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Fixed point theorems for a class of extended JS contraction mappings over a generalized metric space with an application to fixed circle problem

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Abstract

In this paper we prove some generalized fixed point theorems for a class of contractive mappings over an extended JS-generalized metric space. Notions of weakly sensitive and strongly sensitive coefficient functions have been used here in proving fixed point theorems. Examples are given in strengthening the hypothesis of our established theorems. Moreover an application is given to fixed circle problem.

Key words and phrases: *Extended JS-generalized metric space, weakly sensitive and strongly sensitive coefficient functions, extended JS-quasi contraction mapping, fixed point.*

AMS subject classification (2010): *47H10; 54H25.*

1. Introduction

Fixed point theory has been an active research field over the last few decades for its numerous applications in several branches of Mathematics. S. Banach had been the most responsible distinguished person in proving the well known Contraction Principle Theorem [2] in fixed point theory. Since then many researchers had been attracted to this particular field and took attempts in developing fixed point theory by weakening the contractive conditions of mappings (See [15]) or by relaxing the structure of underlying spaces under consideration or with combination of both.

In the year 2014, M. Jleli and B. Samet introduced the concept of a generalized contraction mapping commonly known as JS-contraction [9] which generalizes the concept of famous Banach Contraction mapping. With an introduction of such concept many researchers have been succeeded to obtain several fixed point theorems by using different types of generalized JS-contraction mappings (See [10],[6],[13],[7],[12]).

Recently Roy and Saha have introduced the concept of extended JS-generalized metric space [16] which includes the concepts of many of our known metric like spaces and have been able to prove some fixed point theorems therein.

In this paper we now adhere to a class of some generalized contractive mappings namely extended JS-Ćirić-contraction mapping and extended JS-quasi-contraction mapping. With the help of such mappings we have been able to prove some fixed point theorems over extended JS-generalized metric spaces. Our fixed point theorems generalizes many of our known results. An example is cited in support of the hypothesis of one of our theorems. Furthermore an application to the fixed circle problem is given in relevance of our investigations in fixed point theorems on such spaces.

2. Preliminaries

Definition 2.1. *[1, 5] Let X be a nonempty set and s be a real number satisfying $s \geq 1$. A function $d : X \times X \to \mathbf{R}^+$ is a $b-$metric on X if for all $x, y, z \in X$, the following conditions hold*

1. *$d(x, y) = 0$ iff $x = y$*
2. *$d(x, y) = d(y, x)$ for all $x, y \in X$*
3. *$d(x, z) \leq s[d(x, y) + d(y, z)]$ for all $x, y, z \in X$.*

The space (X, d) is called a $b-$metric space.

Example 2.2. *[11] (a) Let $X = L_p[0, 1]$ be the space of all real functions*

$x(t), t \in [0,1]$ such that $\int_0^1 |x(t)|^p < \infty$ with $0 < p < 1$. Define $d : X^2 \to \mathbf{R}^+$ as:

$$d(x,y) = \left(\int_{t=0}^{1} |x(t) - y(t)|^p dt \right)^{\frac{1}{p}} \tag{2.1}$$

for all $x, y \in X$. Then d is a $b-$metric space with coefficient $s = 2^{1/p}$.

(b) Let $X = l_p(\mathbf{R})$ with $0 < p < 1$ where $l_p(\mathbf{R}) = \{\{x_n\} \subset \mathbf{R} : \sum_{n=1}^{\infty} |x_n|^p < \infty\}$. If we define $d : X^2 \to \mathbf{R}^+$ as:

$$d(x,y) = \left(\sum_{n=1}^{\infty} |x_n - y_n|^p \right)^{\frac{1}{p}} \tag{2.2}$$

for all $x = \{x_n\}, y = \{y_n\} \in X$ then d is a $b-$metric space with coefficient $s = 2^{1/p}$.

Definition 2.3. *[11] Let X be a nonempty set and $\theta : X \times X \to [1, \infty)$. A function $d_\theta : X^2 \to [0, \infty)$ is called an extended $b-$metric if for all $x, y, z \in X$ it satisfies:*
($d_\theta 1$) $d_\theta(x,y) = 0$ if and only if $x = y$,
($d_\theta 2$) $d_\theta(x,y) = d_\theta(y,x)$,
($d_\theta 3$) $d_\theta(x,z) \leq \theta(x,z)[d_\theta(x,y) + d_\theta(y,z)]$.
The pair (X, d_θ) is called an extended $b-$metric space.

Example 2.4. *[11] Let $X = \{1,2,3\}$. Define $\theta : X^2 \to \mathbf{R}^+$ and $d_\theta : X \times X \to \mathbf{R}^+$ as:*

$$\begin{aligned}
&\theta(x,y) = 1 + x + y \\
&d_\theta(1,1) = d_\theta(2,2) = d_\theta(3,3) = 0 \\
&d_\theta(1,2) = d_\theta(2,1) = 80, d_\theta(1,3) = d_\theta(3,1) = 1000, d_\theta(2,3) \\
&= d_\theta(3,2) = 600
\end{aligned} \tag{2.3}$$

Then (X, d_θ) is an extended $b-$metric space.

Example 2.5. *[11] Let $X = C([a,b], \mathbf{R})$ be the space of all continuous real valued functions define on [a,b]. Let us take $\theta(x,y) = \sup_{t\in[a,b]}(|x(t)| + |y(t)| + 2)$ and $d_\theta(x,y) = \sup_{t\in[a,b]} |x(t) - y(t)|^2$ for all $x, y \in X$. Then (X, d_θ) is an extended $b-$metric space.*

In general an extended $b-$metric is not continuous with respect to each of its components.

Example 2.6. *[11] Let $X = \mathbf{N} \cup \infty$ and let $d_\theta : X \times X \to \mathbf{R}$ be defined by:*

$$d_\theta(m,n) = \begin{cases} 0, & \text{if } m = n \\ |\frac{1}{m} - \frac{1}{n}|, & \text{if } m,n \text{ are even or } mn = \infty \\ 5, & \text{if } m,n \text{ are odd and } m \neq n \\ 2, & \text{otherwise} \end{cases} \tag{2.4}$$

Then (X, d_θ) is an extended $b-$metric for $\theta(x,y) = 3$ for all $x, y \in X$ but it is not continuous.

In [8] M. Jleli and B. Samet have given the following definitions in order to define a generalized metric space and to prove some fixed point theorems as cited below.

Let A be a non-empty set and $d : A \times A \to [0, \infty]$ be a mapping. For any $a \in A$, let us define the set

$$C(d, A, a) = \{\{a_n\} \subset A : \lim_{n\to\infty} d(a_n, a) = 0\}. \tag{2.5}$$

Definition 2.7. *[8] Let $d : A \times A \to [0, \infty]$ be a mapping which satisfies the following conditions:*

(i) $d(a,b) = 0$ implies $a = b$ for all $a, b \in A$;

(ii) for every $a, b \in A$, we have $d(a,b) = d(b,a)$;

(iii) if $(a,b) \in A \times A$ and $\{a_n\} \in C(d, A, a)$ then $d(a,b) \leq p \limsup_{n\to\infty} d(a_n, b)$, for some $p > 0$.

The pair (A, d) is called a generalized metric space, usually known as $JS-$metric space.

