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A note on convolution operators on Riesz Bounded variation spaces

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Abstract

We show some estimates and approximation results of operators of convolution type defined on Riesz Bounded variation spaces in $\mathbf{R}^n$*. We also state some embedding results that involve the collection of generalized absolutely continuous functions.*

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1. Introduction

Bounded variation spaces were introduced by C.Jordan in [1] when studied the convergence of Fourier series. Later, F.Riesz introduced in [2] the nowadays known as the Riesz $p-$variation. He showed that f belongs to the Sobolev spaces $W^{1,p}([a.b])$ if an only if

$$v_p(f,I)^p = \sup \sum_k \frac{|f(x_k) - f(x_{k-1})|^p}{(x_k - x_{k-1})^{p-1}} < \infty, \tag{1.1}$$

where the supremun is taken over all the finite partitions $\{[x_{k-1}, x_k]\}_{k\geq 1}$ of some interval I. The latter result is known sometimes as Medelev-Riesz Theorem. Later on, further extensions of this result were given in the one dimensional case. (See for example [3, 4, 5]). Also, subsequent works came out in the multidimensional frame (See [6, 7, 4]) with a simultaneous interest for understanding the behavior of some integral operators in this scale of functions; maximal operators, operators of convolution type, Mellin integral operators and so on (See [8, 12, 10, 11]). In [**?**] it was introduced the φ-variation on the multidimensional setting following the approach of Tonelli. Moreover, some approximation results were studied for the operators of convolution type,

$$T_w f(y) = (K_w * f)(y), \quad y \in \mathbf{R}, \quad w > 0,$$

induced by a family of approximation of identities $\{K_w\}_{w>0}$. It turns out, in virtue of the translation invariance nature of classical Lebesgue and Orlicz spaces that further extensions of the result given by eq:RieszTheo facilitate the study of the modulus of continuity defined on bounded variation spaces. In this note, we show some estimates for operators of convolution type defined on the multidimensional Riesz bounded variation spaces as introduced in [7] (See Section 2). In our opinion, this definition captures in a more general fashion the spirit of the classical Riesz bounded variation by means of the oscilation of an arbitrary function $f : \Omega \mapsto R$, which measures the "size" of the image of a given set $E \subseteq \mathbf{R}$ under f.

Throughout this paper we use the notation AB and BA to mean that there exist constants $c, C > 0$ such that $A \leq cB$ and $B \leq CA$ respectively for some quantities A, B. If the former and the later inequalities hold simultaneously we write $A \approx B$. We denote the oscillation of f on a set $E \subseteq \mathbf{R}$ by

$$osc_E f = \sup_{x\in E} f(x) - \inf_{x\in E} f(x).$$

Observe that we can change the right part of the definition above by $\sup_{x,y\in E}|f(x)-f(y)|$. We denote the Sobolev space $W_1^p(\Omega)$ as the collection of functions f such that the weak derivatives $D_j f$, $1 \le j \le n$, belong to the Lebesgue space $L^p(\Omega)$.

2. Preliminaries

Given $\Omega \subseteq \mathbf{R}$ an open set, let $f : \Omega \mapsto R$ be a measurable function. Following the notation introduced in [7], we say that f has finite Riesz bounded variation on Ω if

$$V_p(f;\Omega) := \sup\left(\sum_{B_k} \frac{osc_{B_k}(f)^p}{r_k^{p-n}}\right)^{1/p} < \infty, \tag{2.1}$$

where the supremun is taken over all the disjoint collection of balls $\{B_k\}$ of radii r_k, contained in Ω. When $V_p(f;\mathbf{R}) < \infty$ we denote $V_p(f) < \infty$ for the sake of simplicity. Besides, if eq:RBVpRn holds we write $f \in RBV_p(\Omega)$. In the extreme case $p = n$ we denote the class of n-Bounded variation function by $BV_n(\Omega)$. For further properties in this scale of functions see [13].

