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Square root stress-sum index for graphs

K. B. Mahesh[1]
R. Rajendra[2] orcid.org/0000-0003-3562-3147
P. Siva Kota Reddy[3] orcid.org/0000-0003-4033-8148

[1]Dr. P. Dayananda Pai- P. Sathisha Pai, Government First Grade College, Mangalore, Carstreet. Mangalore, KA, India. mathsmahesh@gmail.com
[2]Mangalore University, Dept. of Mathematics. Mangalore, KA, India. rrajendrar@gmail.com
[3]JSS Science and Technology University, Sri Jayachamarajendra College of Engineering, Dept. of Mathematics. Mysurui, KA, India. pskreddy@jssstuniv.in

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Abstract:

The stress of a vertex is a centrality index, which has been introduced by Shimbel (1953). The stress of a vertex in a graph is the number of geodesics (shortest paths) passing through it. In this paper, we introduce a new topological index for graphs called square root stress-sum index using stresses of vertices. Further, we establish some inequalities, obtain some results and compute square root stress-sum index for some standard graphs.

Keywords: Geodesic; Stress of a vertex; Topological index.

MSC (2020): 05C09, 05C38.

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1. Introduction

For standard terminology and notion in graph theory, we follow the text-book of Harary [3]. The non-standard will be given in this paper as and when required.

Let $G = (V, E)$ be a graph (finite and undirected). The distance between two vertices u and v in G, denoted by $d(u, v)$ is the number of edges in a shortest path (also called a graph geodesic) connecting them. We say that a graph geodesic P is passing through a vertex v in G if v is an internal vertex of P (i.e., v is a vertex in P, but not an end vertex of P). v in G, $g(u, v)$ denotes the number of geodesics whose end vertices are u and v. The degree of a vertex v in G is denoted by $d(v)$.

The concept of stress of a node (vertex) in a network (graph) has been introduced by Shimbel as a centrality measure in 1953 [6]. This centrality measure has applications in biology, sociology, psychology, etc., (See [4, 5]). The stress of a vertex v in a graph G, denoted by $\mathrm{str}(v)$, is the number of geodesics passing through it. We denote the maximum stress among all the vertices of G by Θ_G and minimum stress among all the vertices of G by θ_G. Further, the concepts of stress number of a graph and stress regular graphs have been studied by K. Bhargava, N.N. Dattatreya, and R. Rajendra in their paper [1].

The reciprocal sum-connectivity index of a graph (see [2]) is defined as

$$RSC(G) = \sum_{uv \in E(G)} \sqrt{d(u) + d(v)}$$

Motivated by the identity sqrt, in this paper, we introduce a new topological index called square root stress-sum index using stresses on vertices. Further, we establish some inequalities, obtain some results and compute stress-sum index for some standard graphs.

2. Square Root Stress-Sum Index for Graphs

Definition 2.1. *The square root stress-sum index $\mathcal{SRS}(G)$ of a simple graph G is defined as*

$$\mathcal{SRS}(G) = \sum_{uv \in E(G)} \sqrt{(u) + (v)} \tag{2.1}$$

Observation: From the Definition 2.1, it follows that, for any graph G,

$$2m\sqrt{\theta_G} \leq \mathcal{SRS}(G) \leq 2m\sqrt{\Theta_G},$$

where m is the number of edges in G.

Example 2.2. *Consider the graph G given in Figure 2.1.*

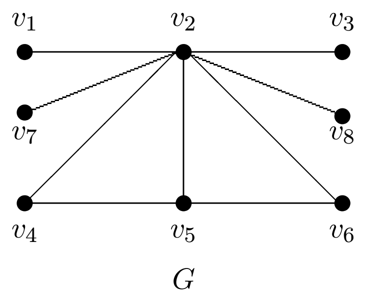

Figure 2.1: A graph G

The stresses of the vertices of G are as follows:

$str(v_1) = str(v_3) = str(v_7) = str(v_8) = 0,$
$str(v_2) = 19,$
$str(v_5) = 1,$
$str(v_4) = str(v_6) = 0.$