Example 2.8. *[17] Let $X = \mathbf{R} \cup \{-\infty, \infty\}$ and $d : X^2 \to [0, \infty]$ be defined by $d(x,y) = |x| + |y|$ for all $x, y \in X$, then clearly conditions (i) and (ii) are satisfied. For any $z \neq 0$, $S(J, X, z) = \emptyset$. If $z = 0$ then for $\{z_n\} \in S(J, X, 0)$, we have*

$$d(x, 0) \leq C \limsup_{n\to\infty} d(x, z_n)$$

for all $x \in X$ and for any $C \geq 1$. Then (iii) is also satisfied. So (X, d) is a $JS-$metric space.

Definition 2.9. *Let X be a nonempty set and $D : X^2 \to [0, \infty)$ be a distance function on X. Also let Ψ_1 be the collection of all functions ψ :*

$(0,\infty) \to (1,\infty)$, Ψ_2 be the collection of all functions $\psi : [0,\infty) \to [1,\infty)$ and $T : X \to X$ be a mapping, then T is

(i) a Ćirić contraction [4] if there exist nonnegative numbers q, r, s, t with $q + r + s + 2t < 1$ such that

$$D(Tx,Ty) \leq qD(x,y) + rD(x,Tx) + sD(y,Ty) + t[D(x,Ty) + D(y,Tx)]$$

for all $x, y \in X$,

(ii) a generalized Reich-contraction if there exists $k \in [0,1)$ such that

$$D(Tx,Ty) \leq k\max\{D(x,y), D(x,Tx), D(y,Ty)\} \text{ for all } x, y \in X,$$

(iii) a quasi-contraction of first kind [3] if there exists $k \in [0,1)$ such that

$$D(Tx,Ty) \leq k\max\{D(x,y), D(x,Tx), D(y,Ty), \frac{D(x,Ty) + D(y,Tx)}{2}\}$$

for all $x, y \in X$,

(iv) a quasi-contraction of second kind [3] if there exists $k \in [0,1)$ such that

$$D(Tx,Ty) \leq k\max\{D(x,y), D(x,Tx), D(y,Ty), D(x,Ty), D(y,Tx)\}$$

for all $x, y \in X$,

(v) a JS-contraction [9] if there exist $\psi \in \Psi_1$ and $q \in [0,1)$ such that

$$\psi(D(Tx,Ty)) \leq \psi(D(x,y))^q \text{ for all } x, y \in X \text{ with } D(Tx,Ty) \neq 0 (\text{or } Tx \neq Ty),$$

(vi) a JS-Ćirić-contraction [6] if there exist $\psi \in \Psi_2$ and $q, r, s, t \in [0,1)$ with $q + r + s + 2t < 1$ such that

$$\psi(D(Tx,Ty)) \leq \psi(D(x,y))^q \psi(D(x,Tx))^r \psi(D(y,Ty))^s \psi(D(x,Ty) + D(y,Tx))^t$$

for all $x, y \in X$,

(vii) a JS-Reich-contraction [10] if there exists $\psi \in \Psi_1$ and $k \in [0,1)$ such that

$$D(Tx,Ty) \leq \psi(\max\{D(x,y), D(x,Tx), D(y,Ty)\})^k$$

for all $x, y \in X$ with $D(Tx,Ty) \neq 0$(or $Tx \neq Ty$),

(viii) a JS-quasi-contraction [12] if there exists $\psi \in \Psi_1$ and $k \in [0,1)$ such that

$$D(Tx,Ty) \leq \psi(\max\{D(x,y), D(x,Tx), D(y,Ty), \frac{D(x,Ty)+D(y,Tx)}{2}\})^k$$

for all $x, y \in X$ with $D(Tx,Ty) \neq 0$(or $Tx \neq Ty$).

Let Θ be the collection of all functions $\theta : (0,\infty) \to (1,\infty)$ satisfying the following conditions:

($\theta 1$) θ is non-decreasing,

($\theta 2$) for each sequence $\{t_n\} \subset (0,\infty)$, $\lim_{n\to\infty} \theta(t_n) = 1$ iff $\lim_{n\to\infty} t_n = 0$,

($\theta 3$) there exists $r \in (0,1)$ and $l \in (0,\infty]$ such that $\lim_{t\to 0^+} \frac{\theta(t)-1}{t^r} = l$,

($\theta 4$) $\theta(a+b) \leq \theta(a)\theta(b)$,

($\theta 5$) θ is continuous.

Now we construct the following subsets of Θ:

$\Theta_1 = \{\theta : (0,\infty) \to (1,\infty) : \theta$ satisfies θ_1, θ_2 and $\theta_3\}$,

$\Theta_2 = \{\theta : (0,\infty) \to (1,\infty) : \theta$ satisfies $\theta_1, \theta_2, \theta_3$ and $\theta_5\}$, and

$\Theta_3 = \{\theta : (0,\infty) \to (1,\infty) : \theta$ satisfies θ_1 and $\theta_5\}$.

Theorem 2.10. *[9] Let (X,d) be a complete metric space and $T : X \to X$ be a JS-contraction for some $\psi \in \Theta_1$. Then T has a unique fixed point in X.*

Theorem 2.11. *[10] Let (X,d) be a complete metric space and $T : X \to X$ be a JS-Reich-contraction for some $\psi \in \Theta_2$. Then T has a unique fixed point in X.*

Theorem 2.12. *[12] Let (X,d) be a complete metric space and $T : X \to X$ be a JS-quasi-contraction for some $\psi \in \Theta_3$. Then T has a unique fixed point in X.*

Theorem 2.13. *[13] Let (X,d) be a complete metric space and $T : X \to X$ be a JS-Ćirić-contraction for some function $\psi : [0,\infty) \to [1,\infty)$ satisfying $\theta_1, \theta_2, \theta_3$ and θ_4 with $\psi(t) = 1$ iff $t = 0$. Then T has a unique fixed point in X.*

Now we give the definition of a newly introduced space called extended JS-generalized metric space. Next we discuss about such spaces (See [16]).

Let X be a nonempty set and $D^e_\theta : X^2 \to [0,\infty]$ be a given mapping. For any $x \in X$, let us define the set

$$S(D^e_\theta, X, x) = \{\{x_n\} \subset X : \lim_{n\to\infty} D^e_\theta(x_n, x) = 0\}. \tag{2.6}$$

Definition 2.14. *Let X be a nonempty set, $\theta : X^2 \to [1, \infty)$ be a function and $D^e_\theta : X^2 \to [0, \infty]$ satisfies the following conditions:*

($D^e_\theta 1$) $D^e_\theta(x, y) = 0$ implies $x = y$ for every $x, y \in X$;

($D^e_\theta 2$) $D^e_\theta(x, y) = D^e_\theta(y, x)$ for all $x, y \in X$;

($D^e_\theta 3$) if $(x, y) \in X^2$ and $\{x_n\} \in S(D^e_\theta, X, x)$, then

$$D^e_\theta(x, y) \leq \theta(x, y) \limsup_{n\to\infty} D^e_\theta(x_n, y).$$

The pair (X, D^e_θ) is called an extended JS-generalized metric space (shortly extn. JS-GMS).

Example 2.15. *Let $X = C[a, b]$ be the space of all real valued continuous functions defined on $[a, b]$. Let us take a mapping $\theta : X^2 \to [1, \infty)$ defined by $\theta(x, y) = \sup_{t\in[a,b]}(1+|x(t)-y(t)|)$ for all $x, y \in X$ and $D^e_\theta : X\times X \to \mathbf{R}$ defined by*

$$D^e_\theta(x, y) = \sup_{t\in[a,b]} (1 + |x(t) - y(t)|)|x(t) - y(t)| \tag{2.7}$$

for all $x, y \in X$. Then (X, D^e_θ) is clearly an extended JS-generalized metric space.