Observe that according to eq:RieszTheo and eq:RBVpRn, by considering $\Omega = I$ a bounded open interval we have that

$$v_p(f,I)V_p(f,I).$$

Hence, the $p-$variation defined in $\mathbf{R}$ by means of the oscillation stands as an extension of the original concept of p variation introduced by Riesz in [2]. Another well know feature in the unidimensional frame is that if $v_p(f;I) < \infty$ then f is a bounded function on I. The same phenomenon occurs in the multidimensional case. Without loss of generality, let us assume that $V_p(f;\Omega) < \infty$ for some open bounded set Ω that contains 0. Then

$$\|f\|_{L_\infty(\Omega)}|\Omega|^{1-n/p} + |f(0)| < \infty.$$

However, if Ω is not bounded f may be not a bounded function. Consider $f(x) = \sqrt{x}$, $I = (1,\infty)$. Clearly f is not bounded in I. On the other hand, by the Riesz-Medvedev Theorem,

$$v_p(f;I) = \sup_{J\subseteq I}\|f'\|_{L_p} < \infty, \quad p > 2.$$

where the supremum is taken over all the bounded intervals J contained in I. Not long ago, Barza and Lind stated a multidimensional Riesz-Medvedev type Theorem. This result gives a simple variational characterization of the Sobolev spaces $W_1^p(\Omega)$.

Lemma 2.1 (Barza-Lind, [7]). *Given $\Omega \subseteq \mathbf{R}$, let $p > n$. Then $f \in W_p^1(\Omega)$ if and only if f can be modified in a set of measure 0 to be continuous and $V_p(f;\Omega) < \infty$. Moreover*

$$\nabla f_{L^p(\Omega)} \approx V_p(f;\Omega).$$

From Lemma 2.1, if Ω is a bounded open set we have that for $1 \le p \le q < \infty$,

$$RBV_q(\Omega) \hookrightarrow RBV_p(\Omega).$$

This is also a straightforward consequence of the Hoĺder inequality. A well known embedding in the unidimensional frame is $RBV_p([a,b]) \subset AC([a,b])$, $p > 1$. We recall an equivalent definition of absolute continuity in the multidimensional setting (see [13]). We say that a function $f : \Omega \mapsto R$ is $n-$absolutely continuous in Ω ($AC^n(\Omega)$) if for any $\delta > 0$ there exists $\varepsilon > 0$ such that for every finite disjoint collection of closed balls $\{B_k\}$ contained in Ω we have

$$\sum_{B_k}(osc_{B_k} f)^n < \varepsilon \quad \text{whenever} \quad \sum_{B_k}|B_k| < \delta.$$

Proposition 2.2. *Ω be an open set and $p > n$. Then*

$$RBV_p(\Omega) \subseteq AC^n(\Omega).$$

Proof. Given $\varepsilon > 0$, choose $\delta < \left(\varepsilon/V_p(f;\Omega)^{p/n}\right)^{n/s}$. We set a finite disjoint family of balls $\{B_k\}$ contained in Ω and $f \in RBV_p(\Omega)$. Consider $s > 1$ such that $1/p + 1/s = 1/n$. By the Hoĺder inequality we obtain

$$\begin{aligned}\textstyle\sum_{B_k}(osc_{B_k} f)^n &\le \left(\textstyle\sum_{B_k} \frac{(osc_{B_k} f)^p}{r_k^{p-n}}\right)^{p/n} \left(\textstyle\sum_{B_k} r_k^n\right)^{s/n} \\ & V_p(f;\Omega)^{p/n} \left(\textstyle\sum_{B_k} |B_k|\right)^{s/n} < \varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we conclude $f \in AC^n(\Omega)$. $\square$

Observe that in the extreme value $p = n$, $RBV_p(\Omega)$ coincides with the bounded variation spaces $BV^n(\Omega)$ introduced in [13]. Furthermore, the following embedding also holds

Proposition 2.3. *Let f be a Lipschitz function in Ω, $|\Omega| < \infty$. Then $f \in RBV_p(\Omega)$.*

Proof. Let B a ball of radius r_B contained in Ω, and consider $x, y \in B$, then $|f(x) - f(y)| \leq C|x - y|$ implies

$$osc_B f r_B,$$

So, for a fixed collection $\{B_k\}$ of balls contained in Ω we have

$$\sum_{B_k} \left(\frac{osc_{B_k} f}{r_k}\right)^p r_k^n \sum_{B_k} |B_k| \leq |\Omega| < \infty$$

