2\}$, $\{b, b^3\}$, $\{b^2\}$, $\{c\}$, $\{ab, a^3b^3\}$, $\{ab^2, a^3b^2\}$, $\{ab^3, a^3b\}$, $\{a^2b, a^2b^3\}$, $\{a^2b^2\}$, $\{bc, b^3c\}$, $\{b^2c\}$, $\{abc, a^3b^3c\}$, $\{ab^2c, a^3b^2c\}$, $\{ab^3c, a^3bc\}$, $\{a^2bc, a^2b^3c\}$, $\{a^2b^2c\}$, $\{ac, a^3c\}$, $\{a^2c\}$ and w=20. Hence $F[C_4^2 \times C_2] \cong F^8 \oplus F_2^{12}$. Thus we have the result.

□

Theorem 6. *Let F be a finite field of characteristic $p > 0$ having $q = p^n$ elements and $G \cong C_4 \times C_2^3$.*

1. *If* $p = 2$. *Then,*

$$U(F[C_4 \times C_2^3]) \cong C_4^n \times C_2^{29n} \times C_{2^n-1}.$$

2. *If* $p \neq 2$. *Then,*

$$U(F[C_4 \times C_2^3]) \cong \begin{cases} C_{p^n-1}^{32}, & if \quad q \equiv 1 \, mod \; 4; \\ C_{p^n-1}^{16} \times C_{p^{2n}-1}^{8}, & if \quad q \equiv -1 \, mod \; 4. \end{cases}$$

Proof. The presentation of $G \cong C_4 \times C_2^3$ is given by

$C_4 \times C_2^3 =< a, b, c, d \mid a^4 = b^2 = c^2 = d^2 = 1, \; ab = ba, bc = cb, dc = cd, ad = da > .$

1. If $p = 2$, then FG is non-semisimple and $|F| = q = 2^n$. It is well known that $U(FG) \cong V(FG) \times F^*$ and $|V(FG)| = 2^{31n}$ as $dim_F J(FG) = 31$. Obviously exponent of $V(FG)$ is 4. Suppose $V(FG) \cong C_4^{l_1} \times C_2^{l_2}$ such that $2^{31n} = 4^{l_1} \times 2^{l_2}$. Now we will compute l_1 and l_2. Set

   $$W = \Big\{\alpha \in \omega(G) : \alpha^2 = 0 \, \text{and there exists} \, \beta \in \omega(G), \text{ such that} \, \alpha = \beta^2\Big\}.$$

   Let $\alpha = \sum_{s=0}^{1} \sum_{k=0}^{1} \sum_{j=0}^{1} \sum_{i=0}^{3} \alpha_{4(j+2(k+2s))+i} a^i b^j c^k d^s \in \omega(G)$ and $\beta = \sum_{s=0}^{1} \sum_{k=0}^{1} \sum_{j=0}^{1} \sum_{i=0}^{3} \beta_{4(j+2(k+2s))+i} a^i b^j c^k d^s$ such that $\alpha = \beta^2$. Now applying the conditions $\alpha^2 = 0$, $\alpha = \beta^2$ and by direct computation, we have $\alpha_i = 0$, for all $i \neq 0, 2$ and $\alpha_0 = \alpha_2$. Thus $W = \Big\{\alpha_0(1 + a^2), \alpha_0 \in F\Big\}$. Therefore $|W| = 2^n$, $l_1 = n$ and $l_2 = 29n$. Hence $V(FG) \cong C_4^n \times C_2^{29n}$ and the result follows.

2. If $p \neq 2$, then $|F| = p^n$. Using the similar arguments as in Theorem 1, $F[C_4 \times C_2^3]$ is semisimple and m=4, $\sum_{i=1}^{r}[D_i : F] = 32$. By observation we have following possibilities for q:

(a) If $q \equiv 1 \; mod \; 4$, then $t = 1$;
(b) If $q \equiv -1 \; mod \; 4$, then $t = 2$.

Now have the following cases:

1. If $q \equiv 1 \, mod \, 4$, then $T = \{1\} \, mod \; 4$. Thus, p-regular F-conjugacy classes are the conjugacy classes of $C_4 \times C_2^3$ and w=32. Hence $F[C_4 \times C_2^3] \cong F^{32}$.

2. If $q \equiv -1\, mod\, 4$, then $T = \{1, 3\}\, mod\ 4$. Thus, p-regular F-conjugacy classes are $\{1\}$, $\{a, a^3\}$, $\{a^2\}$, $\{b\}$, $\{c\}$, $\{d\}$, $\{ab, a^3b\}$, $\{a^2b\}$, $\{ac, a^3c\}$, $\{a^2c\}$, $\{ad, a^3d\}$, $\{a^2d\}$, $\{bc\}$, $\{cd\}$, $\{bd\}$, $\{abc, a^3bc\}$, $\{a^2bc\}$, $\{acd, a^3cd\}$, $\{a^2cd\}$, $\{abd, a^3bd\}$, $\{a^2bd\}$, $\{bcd\}$, $\{abcd, a^3bcd\}$, $\{a^2bcd\}$ and w=24. Hence $F[C_4 \times C_2^3] \cong F^{16} \oplus F_2^8$.