Example 2.16. *Let $X = \mathbf{R}$ be the space of reals. Let us take a mapping $\theta : X^2 \to [1, \infty)$ defined by $\theta(x, y) = 2 + |x| + |y|$ for all $x, y \in X$ and $D^e_\theta : X \times X \to \mathbf{R}$ defined by*

$$D^e_\theta(x, y) = (2 + |x| + |y|)(|x| + |y|) \tag{2.8}$$

for all $x, y \in X$. Then (X, D^e_θ) is precisely an extended JS-generalized metric space.

Remark 2.17. *(i) Any extended $b-$metric space is clearly an extn. JS-GMS. (ii) Every JS-metric space is also an extn. JS-GMS.*

Thus one can observe that extended JS-generalized metric space generalizes all the spaces such as (a) metric space, (b) $b-$metric space, (c) dislocated metric space, (d) modular metric space with the Fatou property, (e) extended $b-$metric space, (f) JS-metric space.

Definition 2.18. *Let (X, D^e_θ) be an extn. JS-GMS. Let $\{x_n\}$ be a sequence in X and $x \in X$.*

(i) $\{x_n\}$ is said to be convergent and converges to x if $\{x_n\} \in S(D^e_\theta, X, x)$.

(ii) $\{x_n\}$ is said to be Cauchy if $\lim_{n,m\to\infty} D^e_\theta(x_n, x_m) = 0$.

(iii) X is called complete if any Cauchy sequence in X is convergent.

Definition 2.19. *Let (X, D^e_θ) and (Y, d^e_θ) be two extn. JS-GMSs. A mapping $T : X \to Y$ is called continuous at a point $a \in X$ if for any $\epsilon > 0$ there exists $\delta_\epsilon > 0$ such that for any $x \in X$, $d^e_\theta(Tx, Ta) < \epsilon$ whenever $D^e_\theta(x, a) < \delta_\epsilon$.*

Proposition 2.20. *Let (X, D^e_θ) be an extn. JS-GMS. The following propositions are given in [16].*

(i) If $\{x_n\}$ converges to both x and y for $x, y \in X$ then $x = y$,

(ii) If $\{x_n\} \in S(D^e_\theta, X, x)$ then $D^e_\theta(x, x) = 0$,

(iii) If a Cauchy sequence $\{x_n\} \subset X$ has a convergent subsequence $\{x_{n_k}\}$ convergent to $x \in X$ such that $\{\theta(x, x_n)\}$ converges to some $h \in \mathbf{R}$ then $\{x_n\}$ is also convergent,

(iv) If T is a self mapping continuous at $a \in X$ then for any sequence $\{x_n\} \in S(D^e_\theta, X, a)$ we have $\{Tx_n\} \in S(D^e_\theta, X, Ta)$.

Remark 2.21. *Since any extended $b-$metric space is also an extended JS-generalized metric space, Example 2.6 shows that an extn. JS-GMS may not be continuous with respect to its components.*

Definition 2.22. *Let (X, D^e_θ) be an extn. JS-GMS. The open and closed balls centered at $x \in X$ and radius $r > 0$ in X are defined respectively as follows:*

$$
\begin{aligned}
B_{D^e_\theta}(x, r) &= \{y \in X : D^e_\theta(y, x) < r\}; \\
B_{D^e_\theta}[x, r] &= \{y \in X : D^e_\theta(y, x) \leq r\}.
\end{aligned} \tag{2.9}
$$

Remark 2.23. *(i) It may happen that in an extn. JS-GMS X, $x \notin B_{D^e_\theta}(x, r)$ for some $r > 0$ and $x \in X$. In Example 2.16 we see that $1 \notin B_{D^e_\theta}(1, 4)$. (ii) Let (X, D^e_θ) be an extn. JS-GMS. Let $\tau = \{\emptyset\} \cup \{U(\neq \emptyset) \subset X$: for any $x \in U$ there exists $r > 0$ such that $B_{D^e_\theta}(x, r) \subset U\}$. Then clearly τ forms a topology on X.*

Definition 2.24. *Let (X, D^e_θ) be an extn. JS-GMS and $F \subset X$. Then F is said to be closed if there exists an open set $U \subset X$ such that $F = U^c$.*

Definition 2.25. *Let (X, D^e_θ) be an extn. JS-GMS and $A \subset X$. Then $diam(A) = \sup\{D^e_\theta(a, b) : a, b \in A\}$.*

Definition 2.26. *In an extn. JS-GMS (X, D^e_θ), a sequence $\{F_n\}$ of subsets of X is said to be decreasing if $F_1 \supset F_2 \supset F_3 \supset \ldots$.*

Theorem 2.27. *Let (X, D^e_θ) be a complete extn. JS-GMS and $\{F_n\}$ be a decreasing sequence of nonempty closed subsets of X such that $diam(F_n) \to 0$ as $n \to \infty$. Then the intersection $\cap_{n=1}^{\infty} F_n$ contains exactly one point.*

Now we are in a position to prove our main Theorems. Before going to our main results we first define some generalized mappings.

3. Main Results

Let X be a nonempty set and $D : X^2 \to [0, \infty)$ be a distance function on X. Also let Ψ be the collection of all functions $\psi : [0, \infty) \to [1, \infty)$.

Definition 3.1. *A mapping $T : X \to X$ is called an extended JS-Ćirić-contraction if there exists some $\psi \in \Psi$ and $q, r, s, t \in [0, 1)$ with $q + r + s + 2t < 1$ such that*

$$\psi(D(Tx, Ty)) \leq \psi(D(x, y))^q \psi(D(x, Tx))^r \psi(D(y, Ty))^s [\psi(D(x, Ty))\psi(D(y, Tx))]^t \tag{3.1}$$

for all $x, y \in X$.

Definition 3.2. *A mapping $T : X \to X$ is called an extended JS-quasi-contraction if there exists some $\psi \in \Psi$ and $k \in [0, 1)$ such that*

$$\psi(D(Tx, Ty)) \leq \psi(M(x, y))^k \tag{3.2}$$

for all $x, y \in X$, where $M(x, y) = \max\{D(x, y), D(x, Tx), D(y, Ty), D(x, Ty), D(y, Tx)\}$.

Definition 3.3. *A mapping $T : X \to X$ is called an extended JS-Reich-contraction if there exists some $\psi \in \Psi$ and $k \in [0, 1)$ such that*

$$\psi(D(Tx, Ty)) \leq \psi(M^*(x, y))^k \tag{3.3}$$

for all $x, y \in X$, where $M^*(x, y) = \max\{D(x, y), D(x, Tx), D(y, Ty)\}$.

Definition 3.4. *Let (X, D^e_θ) be an extended JS-GMS and $T : X \to X$ be a mapping. Then the coefficient function $\theta : X^2 \to [1, \infty)$ is said to be*

(i) weakly sensitive with respect to T if $\sup\{\theta(x, Tx) : x \in X$ and $S(D^e_\theta, X, x) \neq \emptyset\} = 1$;

(ii) strongly sensitive with respect to T if $\sup\{\theta(x, Tx) : x \in X$ and $S(D^e_\theta, X, x) \neq \emptyset\} = 1$ and for any $x_0 \in X$ whenever $\{x_n\} = \{T^n x_0\} \in S(D^e_\theta, X, x)$ then $\theta(x, x_n) = 1$ for all $n \in \mathbf{N}$.

$^{\dagger}If\theta$ is strongly sensitive with respect to T then it is clearly weakly sensitive with respect to T but the converse is not true in general. The next example supports our contention.