. Hence $V_p(f; \Omega) < \infty$. □

Since $osc_E(f+g) \leq osc_E(f)+osc_E(g)$, $E \subseteq \mathbf{R}$, then $V_p(f+g;\Omega)V_p(f;\Omega)+V_p(g;\Omega)$. Besides, $V_p(f;\Omega) = 0$ if and only if f is a constant function for all $x \in \Omega$. Therefore $V_p(\cdot;\Omega)$ is not a norm in $RBV_p(\Omega)$. However, we can endow $RBV_p(\Omega)$ with a norm. The following result shows that RBV_p is closed under the multiplication of functions.

Lemma 2.4. *Let $f, g \in RBV_p(\Omega)$. Then $fg \in RBV_p(\Omega)$.*

Proof. Set a disjoint collection of balls $\{B_k\}$ contained in Ω, since f, g are bounded functions in Ω, we estimate as follows

$$\begin{aligned}
\sum_{B_k} \left(\frac{osc_{B_k}(fg)}{r_k}\right)^p r_k^n &= \sum_{B_k} \left(\frac{\sup_{x,y\in B_k} |(fg)(x)-(fg)(y)|}{r_k}\right)^p r_k^n \\
&\leq \sum_{B_k} \left(\frac{|\|g\|_{L_\infty(\Omega)} osc_{B_k} f + \|f\|_{L_\infty(\Omega)} osc_{B_k} g|}{r_k}\right)^p r_k^n \\
&\|g\|_{L_\infty(\Omega)}^p \sum_{B_k} \left(\frac{osc_{B_k} f}{r_k}\right)^p r_k^n + \|f\|_{L_\infty(\Omega)}^p \sum_{B_k} \left(\frac{osc_{B_k} g}{r_k}\right)^p r_k^n \\
&\|g\|_{L_\infty(\Omega)}^p V_p(f;\Omega)^p + \|f\|_{L_\infty(\Omega)}^p V_p(g;\Omega)^p,
\end{aligned}$$

so

$$V_p(fg;\Omega) \, \|g\|_{L_\infty(\Omega)} V_p(f;\Omega) + \|f\|_{L_\infty(\Omega)} V_p(g;\Omega).$$

Observe that an alternative proof can be performed by Lemma 2.1, the product rule for weak derivatives and the Minkownski inequality. □

Proposition 2.5. *Assume Ω is an open set and $1 < p < \infty$. Then the functional defined by*

$$\cdot_{RBV_p(\Omega)} = \|\cdot\|_{L_\infty(\Omega)} + V_p(\cdot;\Omega)$$

is a norm in $RBV_p(\Omega)$. Moreover, $RBV_p(\Omega)$ is a Banach algebra.

Proof. Clearly $\cdot_{RBV_p(\Omega)}$ is a norm. Let $f, g \in RBV_p(\Omega)$, by Lemma 2.4 we have

$$\begin{aligned} fg_{RBV_p(\Omega)} & \|f\|_{L_\infty}\|g\|_{L_\infty} + \|g\|_{L_\infty}V_p(f;\Omega) + \|f\|_{L_\infty}V_p(g;\Omega) \\ & \leq (\|f\|_{L_\infty} + V_p(f;\Omega))(\|g\|_{L_\infty} + V_p(g;\Omega)) = f_{RBV_p(\Omega)}g_{RBV_p(\Omega)} \end{aligned}$$

□

Remark 2.6. *Assume that $p > n$ and set $\|\cdot\|_{RBV_p(\Omega)} = \|\cdot\|_{L_p(\Omega)} + V_p(\cdot;\Omega)$. By virtue of the Morrey inequality (see [14], Theorem 9.12) and Lemma 2.1, the norms $\cdot_{RBV_p(\Omega)}$ and $\|\cdot\|_{RBV_p(\Omega)}$ are equivalent.*

The next result is a Fatou type Lemma for $V_p(\cdot;\Omega)$.