   Hence we have the result.

□

Theorem 7. *Let F be a finite field of characteristic $p > 0$ having $q = p^n$ elements and $G \cong C_2^5$.*

1. *If $p = 2$. Then, $U(F[C_2^5]) \cong C_2^{31n} \times C_{2^n-1}$.*
2. *If $p \neq 2$. Then,*

$U(F[C_2^5]) \cong C_{p^n-1}^{32}$, *if $q \equiv 1\, mod\, 2$.*

Proof. The presentation of $G \cong C_2^5$ is given by $C_2^5 =< a, b, c, d, e \mid a^2 = b^2 = c^2 = d^2 = e^2 = 1, ab = ba, bc = cb, dc = cd, ed = de, ea = ae >$.

1. If $p = 2$, then FG will be non-semisimple in this case and $|F| = q = 2^n$. Since $G \cong C_2^5$, therefore by Lemma 2, we have $U(FG) \cong C_2^{31n} \times C_{2^n-1}$.
2. If $p \neq 2$, then $|F| = p^n$. Using the similar arguments as in Theorem 1, $F[C_2^5]$ is semisimple and m=2, $\sum_{i=1}^{r}[D_i : F] = 32$. By observation we have $q \equiv 1\ mod\ 2$ and $t = 1$.

Hence $q \equiv 1\, mod\, 2$, implies $T = \{1\}\, mod\ 2$. Thus, p-regular F-conjugacy classes are the conjugacy classes of C_2^5 and w=32. Therefore, $F[C_2^5] \cong F^{32}$ and we have the result. □

References

[1] S. F. Ansari and M. Sahai, “Unit groups of group algebras of groups of order 20”, *Quaestiones Mathematicae*, vol. 44, no. 4, pp. 503-511, 2021.

[2] S. Bhatt and H. Chandra, "Structure of unit group of $F_{p^n}D_{60}$", *Asian-European Journal of Mathematics*, vol. 14, no. 5, Art, ID. 2150075, 2021.

[3] L. Creedon, “The unit group of small group algebras and the minimum counter example to the isomorphism problem”, *International Journal of Pure and Applied Mathematics*, vol. 49, no. 4, pp. 531-537, 2008.

[4] R. A. Ferraz, “Simple components of the center of F G/J(F G)”, *Communications in Algebra*, vol. 36, no. 9, pp. 3191-3199, 2008.

[5] G. Karpilovsky, *Unit groups of classical rings.* Oxford: Clarendon Press, 1988.

[6] G. Karpilovsky, *Group Representations: Introduction to group representations and characters*, vol. 1-B. Amsterdam: North-Holland, 1992.

[7] M. Khan, “Structure of the unit group of FD10”, *Serdica Mathematical Journal*, vol. 35, no. 1, pp. 15-24, 2009.

[8] M. Khan, R. K. Sharma, and J.B. Srivastava, "The Unit Group of FS_4," *Acta Mathematica Hungarica*, vol. 118, no. 1-2, pp. 105-113, 2008.

[9] N. Makhijani, R. K. Sharma, and J. B. Srivastava, “The unit group of algebra of circulant matrices”, *International Journal of Group Theory*, vol. 3, no. 4, pp. 13-16, 2014.

[10] N. Makhijani, R. K. Sharma, and J. B. Srivastava, "The unit group of $F_q[D_{30}]$", *Serdica Mathematical Journal*, vol. 41,no. 2-3, pp.185-198, 2015.

[11] C. P. Milies, and S. K. Sehgal, *An introduction to group rings.* Dordrecht: Kluwer Academic Publishers, 2002.

[12] M. Sahai and S. F. Ansari, “Unit groups of finite group algebras of Abelian groups of order at most 16”, *Asian-European Journal of Mathematics*, vol. 14, no. 03, Art. ID. 2150030, 2021.

[13] R. K. Sharma, J. B. Srivastava, and M. Khan, "The unit group of F A4", *Publicationes Mathematicae Debrecen*, vol. 71, no. 1-2, pp. 21-26, 2007.

[14] G. Tang and Y. Gao,"The unit group of F G of group with order 12", *International journal of pure and applied mathematics*, vol. 73, pp. 143-158, 2011.

[15] G. Tang, Y. Wei and Y. Li, "Unit groups of group algebras of some small groups", *Czechoslovak Mathematical Journal*, vol. 64, pp. 149-157, 2014.

Suchi Bhatt
Department of Mathematics and Scientific Computing,
M. M. M. University of Technology,
Gorakhpur
India
e-mail: 1995suchibhatt@gmail.com

and

Harish Chandra
Department of Mathematics and Scientific Computing,
M. M. M. University of Technology,
Gorakhpur
India
e-mail: hcmsc@mmmut.ac.in
Corresponding author