Example 3.5. *Let $X = [0,1]$ and $T : X \to X$ be defined by $Tx = \frac{x}{2}$ for all $x \in X$. Also let $\theta : X^2 \to [1,\infty)$ be given by $\theta(x,y) = 1 + |y - \frac{x}{2}|$ and $D^e_\theta(x,y) = |x-y|$ for all $x,y \in X$. Then (X, D^e_θ) is clearly an extended JS-GMS. Here we see that for any $x \in X$, $\theta(x,Tx) = 1$. Also $\{\frac{1}{2^n}\} = \{T^n 1\} \in S(D^e_\theta, X, 0)$ but $\theta(0, \frac{1}{2^n}) = 1 + \frac{1}{2^n} > 1$ for all $n \geq 1$. Therefore θ is weakly sensitive with respect to T but not strongly sensitive with respect to T.*

Let Φ be the collection of all functions $\varphi : [0,\infty) \to [1,\infty)$ satisfying the following conditions:

(φ_1) φ is non-deceasing and $\varphi(t) = 1$ iff $t = 0$,

(φ_2) for each sequence $\{t_n\} \subset (0,\infty)$, $\lim_{n\to\infty} \varphi(t_n) = 1$ iff $\lim_{n\to\infty} t_n = 0$,

(φ_3) for any subset $A = \{a_\alpha : \alpha \in \Lambda\}$, Λ is an indexing set, of $[0,\infty)$ with $\sup_{\alpha\in\Lambda} a_\alpha = a < \infty$ and for any $M \geq 1$, if $\varphi(a_\alpha) \leq M$ for all $\alpha \in \Lambda$, then $\varphi(a) \leq M$,

(φ_4) for any sequence $\{x_n\} \subset [0,\infty)$ with $\limsup_{n\to\infty} x_n = x < \infty$, we have $\varphi(x) \leq \limsup_{n\to\infty} \varphi(x_n)$ and $\limsup_{n\to\infty} x_n = \infty$ implies $\limsup_{n\to\infty} \varphi(x_n) = \infty$.

In our next theorems we always assume that D^e_θ is defined from $X \times X$ to $[0,\infty)$.

Theorem 3.6. *Let (X, D^e_θ) be a complete extn. JS-GMS with θ is weakly sensitive with respect to T, where $T : X \to X$ be an extended JS-Reich-contraction for some $\varphi \in \Phi$. If there exists some $x_0 \in X$ such that $\delta(D^e_\theta, T, x_0) = \sup\{D^e_\theta(T^i x_0, T^j x_0) : i,j \in \mathbf{N}\} < \infty$ then T has a unique fixed point in X.*

Proof. Since T satisfies (3.3), for any $n \in \mathbf{N}$ we have

$$\begin{aligned} & \varphi(D^e_\theta(T^{n+i}x_0, T^{n+j}x_0)) \\ \leq\ & \varphi(\max\{D^e_\theta(T^{n-1+i}x_0, T^{n-1+j}x_0), D^e_\theta(T^{n-1+i}x_0, T^{n+i}x_0), \\ & D^e_\theta(T^{n-1+j}x_0, T^{n+j}x_0)\})^k \end{aligned} \tag{3.4}$$

for all $i, j \geq 1$. Let us take $\delta(D^e_\theta, T^{p+1}, x_0) = \sup\{D^e_\theta(T^{p+i}x_0, T^{p+j}x_0) : i, j \in \mathbf{N}\}$ for any non-negative integer p. From (3.4) we get $\varphi(D^e_\theta(T^{n+i}x_0, T^{n+j}x_0)) \leq \varphi(\delta(D^e_\theta, T^n, x_0))^k$ for all $i, j \geq 1$. Since $\delta(D^e_\theta, T^{p+1}, x_0) \leq \delta(D^e_\theta, T, x_0) < \infty$ for any $p \geq 1$, using the property (φ_3) of φ from (3.4) we have $\varphi(\delta(D^e_\theta, T^{n+1}, x_0)) \leq \varphi(\delta(D^e_\theta, T^n, x_0))^k$ and therefore

$$\begin{aligned} \varphi(\delta(D^e_\theta, T^{m+1}, x_0)) &\leq \varphi(\delta(D^e_\theta, T^m, x_0))^k \\ &\quad ... \\ &\leq \varphi(\delta(D^e_\theta, T, x_0))^{k^m} \end{aligned} \tag{3.5}$$

for any $m \in \mathbf{N}$. Therefore $\lim_{m\to\infty} \varphi(\delta(D^e_\theta, T^{m+1}, x_0)) = 1$. From the property (φ_2) of φ it follows that $\lim_{m\to\infty} \delta(D^e_\theta, T^{m+1}, x_0) = 0$. Thus for any $1 \leq n < m$ it implies that $D^e_\theta(T^n x_0, T^m x_0) \leq \delta(D^e_\theta, T^n, x_0)$ tending to 0 as n tending to infinity. Thus $\{T^n x_0\}$ is a Cauchy sequence in X. By the completeness of X there exists some $z \in X$ such that $\{T^n x_0\} \in S(D^e_\theta, X, z)$. Now for any $n \in \mathbf{N}$ we have

$$\begin{aligned} \varphi(D^e_\theta(T^{n+1}x_0, Tz)) &\leq \varphi(\max\{D^e_\theta(T^n x_0, z), D^e_\theta(z, Tz), D^e_\theta(T^n x_0, T^{n+1}x_0)\})^k \\ &= (\max\{\varphi(D^e_\theta(T^n x_0, z)), \varphi(D^e_\theta(z, Tz)), \\ &\qquad \varphi(D^e_\theta(T^n x_0, T^{n+1}x_0))\})^k. \end{aligned} \tag{3.6}$$

Taking $n \to \infty$ we get $\limsup_{n\to\infty} \varphi(D^e_\theta(T^{n+1}x_0, Tz)) \leq \varphi(D^e_\theta(z, Tz))^k$. By the property (φ_4) of φ we have $\varphi(\limsup_{n\to\infty} D^e_\theta(T^{n+1}x_0, Tz)) \leq \varphi(D^e_\theta(z, Tz))^k$. Now

$$\begin{aligned} D^e_\theta(z, Tz) &\leq \theta(z, Tz) \limsup_{n\to\infty} D^e_\theta(T^{n+1}x_0, Tz) \\ &= \limsup_{n\to\infty} D^e_\theta(T^{n+1}x_0, Tz), \end{aligned}$$

which implies that $\varphi(D^e_\theta(z, Tz)) \leq \varphi(\limsup_{n\to\infty} D^e_\theta(T^{n+1}x_0, Tz)) \leq \varphi(D^e_\theta(z, Tz))^k$. Since $k \in [0, 1)$, we get $\varphi(D^e_\theta(z, Tz)) = 1$ and therefore $D^e_\theta(z, Tz) = 0$, implying that $Tz = z$.