Proposition 2.7. *Given Ω an open set, let $\{f_k\}$ be a sequence of bounded functions in Ω. Assume that $f_k(x) \to f(x)$, $k \to \infty$, for every $x \in \Omega$, . Then*

$$V_p(f;\Omega) \leq \liminf_{k\to\infty} V_p(f_k;\Omega). \tag{2.2}$$

Proof. Fix a collection of disjoint balls $\mathcal{D} = \{B\}$ contained in Ω. Then

$$V_p(f_k;\mathcal{D}) = \sum_{B\in\mathcal{D}} \left(\frac{osc_B(f_k)}{r_B}\right)^p r^n \leq V_p(f_k). \tag{2.3}$$

On the other hand, since $\{f_k\}$ is a bounded sequence in every $B \in \mathcal{D}$ and because the continuity of $t \mapsto t^p$, $p > 1$, we obtain
$|V_p(f;\mathcal{D}) - V_p(f_k;\mathcal{D})| \leq \sum_{B\in\mathcal{D}} |osc_B(f_k)^p - osc_B(f)^p| \frac{1}{r_B^{p-n}} \to 0,$
when $k \to \infty$. So,

$$V_p(f;\mathcal{D}) \leq \liminf_{k\to\infty} V_p(f_k;\Omega),$$

for every collection $\mathcal{D}$. Then the result follows. □

3. Convolution Operators on $RBV_p(\mathbf{R})$

In this section we prove some approximation results of Integral operators in $RBV_p(\mathbf{R})$ modular. We recall the definition of approximate identities.

Definition 3.1. *A family of functions $\{K_w\}_{w>0}$ is said to be approximate identities if:*

(i) $K_w \in L^1(\mathbf{R})$ *for any* $w > 0$ *and* $M = \sup_{w>0} K_{w1} < \infty$.

(ii) *For every* $w > 0$,

$$\int_{\mathbf{R}} K_w(x)x = 1.$$

(iii) *For every* $\delta > 0$,

$$\lim_{w\to\infty} \int_{x\geq\delta} K_w(x)x = 0. \tag{3.1}$$

Let us consider the operator T_w given by

$$(T_w f)(x) = \int_{\mathbf{R}} K_w(x-y)f(y)y,$$

which is known as the family of convolution operators associated to $\{K_w\}_{w>0}$. We point out that T_w is well defined some scale of functions, for example $L^p(\mathbf{R})$ and $C_0(\mathbf{R})$. We assume in the next results that $f \in RBV_p(\mathbf{R})$ endowed with the norm $\|\cdot\|_{RBV_p}$.

Proposition 3.2. *Let* T_w *be an operator that satisfies* (i) *and* (ii) *above. If* $f \in RBV_p(\mathbf{R})$, *then*

$$T_w : RBV_p(\mathbf{R}) \hookrightarrow RBV_p(\mathbf{R}).$$

Proof. We show that $V_p(T_w f)\, V_p(f)$. Fix a disjoint collection of balls $\mathcal{F} = \{B_k\}$ of radius r_k. By the Definition 3.1 and by the Jensen inequality we estimate as follows

$$\begin{aligned}
\sigma_p(T_w f;\mathcal{F}) &= \sum_{B_k}\left(\frac{osc_{B_k}(T_w f)}{r_k}\right)^p r_k^n = \sum_{B_k}\left(\frac{\sup_{t,z\in B_k}|T_w f(t)-T_w f(z)|}{r_k}\right)^p r_k^n \\
&= \sum_{B_k}\left(\frac{\sup_{t,z\in B_k}\int_{\mathbf{R}} K_w(y)(f(t-y)-f(z-y))y}{r_k}\right)^p r_k^p \\
&\leq \sum_{B_k}\left(\frac{M\int_{\mathbf{R}}|K_w(y)|osc_{B_k}f(\cdot-y)y}{r_k\int_{\mathbf{R}}|K_w(y)|y}\right)^p r_k^n \\
&\frac{\int_{\mathbf{R}}|K_w(y)|\sum_{B_k}\left(\frac{osc_{B_k}f(\cdot-y)}{r_k}\right)^p r_k^n y}{\int_{\mathbf{R}}|K_w(y)|y} V_p(f;\mathbf{R}) < \infty,
\end{aligned}$$

which ends the proof. □

A function f is said to be Lipchitz continuous at the point x if its upper pointwise Lipchitz constant at x is finite, i.e.