The stress-sum index of G is:

$$\begin{aligned}\mathcal{SRS}(G) &= \sqrt{(v_2)+(v_1)} + \sqrt{(v_2)+(v_3)} + \sqrt{(v_2)+(v_7)} \\ &+\sqrt{(v_2)+(v_8)} + \sqrt{(v_2)+(v_4)} + \sqrt{(v_2)+(v_5)} \\ &+\sqrt{(v_2)+(v_6)} + \sqrt{(v_4)+(v_5)} + \sqrt{(v_5)+(v_6)} \\ &= \sqrt{19+0} + \sqrt{19+0} + \sqrt{19+0} + \sqrt{19+0} + \sqrt{19+0} + \sqrt{19+1} \\ &+\sqrt{19+0} + \sqrt{0+1} + \sqrt{1+0} \\ &= 2 + 6\sqrt{19} + \sqrt{20}.\end{aligned}$$

Proposition 2.3. *Let N be the number of geodesics of length ≥ 2 in a graph G. Then*

$$0 \leq \mathcal{SRS}(G) \leq \sqrt{2N}(|E| - t), \tag{2.2}$$

where t is the number of edges with end vertices having zero stress in G.

Proof. If N is the number of all geodesics of length ≥ 2 in a graph G, then by the definition of stress of a vertex, for any vertex v in G, $0 \leq \mathrm{str}(v) \leq N$. Hence by the Definition 2.1, we have

$$0 \leq \mathcal{SRS}(G) \leq \sqrt{2N}(|E| - t), \tag{2.3}$$

where t is the number of edges with end vertices having zero stress in G. $\square$

Corollary 2.4. *If there is no geodesic of length ≥ 2 in a graph G, then $\mathcal{SRS}(G) = 0$. Moreover, for a complete graph K_n, $\mathcal{SRS}(K_n) = 0$.*

Proof. If there is no geodesic of length ≥ 2 in a graph G, then $N = 0$. Hence, by the Proposition 2.3, we have $\mathcal{SRS}(G) = 0$.

In K_n, there is no geodesic of length ≥ 2 and so $\mathcal{SRS}(K_n) = 0$. $\square$

Theorem 2.5. *For a graph G, $\mathcal{SRS}(G) = 0$ if and only if neighbours of every vertex induce a complete subgraph of G.*

Proof. Suppose that $\mathcal{SRS}(G) = 0$. Then by the Definition 2.1(Eq.srs), $\sqrt{(u) + (v)} = 0$, $\forall uv \in E(G)$ and so $(u) + (v) = 0$, $\forall uv \in E(G)$. Hence $(v) = 0$, $\forall v \in V(G)$. Let $v \in V(G)$. We need to show that neighbours of v induce a complete subgraph of G. If v is a pendant vertex, then there is nothing to prove. Suppose that v is not a pendant vertex. We claim that any two neighbouring vertices are adjacent in G. If there are two neighbours u and w of v that are not adjacent in G, then uvw is a graph geodesic passing through v, which implies $(v) \geq 1$, a contradiction. Hence our claim holds. Thus neighbours of v induce a complete subgraph of G. Since v is arbitrary in $V(G)$, the neighbours of every vertex induce a complete subgraph of G.

Conversely, suppose that neighbours of every vertex in G induce a complete subgraph of G. Let $v \in V(G)$. Since neighbours of v induce a complete subgraph of G, any two neighbouring vertices are adjacent and so there is no geodesic of length ≥ 2 passing through v. Since v is an arbitrary vertex in G, by the Corollary 2.4, it follows that $\mathcal{SRS}(G) = 0$. $\square$

