Proyecciones Journal of Mathematics
Vol. 40, N^o 6, pp. 1357-1365, December 2021.
Universidad Católica del Norte
Antofagasta - Chile

10.22199/issn.0717-6279-4360

Linear maps on $\mathcal{B}(\mathcal{H})$ preserving some operator properties

Abolfazl Niazi Motlagh
University of Bojnord, Iran
Abasalt Bodaghi
Islamic Azad University, Iran
and
Somaye Grailoo Tanha
Esfarayen University of Technology, Iran
Received : June 2020. Accepted : June 2021

Abstract

In this paper, for a complex Hilbert space $\mathcal{H}$ with $\dim \mathcal{H} \geq 2$, we study the linear maps on $\mathcal{B}(\mathcal{H})$, the bounded linear operators on $\mathcal{H}$, that preserves projections and idempotents. As a result, we characterize the linear maps on $\mathcal{B}(\mathcal{H})$ that preserves involutions in both directions.

Subjclass [2010]: *Primary 47B49; Secondary 16W10.*

Keywords: *Linear map, standard operator algebra, involution.*

1. Introduction

Linear preserver problems (LPP) is an active research topic in matrix theory, operator spaces and operator algebras and has attracted the attention of many mathematicians in the last few decades [1, 2, 5, 6] (and references therein). In a purely algebraic point of view, Martindale in [8] started to study multiplicative bijections on rings and proved that every multiplicative bijection from a prime ring containing a nontrivial idempotent onto an arbitrary ring is necessarily additive. This result shows that the multiplicative structure of a ring can determine its ring structure.

For an associative algebra $\mathcal{A}$, the Jordan product on $\mathcal{A}$ is defined by $A \circ B = \frac{1}{2}(AB + BA)$ for all $A, B \in \mathcal{A}$. Note that this product is commutative which is not associative. This means that $\mathcal{A}$ with this product is a Jordan algebra; see [3] for more details about these objects.

Let $\mathcal{A}$ and $\mathcal{B}$ be algebras. A mapping $\phi : \mathcal{A} \longrightarrow \mathcal{B}$ is Jordan multiplicative if for each $A, B \in \mathcal{A}$, it satisfies one of the following equations

$$\phi\Big(\frac{AB+BA}{2}\Big) = \frac{1}{2}\Big(\phi(A)\phi(B) + \phi(B)\phi(A)\Big), \tag{1.1}$$

$$\phi(AB+BA) = \phi(A)\phi(B) + \phi(B)\phi(A), \tag{1.2}$$

$$\phi(ABA) = \phi(A)\phi(B)\phi(A), \tag{1.3}$$

It is easy to see that if ϕ is additive, then (1.1) and (1.2) are equivalent and imply (1.3). Moreover, for unital algebras $\mathcal{A}, \mathcal{B}$ such that ϕ is additive and unital, then the above three forms of Jordan multiplicativity are equivalent.

Throughout this paper, $\mathcal{H}$ is a Hilbert space with $dim(\mathcal{H}) > 1$, $B(\mathcal{H})$ is the C^*-algebra of all bounded linear operators acting on $\mathcal{H}$. Furthermore, the real linear space of all bounded self adjoint operators on $\mathcal{H}$ will be denoted by $\mathcal{B}_s(\mathcal{H})$. One can easily observe that $\mathcal{B}_s(\mathcal{H})$ is closed under the Jordan product. In fact, it is a special Jordan algebra over the field of real numbers.

We remember that each self-adjoint rank one operator on $\mathcal{H}$ is of the form $ax \otimes x$ for some $x \in \mathcal{H}$ and some $a \in \mathbf{R}$ and also rank one projections are exactly of the form $x \otimes x$ for some unit vector $x \in \mathcal{H}$. Moreover, each self-adjoint finite rank operator is a real linear combination of pairwise orthogonal rank one projections. There is a natural order on the set of all projections of $\mathcal{B}(\mathcal{H})$, namely for two projections $P, Q \in \mathcal{B}(\mathcal{H})$ we say $P \leq Q$ if $PQ = QP = P$. We also say two projections P and Q are *orthogonal*, if $PQ = QP = 0$ and we denote it by $P \perp Q$. In addition, a conjugate linear bijective map U on $\mathcal{H}$ is said to be *anti-unitary* if $\langle Ux, Uy\rangle = \langle y, x\rangle$ for all $x, y \in \mathcal{H}$. In the sequel, I stands for the identity operator on $\mathcal{H}$. An element $T \in \mathcal{B}(\mathcal{H})$ is called *involution* if $T^2 = I$. Recall that an operator S on a Hilbert space $\mathcal{H}$ is *quasi-unitary* if $SS^* = S^*S = S + S^*$.