Let u be any fixed point of T. Then

$$\begin{aligned} \varphi(D^e_\theta(u, u)) &= \varphi(D^e_\theta(Tu, Tu)) \\ &\leq \varphi(\max\{D^e_\theta(u, u), D^e_\theta(u, Tu), D^e_\theta(u, Tu)\})^k \\ &= \varphi(D^e_\theta(u, u))^k. \end{aligned} \tag{3.7}$$

Therefore $\varphi(D^e_\theta(u,u)) = 1$ and using the property (φ_1) of φ we get $D^e_\theta(u,u) = 0$. Also we have

$$\begin{aligned} \varphi(D^e_\theta(z,u)) &= \varphi(D^e_\theta(Tz,Tu)) \\ &\leq \varphi(\max\{D^e_\theta(z,u), D^e_\theta(z,Tz), D^e_\theta(u,Tu)\})^k \\ &= \varphi(D^e_\theta(z,u))^k, \end{aligned} \tag{3.8}$$

which implies that $D^e_\theta(z,u) = 0$ and hence $z = u$. Thus T has a unique fixed point in X. □

Theorem 3.7. *Let (X, D^e_θ) be a complete extn. JS-GMS with θ is strongly sensitive with respect to T, where $T : X \to X$ be an extended JS-quasi-contraction for some $\varphi \in \Phi$. If there exists some $x_0 \in X$ such that $\delta(D^e_\theta, T, x_0) < \infty$ then T has a unique fixed point in X.*

Proof. Here T satisfies (3.2) therefore for any $n \in \mathbf{N}$ we have

$$\begin{aligned} &\varphi(D^e_\theta(T^{n+i}x_0, T^{n+j}x_0)) \\ \leq\ &\varphi(\max\{D^e_\theta(T^{n-1+i}x_0, T^{n-1+j}x_0), D^e_\theta(T^{n-1+i}x_0, T^{n+i}x_0), \\ &D^e_\theta(T^{n-1+j}x_0, T^{n+j}x_0), \\ &D^e_\theta(T^{n-1+i}x_0, T^{n+j}x_0), D^e_\theta(T^{n-1+j}x_0, T^{n+i}x_0)\})^k \end{aligned} \tag{3.9}$$

for all $i, j \geq 1$. Let us take $\delta(D^e_\theta, T^{p+1}, x_0) = \sup\{D^e_\theta(T^{p+i}x_0, T^{p+j}x_0) : i, j \in \mathbf{N}\}$ for any non-negative integer p. From (3.9) we get $\varphi(D^e_\theta(T^{n+i}x_0, T^{n+j}x_0)) \leq \varphi(\delta(D^e_\theta, T^n, x_0))^k$ for all $i, j \geq 1$. Since $\delta(D^e_\theta, T^{p+1}, x_0) \leq \delta(D^e_\theta, T, x_0) < \infty$ for any $p \geq 1$, using the property (φ_3) of φ from (3.9) we have $\varphi(\delta(D^e_\theta, T^{n+1}, x_0)) \leq \varphi(\delta(D^e_\theta, T^n, x_0))^k$. Proceeding in a similar fashion as above we see that $\{T^n x_0\}$ is convergent and converges to some $z \in X$ (say).

Now for any $m \in \mathbf{N}$, we have

$$\begin{aligned} D^e_\theta(T^m x_0, z) &= D^e_\theta(z, T^m x_0) \\ &\leq \theta(z, T^m x_0) \limsup_{k\to\infty} D^e_\theta(T^{m+k}x_0, T^m x_0) \\ &= \limsup_{k\to\infty} D^e_\theta(T^{m+k}x_0, T^m x_0) \\ &\leq \delta(D^e_\theta, T^m, x_0). \end{aligned} \tag{3.10}$$

Which implies that

$$\begin{aligned}\varphi(D^e_\theta(T^m x_0, z)) &\leq \varphi(\delta(D^e_\theta, T^m, x_0))\\ &\quad ...\\ &\leq \varphi(\delta(D^e_\theta, T, x_0))^{k^{m-1}} \text{ for all } m \geq 1.\end{aligned} \tag{3.11}$$

For a fixed $n \in \mathbf{N}$ we have

$$\begin{aligned}\varphi(D^e_\theta(T^{n+1}x_0, Tz)) &\leq \varphi(\max\{D^e_\theta(T^n x_0, z), D^e_\theta(T^n x_0, T^{n+1}x_0), D^e_\theta(z, Tz),\\ &\quad D^e_\theta(z, T^{n+1}x_0), D^e_\theta(T^n x_0, Tz)\})^k\\ &= (\max\{\varphi(D^e_\theta(T^n x_0, z)), \varphi(D^e_\theta(T^n x_0, T^{n+1}x_0)),\\ &\quad \varphi(D^e_\theta(z, Tz)),\\ &\quad \varphi(D^e_\theta(z, T^{n+1}x_0)), \varphi(D^e_\theta(T^n x_0, Tz))\})^k\\ &= \max\{\varphi(D^e_\theta(T^n x_0, z))^k, \varphi(D^e_\theta(T^n x_0, T^{n+1}x_0))^k,\\ &\quad \varphi(D^e_\theta(z, Tz))^k,\\ &\quad \varphi(D^e_\theta(z, T^{n+1}x_0))^k, \varphi(D^e_\theta(T^n x_0, Tz))^k\}\\ &\leq \max\{\varphi(\delta(D^e_\theta, T, x_0))^{k^n}, \varphi(D^e_\theta(z, Tz))^k,\\ &\quad \varphi(\delta(D^e_\theta, T, x_0))^{k^{n+1}},\\ &\quad \varphi(D^e_\theta(T^n x_0, Tz))^k\}\\ &= \max\{\varphi(\delta(D^e_\theta, T, x_0))^{k^n}, \varphi(D^e_\theta(z, Tz))^k, \varphi(D^e_\theta(T^n x_0, Tz))^k\}.\end{aligned} \tag{3.12}$$

In a similar way we get

$$\varphi(D^e_\theta(T^n x_0, Tz)) \leq \max\{\varphi(\delta(D^e_\theta, T, x_0))^{k^{n-1}}, \varphi(D^e_\theta(z, Tz))^k, \varphi(D^e_\theta(T^{n-1}x_0, Tz))^k\}. \tag{3.13}$$

From (3.12) and (3.13) it follows that

$$\varphi(D^e_\theta(T^{n+1}x_0, Tz)) \leq \max\{\varphi(\delta(D^e_\theta, T, x_0))^{k^n}, \varphi(D^e_\theta(z, Tz))^k, \varphi(D^e_\theta(T^{n-1}x_0, Tz))^{k^2}\}. \tag{3.14}$$

Proceeding in a similar way we get

$$\varphi(D^e_\theta(T^{n+1}x_0, Tz)) \leq \max\{\varphi(\delta(D^e_\theta, T, x_0))^{k^n}, \varphi(D^e_\theta(z, Tz))^k, \varphi(D^e_\theta(x_0, Tz))^{k^{n+1}}\}. \tag{3.15}$$

Taking $n \to \infty$ it follows that $\limsup_{n\to\infty} \varphi(D^e_\theta(T^{n+1}x_0, Tz)) \leq \varphi(D^e_\theta(z, Tz))^k$. By a similar calculation as in Theorem 3.7 we have $\varphi(D^e_\theta(z, Tz)) = 1$ since $k < 1$. Which implies that $D^e_\theta(z, Tz) = 0$ and therefore $Tz = z$.

Let u be any fixed point of T. Then

$$\begin{aligned}\varphi(D^e_\theta(u,u)) &= \varphi(D^e_\theta(Tu,Tu))\\ &\leq \varphi(\max\{D^e_\theta(u,u), D^e_\theta(u,Tu), D^e_\theta(u,Tu),\\ &\quad D^e_\theta(u,Tu), D^e_\theta(u,Tu)\})^k\\ &= \varphi(D^e_\theta(u,u))^k. \qquad (3.16)\end{aligned}$$

Therefore $\varphi(D^e_\theta(u,u)) = 1$ and using the property (φ_1) of φ we get $D^e_\theta(u,u) = 0$. Also we have

$$\begin{aligned}\varphi(D^e_\theta(z,u)) &= \varphi(D^e_\theta(Tz,Tu))\\ &\leq \varphi(\max\{D^e_\theta(z,u), D^e_\theta(z,Tz), D^e_\theta(u,Tu),\\ &\quad D^e_\theta(z,Tu), D^e_\theta(u,Tz)\})^k\\ &= \varphi(D^e_\theta(z,u))^k, \qquad (3.17)\end{aligned}$$

which implies that $D^e_\theta(z,u) = 0$ and hence $z = u$. Thus T has a unique fixed point in X. □

As a consequence of Theorem 3.8 we get the next Theorem.