Proposition 2.6. *For the complete bipartite $K_{r,s}$,*

$$\mathcal{SRS}(K_{r,s}) = \frac{rs}{\sqrt{2}}\sqrt{s(s-1) + r(r-1)}.$$

Proof. Let $V_1 = \{v_1, \ldots, v_r\}$ and $V_2 = \{u_1, \ldots, u_s\}$ be the partite sets of $K_{r,s}$. We have,

$$(v_i) = \frac{s(s-1)}{2} \text{ for } 1 \leq i \leq r \tag{2.4}$$

and

$$(u_j) = \frac{r(r-1)}{2} \text{ for } 1 \leq j \leq s. \tag{2.5}$$

Using 2.5 and 2.6 in the Definition 2.1, we have

$$\begin{aligned} \mathcal{SRS}(K_{r,s}) &= \textstyle\sum_{uv \in E(G)} \sqrt{(u) + (v)} \\ &= \textstyle\sum_{1 \leq i \leq r,\ 1 \leq j \leq r} \sqrt{(v_i) + (u_j)} \\ &= \textstyle\sum_{1 \leq i \leq r,\ 1 \leq j \leq s} \left[\sqrt{\frac{s(s-1)}{2} + \frac{r(r-1)}{2}}\right] \\ &= rs \left[\sqrt{\frac{s(s-1)}{2} + \frac{r(r-1)}{2}}\right] \\ &= \frac{rs}{\sqrt{2}} \sqrt{s(s-1) + r(r-1)}. \end{aligned}$$

□

Proposition 2.7. *If $G = (V, E)$ is a k-stress regular graph, then*

$$\mathcal{SRS}(G) = \sqrt{2k}\ |E|.$$

Proof. Suppose that G is a k-stress regular graph. Then

$$(v) = k \text{ for all } v \in V(G).$$

By the Definition 2.1, we have

$$\begin{aligned} \mathcal{SRS}(G) &= \textstyle\sum_{uv \in E(G)} \sqrt{(u) + (v)} \\ &= \textstyle\sum_{uv \in E(G)} \sqrt{k + k} \\ &= \sqrt{2k}\ |E|. \end{aligned}$$

□

Corollary 2.8. *For a cycle C_n,*

$$\mathcal{SRS}(C_n) = \begin{cases} \dfrac{n(n-1)(n-3)}{4}, & \text{if } n \text{ is odd;} \\ \dfrac{n^2(n-2)}{4}, & \text{if } n \text{ is even.} \end{cases}$$

Proof. For any vertex v in C_n, we have,

$$(v) = \begin{cases} \dfrac{(n-1)(n-3)}{8}, & \text{if } n \text{ is odd;} \\ \dfrac{n(n-2)}{8}, & \text{if } n \text{ is even.} \end{cases}$$

Hence C_n is

$$\begin{cases} \dfrac{(n-1)(n-3)}{8}\text{-stress regular}, & \text{if } n \text{ is odd;} \\ \dfrac{n(n-2)}{8}\text{-stress regular}, & \text{if } n \text{ is even.} \end{cases}$$

Since C_n has n vertices and n edges, by the Proposition 2.7, we have

$$\begin{aligned} \mathcal{SRS}(C_n) &= n \times \begin{cases} \sqrt{2 \cdot \dfrac{(n-1)(n-3)}{8}}, & \text{if } n \text{ is odd;} \\ \sqrt{2 \cdot \dfrac{n(n-2)}{8}}, & \text{if } n \text{ is even.} \end{cases} \\ &= \begin{cases} \dfrac{n}{2}\sqrt{(n-1)(n-3)}, & \text{if } n \text{ is odd;} \\ \dfrac{n}{2}\sqrt{n(n-2)}, & \text{if } n \text{ is even.} \end{cases} \end{aligned}$$ □

Proposition 2.9. *Let T be a tree on n vertices. Then*

$$\mathcal{SRS}(T) = \sum_{uv\in J} \sqrt{\sum_{1\le i<j\le m(u)} |C_i^u||C_j^u| + \sum_{1\le i<j\le m(v)} |C_i^v||C_j^v|} + \sum_{w\in Q} \sqrt{\sum_{1\le i<j\le m(w)} |C_i^w||C_j^w|}.$$

where J is the set of internal(non-pendant) edges in T, Q denotes the set of all vertices adjacent to pendant vertices in T, and the sets $C_1^v, \dots, C_m^v$ denotes the vertex sets of the components of $T - v$ for an internal vertex v of degree $m = m(v)$.