Let $\dim \mathcal{H} \geq 2$ and $\Psi : \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{B}(\mathcal{H})$ be a surjective linear mapping satisfies the implication

$(\mathrm{S}{\circ}T)^2 - (S \circ T) = 0 \Rightarrow (\Psi(S) \circ \Psi(T))^2 - (\Psi(S) \circ \Psi(T)) = 0.$

for all $S, T \in \mathcal{B}(\mathcal{H})$. We show that if $\Psi(\mathcal{B}_s(\mathcal{H})) = \mathcal{B}_s(\mathcal{H})$, then there exist a unitary or anti-unitary operator U on $\mathcal{H}$ and a constant λ with $\lambda \in \{-1, 1\}$ such that $\Psi(S) = \lambda USU^*$ for every $S \in \mathcal{B}(\mathcal{H})$.

2. Main Results

Suppose that $\mathcal{H}$ is a Hilbert space and $\Psi : \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{B}(\mathcal{H})$ is a linear map which preserves projections in both directions. In other words,

$$P^2 = P \Longleftrightarrow \Psi(P)^2 = \Psi(P). \tag{2.1}$$

Next, we indicate an elementary lemma which is useful in the proofs of main results.

Lemma 2.1. *Let $\mathcal{H}$ be a complex Hilbert space and $\Psi : \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{B}(\mathcal{H})$ be a linear map that preserves projections in both directions. Then*

(i) Ψ preserves the orthogonality of projections;

(ii) Ψ preserves the order of projections;

(iii) Ψ preserves rank-1 projections as well as orthogonality of rank-1 projections.

Proof. (i) Assume that P and Q are two mutually orthogonal projections. Then, both $P + Q$ and $\Psi(P) + \Psi(Q)$ are orthogonal projection. Hence, $(\Psi(P) + \Psi(Q))^2 = \Psi(P) + \Psi(Q)$. Consequently

$$\Psi(P)\Psi(Q) + \Psi(Q)\Psi(P) = 0. \tag{2.2}$$

It follows from (2.2) that $\Psi(P)\Psi(Q)+\Psi(P)\Psi(Q)\Psi(P) = 0$ and $\Psi(Q)\Psi(P)+\Psi(P)\Psi(Q)\Psi(P) = 0$.

The last equations yield

$$\Psi(P)\Psi(Q) = \Psi(Q)\Psi(P). \tag{2.3}$$

Equations (2.2) and (2.3) imply that $\Psi(P)\Psi(Q) = \Psi(Q)\Psi(P) = 0$. (ii) Let P and Q be two projections in $\mathcal{B}(\mathcal{H})$ such that $P \le Q$. Since $PQ = QP = P$, we conclude that $P \perp (P - Q)$ and $P \perp (Q - P)$. The part (i) implies $\Psi(P) \perp (\Psi(P) - \Psi(Q))$ and $\Psi(P) \perp (\Psi(Q) - \Psi(P))$ and so $\Psi(P)\Psi(Q) = \Psi(Q)\Psi(P) = \Psi(P)$ which means that $\Psi(P) \le \Psi(Q)$. (iii) Set $\Psi(E) = e \otimes e$ for some unit vector $e \in \mathcal{H}$. We know that E is a non-zero projection. Suppose contrary to our claim, that $rankE \ge 2$. Then, there exists two unit vectors f_1 and f_2 in $\mathcal{H}$ with $f_1 \perp f_2$ such that $E \ge f_1 \otimes f_1, E \ge f_2 \otimes f_2$. It follows part (i) that the last equalities are equivalent to

$$e \otimes e \ge \Psi^{-1}(f_1 \otimes f_1) \; and \; e \otimes e \ge \Psi^{-1}(f_2 \otimes f_2).$$

On the other hand, $\Psi^{-1}(f_1 \otimes f_1) \perp \Psi^{-1}(f_2 \otimes f_2)$ which is a contradiction. Thus, $rankE = 1$. □

For a Banach space $\mathcal{X}$, we denote the dual space of $\mathcal{X}$ and the set of all finite-rank operators on $\mathcal{X}$ by $\mathcal{X}'$ and $F(\mathcal{X})$, respectively. Taghvi and Hosseinzadeh in [6] proved the following theorem.