Theorem 3.8. *Let (X, D^e_θ) be a complete extn. JS-GMS with θ is strongly sensitive with respect to T, where $T : X \to X$ be an extended JS-Ćirić-contraction for some $\varphi \in \Phi$. If there exists some $x_0 \in X$ such that $\delta(D^e_\theta, T, x_0) < \infty$ then T has a unique fixed point in X.*

Proof. Since T is an extended JS-Ćirić-contraction, it follows that there exists some $\psi \in \Psi$ and $q, r, s, t \in [0, 1)$ with $q + r + s + 2t < 1$ such that

$$\begin{aligned}\psi(D^e_\theta(Tx,Ty)) &\leq \psi(D^e_\theta(x,y))^q \psi(D^e_\theta(x,Tx))^r \psi(D^e_\theta(y,Ty))^s\\ &\quad [\psi(D^e_\theta(x,Ty))\psi(D^e_\theta(y,Tx))]^t\\ &\leq \psi(M^e_\theta(x,y))^{q+r+s+2t}, \qquad (3.18)\end{aligned}$$

where $M^e_\theta(x,y) = \max\{D^e_\theta(x,y), D^e_\theta(x,Tx), D^e_\theta(y,Ty), D^e_\theta(x,Ty), D^e_\theta(y,Tx)\}$ for all $x, y \in X$. Therefore T is an extended JS-quasi-contraction for $\psi \in \Psi$. Thus all the conditions of Theorem 3.8 are satisfied and therefore T has a unique fixed point in X. □

Now we prove the following proposition.

Proposition 3.9. *Let Δ^* be the collection of all functions $\psi : [0,\infty) \to [1,\infty)$ satisfying the following conditions:*

(i) ψ is non-decreasing and $\psi(t) = 1$ if and only if $t = 0$;

(ii) for each sequence $\{t_n\} \subset (0, \infty)$, $\lim_{n\to\infty} \psi(t_n) = 1$ *iff* $\lim_{n\to\infty} t_n = 0$;

(iii) $\psi(a+b) \leq \psi(a)\psi(b)$ *for all* $a, b > 0$.

(iv) $\lim_{t\to\infty} \psi(t) = \infty$. *Then* $\Delta^* \subset \Phi$.

Proof. There are nothing to prove for the first and second conditions.

Let $A = \{a_\alpha : \alpha \in \Lambda\} \subset [0, \infty)$ be such that $a = \sup_{\alpha\in\Lambda} a_\alpha < \infty$ and for some $M \geq 1$, $\psi(a_\alpha) \leq M$ for all $\alpha \in \Lambda$. We have to show that $\psi(a) \leq M$.

If $a = 0$ then it holds clearly. So we take $a > 0$. Then for any $\epsilon > 0$ there exists some $a_\alpha \in A$ such that $a - \epsilon < a_\alpha$. If $a_\alpha = 0$ then $\psi(a) \leq \psi(\epsilon) \leq \psi(\epsilon)M$. Also if $a_\alpha > 0$ then we have $\psi(a) \leq \psi(\epsilon + a_\alpha) \leq \psi(\epsilon)\psi(a_\alpha) \leq \psi(\epsilon)M$. Thus by taking $\epsilon \to 0$ we have $\psi(a) \leq M$.

For the fourth condition let us first show that for any $\{t_n\} \subset [0, \infty)$ with $t = \lim_{n\to\infty} t_n < \infty$, $\psi(t) \leq \limsup_{n\to\infty} \psi(t_n)$. So let $\{x_n\} \subset [0, \infty)$ with $x = \lim_{n\to\infty} x_n < \infty$. Then for any $\epsilon > 0$ there exists some $N \in \mathbf{N}$ such that $|x_n - x| < \epsilon$ whenever $n \geq N$.

If $x = 0$ then $\psi(x) = 1$. Then clearly $\psi(x) \leq \limsup_{n\to\infty} \psi(x_n)$. So without loss of generality let us take $x > 0$. Then we can assume that $x_n > 0$ for all $n \geq 1$ and we have $x - \epsilon < x_n < x + \epsilon$ for all $n \geq N$ implying that $\psi(x) \leq \psi(x_n + \epsilon) \leq \psi(\epsilon)\psi(x_n)$ for all $n \geq N$. Therefore

$$\begin{aligned} \frac{\psi(x)}{\psi(\epsilon)} &\leq \psi(x_n) \text{ for all } n \geq N \\ \Rightarrow \frac{\psi(x)}{\psi(\epsilon)} &\leq \limsup_{n\to\infty} \psi(x_n). \end{aligned} \tag{3.19}$$

Taking $\epsilon \to 0^+$ we have $\psi(x) \leq \limsup_{n\to\infty} \psi(x_n)$. Now we come to our main proof of condition (φ_4). For this let us take a sequence $\{y_n\} \subset [0, \infty)$ with $y = \limsup_{n\to\infty} y_n < \infty$. Without loss of generality let us take $y > 0$ and $y_n > 0$ for all $n \geq 1$. Then for a chosen $\epsilon > 0$ there exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that $y - \epsilon < \lim_{k\to\infty} y_{n_k}$.

If $\lim_{k\to\infty} y_{n_k} = 0$ then $\psi(y) \leq \psi(\epsilon) \leq \psi(\epsilon) \limsup_{n\to\infty} \psi(y_n)$. Also if $\lim_{k\to\infty} y_{n_k} > 0$ then $\psi(y) \leq \psi(\epsilon)\psi(\lim_{k\to\infty} y_{n_k}) \leq \psi(\epsilon) \limsup_{k\to\infty} \psi(y_{n_k}) \leq \psi(\epsilon) \limsup_{n\to\infty} \psi(y_n)$. Taking $\epsilon \to 0^+$ we get $\psi(y) \leq \limsup_{n\to\infty} \psi(y_n)$.

Also let $\{x_n\} \subset [0, \infty)$ be such that $\limsup_{n\to\infty} x_n = \infty$. So either there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\lim_{k\to\infty} x_{n_k} = \infty$ or for any $m \geq 1$ there exists $\{x_{m_k}\} \subset \{x_n\}$ such that $m \leq \lim_{k\to\infty} x_{m_k} < \infty$.