Proof. We know that a pendant vertex in T has zero stress. Let v be an internal vertex of T of degree $m = m(v)$. Let $C_1^v, \dots, C_m^v$ be the components of $T - v$. Since there is only one path between any two vertices in a tree, it follows that,

$$\text{str}(v) = \sum_{1\le i<j\le m} |C_i^v||C_j^v|$$

Let J denotes the set of internal(non-pendant) edges, and P denotes pendant edges and Q denotes the set of all vertices adjacent to pendant vertices in T. Then using Qstress in the Definition 2.1, we have

$$\begin{aligned}
\mathcal{SRS}(T) = &\ \textstyle\sum_{uv\in J}\sqrt{(u)+(v)}+\sum_{uv\in P}\sqrt{(u)+(v)}\\
= &\ \textstyle\sum_{uv\in J}\sqrt{(u)+(v)}+\sum_{w\in Q}\sqrt{(w)}\\
= &\ \textstyle\sum_{uv\in J}\sqrt{\sum_{1\le i<j\le m(u)}|C_i^u||C_j^u|+\sum_{1\le i<j\le m(v)}|C_i^v||C_j^v|}\\
&\ \qquad\textstyle+\sum_{w\in Q}\sqrt{\sum_{1\le i<j\le m(w)}|C_i^w||C_j^w|}.
\end{aligned}\qquad\square$$

Corollary 2.10. *For the path P_n on n vertices*

$$\mathcal{SRS}(P_n)=\sum_{i=1}^{n-1}\sqrt{2in-2i^2-n}.$$

Proof. The proof of this corollary follows by above Proposition 2.9. We follow the proof of the Proposition 2.9 to compute the index. Let P_n be the path with vertex sequence $v_1, v_2, \ldots, v_n$ (shown in Figure 2.2).

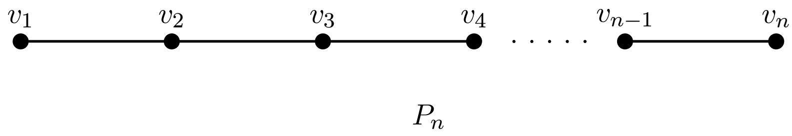

Figure 2.2: The path P_n on n vertices.

We have,

$$\mathrm{str}(v_i)=(i-1)(n-i),\qquad 1\le i\le n.$$

Then

$$\begin{aligned}
\mathcal{SRS}(P_n) &= \textstyle\sum_{uv\in E(P_n)}\sqrt{(u)+(v)}\\
&= \textstyle\sum_{i=1}^{n-1}\sqrt{(v_i)+(v_{i+1})}\\
&= \textstyle\sum_{i=1}^{n-1}\sqrt{(i-1)(n-i)+(i)(n-i-1)}\\
&= \textstyle\sum_{i=1}^{n-1}\sqrt{2in-2i^2-n}.
\end{aligned}\qquad\square$$

Proposition 2.11. *Let $Wd(n,m)$ denotes the windmill graph constructed for $n\ge 2$ and $m\ge 2$ by joining m copies of the complete graph K_n at a shared universal vertex v (a universal vertex of a graph is a vertex that is adjacent to all other vertices of the graph). Then*

$$\mathcal{SRS}(Wd(n,m))=m(n-1)^2\sqrt{\frac{m(m-1)}{2}}.$$

Hence, for the friendship graph F_k on $2k+1$ vertices,

$$\mathcal{SRS}(F_k)=4k\sqrt{\frac{k(k-1)}{2}}.$$

Proof. Clearly the stress of any vertex other than universal vertex is zero in $Wd(n,m)$, because neighbours of that vertex induces a complete subgraph of $Wd(n,m)$. Also, since there are m copies of K_n in $Wd(n,m)$ and their vertices are adjacent to v, it follows that, the only geodesics passing through v are of length 2 only. So, $(v) = \dfrac{m(m-1)(n-1)^2}{2}$. Note that there are $m(n-1)$ edges incident on v and the edges that are not incident on v have end vertices of stress zero. Hence by the Definition 2.1, we have

$$\begin{aligned}
\mathcal{SRS}(Wd(n,m)) &= m(n-1)\sqrt{(v)} \\
&= m(n-1)\sqrt{\frac{m(m-1)(n-1)^2}{2}} \\
&= m(n-1)^2\sqrt{\frac{m(m-1)}{2}}.
\end{aligned}$$