Theorem 2.2. *Let $\mathcal{X}$ and $\mathcal{Y}$ be complex infinite-dimensional Banach spaces and $\phi : B(\mathcal{X}) \longrightarrow B(\mathcal{Y})$ be a linear map. If ϕ preserves idempotent operators in both directions, then one of the following assertions holds.*

(i) There exists a bijective bounded linear or conjugate linear operator $A : \mathcal{X} \longrightarrow \mathcal{Y}$ such that $\phi(T) = ATA^{-1}$ for all $T \in F(\mathcal{X})$;

(ii) there exists a bijective bounded linear or conjugate linear operator $A : \mathcal{X}' \longrightarrow \mathcal{Y}$ such that $\phi(T) = AT'A^{-1}$ for all $T \in F(\mathcal{X})$.

We recall that a linear map $\Psi : \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{B}(\mathcal{H})$ preserves idempotents in both directions provided that

$$T^2 = T \Leftrightarrow \Psi(T)^2 = \Psi(T). \tag{2.4}$$

Proposition 2.3. *Let $\mathcal{H}$ be a complex Hilbert space and $\Psi : \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{B}(\mathcal{H})$ be a linear map that for every $T \in \mathcal{B}(\mathcal{H})$ satisfies equation (2.4). Then, Ψ is injective.*

Proof. Suppose that there exists $T \in \mathcal{B}(\mathcal{H})$ such that $\Psi(T) = 0$. Then, $\Psi(rT) = 0$ for all $r \in \mathbf{R}$. Hence, $\Psi(I-rT) = \Psi(I)$ and so from $\Psi(I-rT)^2 = \Psi(I)^2 = \Psi(I) = \Psi(I - rT)$, we can conclude that $I - rT$ satisfies equation (2.4) for all scalar $r \in \mathbf{R}$. Therefore, $(I - rT)^2 = I - rT$ holds for every scalar $r \in \mathbf{R}$. Taking $r = 1$ and $r = 2$ successively, we get $3T = 2T$. Hence, $T = 0$ and thus Ψ is injective. $\square$

Corollary 2.4. *Let $\mathcal{H}$ be a complex Hilbert space and $\Psi : \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{B}(\mathcal{H})$ be a surjective linear map that for every projection $P \in \mathcal{B}(\mathcal{H})$ satisfies in statement (2.4). Then*

(i) Ψ preserves the orthogonality of projections in both directions;

(ii) Ψ preserves the order of projections in both directions;

(iii) Ψ preserves rank-1 projections as well as orthogonality of rank-1 projections in both directions;

(iv) $\Psi(I) = I$.

Proof. The parts (i), (ii) and (iii) follow immediately from Lemma 2.1 and Proposition 2.3. For (iv), suppose there exists $T \in \mathcal{B}(\mathcal{H})$ such that $\Psi(T) = I$. Since $\Psi(T)^2 = \Psi(T) = I$, we can conclude that T satisfies equation (2.4). Assume now that $T \neq I$ and consider the operator $I - T$. It is easy to see $I - T$ satisfies equation (2.4). Hence, $\Psi(I-T)^2 = \Psi(I-T)$ and thus

$$\Psi(I) - \Psi(T) = \Psi(I)^2 - \Psi(I)\Psi(T) - \Psi(T)\Psi(I) + \Psi(T)^2.$$

It follows from the above equation that

$$\Psi(I) - I = \Psi(I) - \Psi(I)I - I\Psi(I) + I$$

Consequently, $\Psi(I) - I = -\Psi(I) + I$ and therefore $\Psi(I) = \Psi(T) = I$ $\square$

Lemma 2.5. *Let $\mathcal{H}$ be a complex Hilbert space and $\Psi : \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{B}(\mathcal{H})$ be a surjective linear map that for every $T \in \mathcal{B}(\mathcal{H})$ satisfies equation (2.4). Then, Ψ preserve projections in both directions.*

Proof. Let P be a projection in $\mathcal{B}(\mathcal{H})$. Then, both $I-P$ and $\Psi(I-P) = I - \Psi(P)$ satisfy equation (2.4). Consequently, $\Psi(P)^2 = \Psi(P)$. Since $\Psi(P)$ is self-adjoint, it is a projection. Hence, Ψ preserves projection in one direction. Now let $\Psi(P)$ be a projection. A similar argument for Ψ^{-1} guaranties that Ψ preserves projections in both directions. $\square$

We now are ready to state one of our main result of the paper.