For the first case we have for all $G > 0$ there exists $N \in \mathbf{N}$ such that $x_{n_k} > G$ for all $k \geq N$. Thus $\psi(G) \leq \psi(x_{n_k})$ for all $k \geq N$. So $\psi(G) \leq \limsup_{k\to\infty} \psi(x_{n_k}) \leq \limsup_{n\to\infty} \psi(x_n)$ for all $G > 0$. Since $\lim_{t\to\infty} \psi(t) =$

∞ we have $\limsup_{n\to\infty} \psi(x_n) = \infty$.
Also for the second case we have
$\psi(m) \leq \psi(\lim_{k\to\infty} x_{m_k}) \leq \limsup_{k\to\infty} \psi(x_{m_k}) \leq \limsup_{n\to\infty} \psi(x_n)$ for all $m \in \mathbf{N}$. Therefore similarly as in the first case we get
$\limsup_{n\to\infty} \psi(x_n) = \infty$. Hence $\Delta^* \subset \Phi$. $\square$

Remark 3.10. *It is to be noted that $\Delta^* \underset{\neq}{\subset} \Phi$, since for any fixed $n \geq 1$ the function ϕ_n defined by $\phi_n(t) = e^{t^n e^t}$ for all $t \in [0, \infty)$ does not belong to Δ^*.*

In Theorem 4 of [7] the authors assume only first three conditions of Δ^*. Let us denote the collection of all functions ψ satisfying only first three conditions of Δ^* as Δ. Then $\Delta^* \subset \Delta$.

Remark 3.11. *If we take $\psi(t) = \begin{cases} e^t, & \text{if } \ t \in [0,1) \\ e, & \text{if } \ t \geq 1 \end{cases}$ then $\psi \in \Delta$ but clearly it does not belongs to Δ^*. Thus $\Delta^* \underset{\neq}{\subset} \Delta$.*

Our Theorem 3.9 generalizes Theorem 4 of [7] partially.

Theorem 3.12. *Let (X, d) be a complete metric space and $T : X \to X$ be a JS-Ćirić-contraction with $\psi \in \Delta^*$. Then T has a unique fixed point in X.*

Proof. Since T is a JS-Ćirić-contraction with $\psi \in \Delta^*$, then by using the property (iii) of ψ we get

$$\begin{aligned} & \psi(d(Tx, Ty)) \\ \leq \ & \psi(d(x, y))^q \psi(d(x, Tx))^r \psi(d(y, Ty))^s \psi(d(x, Ty) + d(y, Tx))^t \\ \leq \ & \psi(d(x, y))^q \psi(d(x, Tx))^r \psi(d(y, Ty))^s \psi(d(x, Ty))^t \psi(d(y, Tx))^t \\ & \text{ for all } x, y \in X, \end{aligned}$$

where $q, r, s, t \in [0, 1)$ with $q + r + s + 2t < 1$. Therefore T is an extended JS-Ćirić-contraction with $\psi \in \Delta^* \subset \Phi$. Also clearly a metric space X is an extended JS-GMS with $\theta(x, y) = 1$ for all $x, y \in X$. Hence all the conditions of Theorem 3.9 are satisfied and T has a unique fixed point in X. $\square$

Now we give an example in support of our Theorem 3.8.

Example 3.13. *Let* $X = \{\tau_n \in \mathbf{N}\}$ *and* $D^e_\theta : X \times X \to [0, \infty)$ *be defined by* $D^e_\theta(x, y) = |x - y|$ *for all* $x, y \in X$*, where* $\tau_n = \frac{n(n+1)(n+2)}{3}$ *for all* $n \in \mathbf{N}$*. Let us define a mapping* $T : X \to X$ *by* $T\tau_1 = T\tau_2 = \tau_1$ *and* $T\tau_n = \tau_{n-1}$ *for all* $n \geq 3$*.*

Now we show that T *is an extended* $JS-$*quasi-contraction for* $\phi \in \Phi$ *defined by* $\phi(t) = e^{t^2 e^t}$ *for all* $t \in [0, \infty)$*. For this, it suffices to show that there exists* $k \in (0, 1)$ *such that for all* $x, y \in X$

$$\frac{D^e_\theta(Tx, Ty)^2}{M^e_\theta(x, y)^2} e^{D^e_\theta(Tx,Ty) - M^e_\theta(x,y)} \leq k, \text{ where}$$

$$M^e_\theta(x, y) = \max\{D^e_\theta(x, y), D^e_\theta(x, Tx), D^e_\theta(y, Ty), D^e_\theta(x, Ty), D^e_\theta(y, Tx)\}. \tag{3.20}$$

Now we consider the following two cases.

Case-I: *In this case we take* $\tau_n, \tau_m \in X$ *where* $n = 1$ *and* $m > 2$*. Then* $D^e_\theta(T\tau_1, T\tau_m) = D^e_\theta(\tau_1, \tau_{m-1}) = [\frac{m(m^2-1)}{3} - 2]$*.*
Also $M^e_\theta(\tau_1, \tau_m) = \max\{D^e_\theta(\tau_1, \tau_m), D^e_\theta(\tau_1, T\tau_1), D^e_\theta(\tau_m, T\tau_m), D^e_\theta(\tau_1, T\tau_m), D^e_\theta(\tau_m, T\tau_1)\} = \max\{D^e_\theta(\tau_1, \tau_m), D^e_\theta(\tau_{m-1}, \tau_m), D^e_\theta(\tau_1, \tau_{m-1})\} = D^e_\theta(\tau_1, \tau_m) = [\frac{m(m+1)(m+2)}{3} - 2]$*. Therefore* $D^e_\theta(T\tau_1, T\tau_m) - M^e_\theta(\tau_1, \tau_m) = -m(m+1)$ *and thus*

$$\begin{aligned} &\frac{D^e_\theta(T\tau_1, T\tau_m)^2}{M^e_\theta(\tau_1, \tau_m)^2} e^{D^e_\theta(T\tau_1,T\tau_m) - M^e_\theta(\tau_1,\tau_m)} \\ = &\frac{(m(m^2-1) - 6)^2}{(m(m+1)(m+2) - 6)^2} e^{-m(m+1)} < e^{-4}. \end{aligned} \tag{3.21}$$

Case-II: *In this case we take* $\tau_n, \tau_m \in X$ *where* $m > n > 1$*. Then* $D^e_\theta(T\tau_n, T\tau_m) = D^e_\theta(\tau_{n-1}, \tau_{m-1}) = [\frac{m(m^2-1)}{3} - \frac{n(n^2-1)}{3}]$*.*
Now $M^e_\theta(\tau_n, \tau_m) = \max\{D^e_\theta(\tau_n, \tau_m), D^e_\theta(\tau_n, T\tau_n), D^e_\theta(\tau_m, T\tau_m), D^e_\theta(\tau_n, T\tau_m), D^e_\theta(\tau_m, T\tau_n)\} = \max\{D^e_\theta(\tau_n, \tau_m), D^e_\theta(\tau_{n-1}, \tau_n), D^e_\theta(\tau_{m-1}, \tau_m), D^e_\theta(\tau_n, \tau_{m-1}), D^e_\theta(\tau_m, \tau_{n-1})\} = D^e_\theta(\tau_{n-1}, \tau_m) = [\frac{m(m+1)(m+2)}{3} - \frac{n(n^2-1)}{3}]$*. Thus* $D^e_\theta(T\tau_n, T\tau_m) - M^e_\theta(\tau_n, \tau_m) = -m(m+1)$ *and therefore*

$$\begin{aligned} &\frac{D^e_\theta(T\tau_n, T\tau_m)^2}{M^e_\theta(\tau_n, \tau_m)^2} e^{D^e_\theta(T\tau_n,T\tau_m) - M^e_\theta(\tau_n,\tau_m)} \\ = &\frac{(m(m^2-1) - n(n^2-1))^2}{(m(m+1)(m+2) - n(n^2-1))^2} e^{-m(m+1)} < e^{-4}. \end{aligned} \tag{3.22}$$

Since $e^{-4} < 1$ *it follows that* T *is an extended* $JS-$*quasi-contraction with* $\phi(t) = e^{t^2 e^t}$*. Here we see that* (X, D^e_θ) *is a complete extn. JS-GMS with* θ

is strongly sensitive with respect to T, $\delta(D^e_\theta, T, \tau_2) < \infty$ *and* T *has a unique fixed point in* X.