Since the friendship graph F_k on $2k+1$ vertices is nothing but $Wd(3,k)$, it follows that

$$\mathcal{SRS}(F_k) = 4k\sqrt{\frac{k(k-1)}{2}}.$$

□

Proposition 2.12. *Let W_n denote the wheel graph constructed on $n \geq 4$ vertices. Then*

$$\mathcal{SRS}(W_n) = (n-1)\times \begin{cases} \sqrt{\dfrac{(5n-6)\,(n-4)}{8}} + \sqrt{\dfrac{(n-2)(n-4)}{4}}, & \text{if } n \text{ is even;} \\ \sqrt{\dfrac{(n-1)\,(5n-19)}{8}} + \sqrt{\dfrac{(n-1)(n-3)}{4}}, & \text{if } n \text{ is odd.} \end{cases}$$

Proof. In W_n with $n \geq 4$, there are $(n-1)$ peripheral vertices and one central vertex, say v. It is easy to see that

$$(v) = \frac{(n-1)(n-4)}{2} \tag{2.6}$$

Let p be a peripheral vertex. Since v is adjacent to all the peripheral vertices in W_n, there is no geodesic passing through p and containing v. Hence contributing vertices for (p) are the remaining peripheral vertices. So, by denoting the cycle $W_n - p$ (on $n-1$ vertices) by C_{n-1}, we have

$$
\begin{aligned}
{}_{W_n}(p) &= {}_{W_n - v}(p) \\
&= {}_{C_{n-1}}(p) \\
&= \begin{cases} \dfrac{(n-2)(n-4)}{8}, & \text{if } n-1 \text{ is odd;} \\ \dfrac{(n-1)(n-3)}{8}, & \text{if } n-1 \text{ is even,} \end{cases} \\
&= \begin{cases} \dfrac{(n-2)(n-4)}{8}, & \text{if } n \text{ is even;} \\ \dfrac{(n-1)(n-3)}{8}, & \text{if } n \text{ is odd.} \end{cases}
\end{aligned}
$$

Let us denote the set of all radial edges in W_n by R, and the set of all peripheral edges by Q. Note that there are $(n-1)$ radial edges and $(n-1)$ peripheral edges in W_n. Using cntr and peri in the Definition 2.1, we have

$$
\begin{aligned}
\mathcal{SRS}(W_n) &= \textstyle\sum_{xy\in R} \sqrt{(x)+(y)} + \sum_{xy\in Q} \sqrt{(x)+(y)} \\
&= (n-1)\sqrt{(v)+(p)} + (n-1)\sqrt{2\cdot(p)} \\
&= (n-1)\begin{bmatrix} \sqrt{\dfrac{(n-1)(n-4)}{2} + \dfrac{(n-2)(n-4)}{8}}, \text{if } n \text{ is even;} \\ \sqrt{\dfrac{(n-1)(n-4)}{2} + \dfrac{(n-1)(n-3)}{8}}, \text{if } n \text{ is odd.} \end{bmatrix} \\
&\quad + \begin{cases} \sqrt{\dfrac{(n-2)(n-4)}{4}}, \text{if } n \text{ is even;} \\ \sqrt{\dfrac{(n-1)(n-3)}{4}}, \text{if } n \text{ is odd.} \end{cases} \\
&= (n-1) \times \begin{cases} \sqrt{\dfrac{(5n-6)\,(n-4)}{8}} + \sqrt{\dfrac{(n-2)(n-4)}{4}}, \text{if } n \text{ is even;} \\ \sqrt{\dfrac{(n-1)\,(5n-19)}{8}} + \sqrt{\dfrac{(n-1)(n-3)}{4}}, \text{if } n \text{ is odd.} \end{cases}
\end{aligned}
$$

□

Conclusion

Based on vertex degrees, a large number of topological indices have been defined and studied by several authors. We have introduced a new topological index for graphs called square root stress-sum index using stresses of vertices. Further, we established some inequalities, obtained some results and computed the stress-sum index for some standard graphs. The characterizations between properties of graphs and this index will be reported in a subsequent paper.

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