Theorem 2.6. *Let $\mathcal{H}$ be a complex Hilbert space such that $\dim \mathcal{H} \geq 3$ and $\Psi : \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{B}(\mathcal{H})$ be a surjective linear map that for every $T \in \mathcal{B}(\mathcal{H})$ satisfies equation (2.4) and the equality $\Psi(\mathcal{B}_s(\mathcal{H})) = \mathcal{B}_s(\mathcal{H})$. Then, there exist a unitary or anti-unitary operator U on $\mathcal{H}$ such that for every $T \in \mathcal{B}_s(\mathcal{H})$*

$$\Psi(T) = UTU^*. \tag{2.5}$$

Proof. From Lemma 2.1, we know that Ψ is a bijection on the set of all rank-1 projections and preserves orthogonality in both directions. Since $\dim \mathcal{H} \geq 3$, it follows from the Uhlhorn's theorem in [7] that there is a unitary or anti-unitary operator U on $\mathcal{H}$ such that $\Psi(E) = UEU^*$ for any rank-1 projection E. Without loss of generality, we may assume that $\Psi(E) = E$ for every rank-1 projection E. Otherwise, we consider a map $\phi(A) = U^*\Psi(A)U$ for all $A \in \mathcal{B}_s(\mathcal{H})$. Then, ϕ satisfies equation (2.4) and the equality $\phi(E) = E$ holds for every rank-1 projection E. In this case, we have $\Psi(E) = E$ for every finite rank projection E. Let P be an infinite rank projection. Then

$$P = \sup\{E : E \leq P, \ \ E \text{ is a finite rank projection}\}.$$

Since Ψ preserves the order of projections in both directions, we conclude that

$$\Psi(P) = \sup\{F : F \leq P, \ F \text{ is a finite rank projection}\} = P.$$

Hence, $\Psi(P) = P$ holds for every projection P. It now follows that $\Psi(X) = S$ for any $X \in \mathcal{B}_s(\mathcal{H})$ since S is a real linear combination of eight projections from Theorem 3 in [4] and Ψ is linear. This completes the proof. $\square$

As a direct consequence of Theorem (2.6), we describe te surjective unital linear maps on $\mathcal{B}(\mathcal{H})$ that preserve involutions in both directions.

Corollary 2.7. *Let $\mathcal{H}$ be a complex Hilbert space such that $\dim \mathcal{H} \geq 3$ and $\Psi : \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{B}(\mathcal{H})$ be a unital surjective linear map that preserve* *involutions in both directions and $\Psi(\mathcal{B}_s(\mathcal{H})) = \mathcal{B}_s(\mathcal{H})$. Then, there exist a unitary or anti-unitary operator U on $\mathcal{H}$ such that $\Psi(T) = UTU^*$ for all $T \in \mathcal{B}_s(\mathcal{H})$.*

Proof. We firstly prove that Ψ is injective. Suppose there exists $T \in \mathcal{B}(\mathcal{H})$ such that $\Psi(T) = 0$. Then, $\Psi(T + I) = I$ and $\Psi(I - T) = I$. By our assumptions, $T + I$ and $I - T$ are involutions and so $2T = T^2 + I = -2T$. Thus, $T = -T$ which ensures us that $T = 0$. This means that Ψ is bijective. If $T \in \mathcal{B}(\mathcal{H})$ satisfies the equation $T^2 = I$, then T is an involution and hence $\Psi(T)$ is an involution operator. Therefore, $\Psi(T)^2 = \Psi(T)$. A similar argument for the reverse direction proves that Ψ satisfies equation (2.4). It now the results follows from Theorem (2.6). □

Here, we recall a result from [2] as follows.

Theorem 2.8. *Let $\mathcal{H}$ be a Hilbert space with $dim\mathcal{H} > 1$ and $\phi : B_s(\mathcal{H}) \longrightarrow B_s(\mathcal{H})$ be a bijection. Then, the following statements are equivalent.*

(i) $\phi(A^2 \circ B) = \phi(A)^2 \circ \phi(B)$;

(ii) There exists a unitary or conjugate unitary operator U on $\mathcal{H}$ such that $\phi(A) = \epsilon UAU^*$ for all $A \in B_s(\mathcal{H})$, where $\epsilon \in \{-1, 1\}$.