4. Applications to fixed-circle problems

In this section we give some existence and uniqueness theorems for fixed circles of self mappings (One can refer to [14]) on an extended JS-GMS. The notions of a circle and a fixed circle is given as follows:

Definition 4.1. *Let* (X, D^e_θ) *be an extn. JS-GMS and* $C^{D^e_\theta}_{x_0,r} = \{x \in X : D^e_\theta(x, x_0) = r\}$ *be a circle with center at* $x_0 \in X$ *and radius* $r > 0$. *This circle is said to be a fixed circle of a mapping* $T : X \to X$ *if* $Tx = x$ *for every* $x \in C^{D^e_\theta}_{x_0,r}$.

Now we present an existence theorem for a fixed circle of a self mapping T.

Theorem 4.2. *Let* (X, D^e_θ) *be an extn. JS-GMS and* $C^{D^e_\theta}_{x_0,r}$ *be a circle on* X. *Let us define the mapping*

$$\mathcal{F} : X \to [0, \infty),\ \mathcal{F}(x) = D^e_\theta(x, x_0) \tag{4.1}$$

for all $x \in X$. *If there exists a self mapping* $T : X \to X$ *satisfying* $(C1)$ $\frac{D^e_\theta(x,Tx)}{\theta(x,Tx)} \leq [\mathcal{F}(x) + \mathcal{F}(Tx) - 2r]$ *and* $(C2)$ $D^e_\theta(Tx, x_0) \leq r$ *for all* $x \in C^{D^e_\theta}_{x_0,r}$, *then the circle* $C^{D^e_\theta}_{x_0,r}$ *is a fixed circle of* T.

Proof. Let us take an arbitrary element $x \in C^{D^e_\theta}_{x_0,r}$. We show that $Tx = x$. Using the condition $(C1)$ we obtain

$$\begin{aligned} D^e_\theta(x, Tx) &\leq \theta(x, Tx)[\mathcal{F}(x) + \mathcal{F}(Tx) - 2r] \\ &= \theta(x, Tx)[D^e_\theta(x, x_0) + D^e_\theta(Tx, x_0) - 2r] \\ &= \theta(x, Tx)[D^e_\theta(Tx, x_0) - r] \end{aligned} \tag{4.2}$$

From the condition $(C2)$ we have two cases. If $D^e_\theta(Tx, x_0) < r$ then (4.2) leads us to a contradiction. Therefore $D^e_\theta(Tx, x_0) = r$ and we have $D^e_\theta(x, Tx) \leq \theta(x, Tx)[D^e_\theta(Tx, x_0) - r] = \theta(x, Tx)[r - r] = 0$ implying that $D^e_\theta(x, Tx) = 0$ that is $Tx = x$. Hence the circle $C^{D^e_\theta}_{x_0,r}$ is a fixed circle of T. $\square$

Example 4.3. *Let $X = [0,2]$, $D^e_\theta(x,y) = (x-y)^2$ and $\theta(x,y) = 2$ for all $x, y \in X$. Then clearly (X, D^e_θ) is an extn. JS-GMS. Also let $T : X \to X$ be defined by*

$$T(x) = \begin{cases} 1 - \frac{1}{\sqrt{2}}, & \text{if } x \in [0,1) \\ 1, & \text{if } x = 1 \\ 1 + \frac{1}{\sqrt{2}}, & \text{if } x \in (1,2] \end{cases} \tag{4.3}$$

Then T satisfies all the conditions of Theorem 4.2 for the circle $C^{D^e_\theta}_{1,\frac{1}{2}}$ and thus $C^{D^e_\theta}_{1,\frac{1}{2}}$ is a fixed circle of T.

Example 4.4. *Let $X = [0,1]$, $D^e_\theta(x,y) = (x-y)^2$ and $\theta(x,y) = 2$ for all $x, y \in X$. Then clearly (X, D^e_θ) is an extn. JS-GMS. Also let $T : X \to X$ be defined by*

$$T(x) = \begin{cases} x, & \text{if } x \in \{\frac{1}{6}, \frac{5}{6}\} \\ \frac{2}{5}, & \text{otherwise} \end{cases} \tag{4.4}$$

Then T satisfies all the conditions of Theorem 4.2 for the circle $C^{D^e_\theta}_{\frac{1}{2},\frac{1}{9}}$ and thus $C^{D^e_\theta}_{\frac{1}{2},\frac{1}{9}}$ is a fixed circle of T.

Theorem 4.5. *Let (X, D^e_θ) be an extn. JS-GMS with D^e_θ finite and $T : X \to X$ be a mapping satisfying the conditions $(C1)$ and $(C2)$ given in Theorem 4.2. If T satisfies the contractive condition*

$$\varphi(D^e_\theta(Tx, Ty)) \leq \varphi(M^e_\theta(x,y))^k \tag{4.5}$$

for all $x \in C^{D^e_\theta}_{x_0,r}$ and $y \in X \backslash C^{D^e_\theta}_{x_0,r}$, where $C^{D^e_\theta}_{x_0,r}$ is a fixed circle of T, $k \in [0,1)$, $\varphi \in \Phi$ and $M^e_\theta(x,y) = \max\{D^e_\theta(x,y), D^e_\theta(x,Tx), D^e_\theta(y,Ty), D^e_\theta(x,Ty), D^e_\theta(y,Tx)\}$ for any $x, y \in X$, then T has a unique fixed circle in X.

Proof. Since T satisfies the conditions $(C1)$ and $(C2)$, it is guaranteed that T has atleast one fixed circle in X. Let us assume that T has two different fixed circles $C^{D^e_\theta}_{x_0,r_0}$ and $C^{D^e_\theta}_{x_1,r_1}$ in X. Then either $C^{D^e_\theta}_{x_0,r_0} \backslash C^{D^e_\theta}_{x_1,r_1} \neq \emptyset$ or $C^{D^e_\theta}_{x_1,r_1} \backslash C^{D^e_\theta}_{x_0,r_0} \neq \emptyset$. Without loss of generality let us take $C^{D^e_\theta}_{x_1,r_1} \backslash C^{D^e_\theta}_{x_0,r_0} \neq \emptyset$. Then we can choose some $u \in C^{D^e_\theta}_{x_0,r_0}$ and $v \in C^{D^e_\theta}_{x_1,r_1} \backslash C^{D^e_\theta}_{x_0,r_0}$ and by applying contraction condition (4.5) we have

$$\begin{aligned} \varphi(D^e_\theta(u,v)) &= \varphi(D^e_\theta(Tu,Tv)) \\ &\leq \varphi(M^e_\theta(u,v))^k \\ &= \varphi(\max\{D^e_\theta(u,v), D^e_\theta(u,Tu), D^e_\theta(v,Tv), D^e_\theta(u,Tv), D^e_\theta(v,Tu)\})^k \\ &= \varphi(D^e_\theta(u,v))^k. \qquad (4.6) \end{aligned}$$

Which implies that $\varphi(D_\theta^e(u,v)) = 1$ that is $D_\theta^e(u,v) = 0 \Rightarrow u = v$, a contradiction. Therefore our assumption is wrong and T has a unique fixed circle in X. □

5. Declaration

I had submitted this manuscript to the journal when I was a research scholar in the Department of Mathematics, The University of Burdwan under the supervision of Prof. Mantu Saha; but now my address/affiliation changed. Therefore I have changed my (author) address to the current one.

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