Motivated by the above result, we present the next theorem.

Theorem 2.9. *Let $\mathcal{H}$ be a complex Hilbert space such that $\dim \mathcal{H} \geq 2$ and $\Psi : \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{B}(\mathcal{H})$ be a surjective linear map that for every $S, T \in \mathcal{B}(\mathcal{H})$ satisfies the equation*

$$(S \circ T)^2 = (S \circ T) \Rightarrow (\Psi(S) \circ \Psi(T))^2 = (\Psi(S) \circ \Psi(T)). \tag{2.6}$$

If $\Psi(\mathcal{B}_s(\mathcal{H})) = \mathcal{B}_s(\mathcal{H})$, then there exists a unitary or anti-unitary operator U on $\mathcal{H}$ and a constant λ with $\lambda \in \{-1, 1\}$ such that $\Psi(S) = \lambda USU^*$ for every $S \in \mathcal{B}(\mathcal{H})$.

Proof. We prove that Ψ preserves operator pairs whose their Jordan products are non-zero projections. Assume that $S, T \in \mathcal{B}(\mathcal{H})$ such that $S \circ T$ is a non zero projection. Then, $2(S \circ T)$ satisfies equation (2.1) and hence $2(\Psi(S) \circ \Psi(T))$ fulfills equation (2.1) as well. This implies

$$(\Psi(S) \circ \Psi(T))^2 = \Psi(S) \circ \Psi(T).$$

In virtue of the fact that $\Psi(S) \circ \Psi(T)$ is self-adjoint ensures us that $\Psi(S) \circ \Psi(T)$ is a projection. Consequently, Ψ preserves operator pairs whose their Jordan products are non zero projections. Now, from Theorem 1 of [9], we conclude that when $\dim \mathcal{H} \geq 2$ there exist a unitary or anti-unitary operator U on $\mathcal{H}$ and a constant λ with $\lambda \in \{-1, 1\}$ such that $\Psi(S) = \lambda USU^*$ for every $S \in \mathcal{B}_s(\mathcal{H})$. This finishes the proof. $\square$

References

[1] M. Bresăr and P. Sěmrl, “Linear preservers on B(X)”, *Banach center publications*, vol. 38, pp. 49-58, 1997.

[2] S. Ghorbanipour and S. Hejazian, “Maps preserving some multiplcative structures on standard Jordan operator algebras”, *Journal of the Korean Mathematical Society*, vol. 54, no. 2, pp. 563-574, 2017.

[3] H. Hanche-Olsen and E. Størmer, *Jordan operator algebras*. Boston, MA: Pitman, 1984.

[4] C. Pearcy and D. Topping, “Sums of small numbers of idempotents”, *Michigan mathematical journal*, vol. 14, no. 4, pp. 453-465, 1967.

[5] B. Russo and H. A. Dye, “A note on unitary operators in C -algebras”, *Duke mathematical journal*, vol. 33, no. 2, pp. 413-416, 1966.

[6] A. Taghvi and R. Hosseinzadeh, “Linear maps preserving idempotent operators”, *Bulletin of the Korean Mathematical Society*, vol. 47, no. 4, pp. 787-792, 2010.

[7] U. Uhlhorn, “Representation of symmetry transformations in quantum mechanics”, *Arkiv fysik*, vol. 23, pp. 307-340, 1963.

[8] W. S. Martindale, “When are multiplicative mappings additive?”, *Proceedings of the American Mathematical Society*, vol. 21, no. 3, pp. 695-698, 1969.

[9] M. Wang and G. Ji, “Linear maps preserving projections of Jordan products on the space of self-adjoint operators”, *Journal of mathematical research with applications*, vol. 32, no. 2, pp. 235-240, 2012.

Abolfazl Niazi Motlagh
Department of Mathematics,
Faculty of basic sciences,
University of Bojnord,
P. O. Box 1339, Bojnord,
Iran
e-mail: a.niazi@ub.ac.ir

Abasalt Bodaghi
Department of Mathematics,
Garmsar Branch,
Islamic Azad University,
Garmsar,
Iran
e-mail: abasalt.bodaghi@gmail.com
Corresponding author

and

Somaye Grailoo Tanha
Esfarayen University of Technology,
Esfarayen, North Khorasan,
Iran
e-mail: grailotanha@gmail.com