$$L_f(x) := \limsup_{y\to x}\frac{|f(x)-f(y)|}{|x-y|} < \infty$$

Evidently, if $f \in Lip\,(\Omega)$ then f is a Lipchitz continuous function. On the other hand, when Ω is a quasiconvex set every Lipchitz continuous function is also a Lipchitz function (See [15], Lemma 2.2). As a consequence of the Vitalli covering Lemma, L_f satisfies the weak estimate

$$(3.2) \qquad |\{x \in \mathbf{R} : L_f(x) > t\}| \left(\frac{V_p(f)}{t}\right)^p, \quad t > 0, \quad p > n.$$

cf. [7].

Corollary 3.3. *Let $f \in RBV_p\,(\mathbf{R})$ and $p > n$. Then T_w is Lipchitz continuous almost everywhere.*

Proof. In fact, the result $|\mathbf{R} \setminus \{x \in \mathbf{R} : L_{(T_w f)}(x) < \infty\}| = 0$ follows from the weak type inequality,

$$(3.3) \qquad |\{x \in \mathbf{R} : L_{(T_w f)}(x) > t\}| \left(\frac{V_p(f)}{t}\right)^p,$$

which is interesting in its own right. The inequality (3.3) is immediate taking into account Proposition 3.2 and (3.2). □

Given $f \in RBV_p\,(\mathbf{R})$, the V_p-modulus of continuity of order α, $\delta > 0$, is given by

$$\omega_p(f, \delta) := \sup_{|y| \leq \delta} V_p\left(\tau_y f - f\right),$$

where $(\tau_y f)(x) = f(x - y)$ and $x, y \in \mathbf{R}$.

Proposition 3.4. *Let $f \in RBV_p\,(\mathbf{R})$. Then*

$$\lim_{\delta \to 0^+} \omega_p(f, \delta) = 0.$$

Proof. By Lemma 2.1, $D_j f \in L_p\,(\mathbf{R})$, $j = 1, \cdots, n$. Since $L_p\,(\mathbf{R})$ is translation invariant we have

$$V_p\left(\tau_y f - f\right)^p \sum_{j=1}^{n} \int_{\mathbf{R}} |\tau_y D_j f(x) - D_j f(x)|^p x \to 0,$$

when $|y| \to 0$. So, the result follows. □

The next result states the continuity of T_w under the $p-$variation.

THEOREM 3.5. Given $f \in RBV_p(\mathbf{R})$, *let* $\{K_w\}_{w>0}$ *be a sequence of aproximation of identities. Then*

$$\lim_{w\to\infty} V_p\left(T_w f - f\right) = 0.$$

Proof. Let $\mathcal{F}$ a fixed disjoint collection of balls. By the generalized Jensen inequality we have

$$\begin{aligned}
\sigma_p\left(T_w f - f; \mathcal{F}\right)^p &= \sum_{B_k} \left(\frac{osc_{B_k}(\int_{\mathbf{R}} K_w(y)[f(\cdot - y) - f(\cdot)]y)}{r_k}\right)^p r_k^n \\
&\leq \sum_{B_k} \left(\frac{M\int_{\mathbf{R}} |K_w(y)| osc_{B_k}(f(\cdot - y) - f(\cdot))y}{r_k \int_{\mathbf{R}} |K_w(y)|y}\right)^p r_k^n \\
&\sum_{B_k} \frac{\int_{\mathbf{R}} |K_w(y)| osc^p_{B_k}(f(\cdot - y) - f(\cdot))y}{r_k^{p-n}\int_{\mathbf{R}} |K_w(y)|y} \\
&\int_{|y|>\delta} |K_w(y)| \sum_{B_k} \left(\frac{osc_{B_k}(f(\cdot - y) - f(\cdot))}{r_k}\right)^p r_k^n y \\
&+ \int_{|y|\leq\delta} |K_w(y)| \sum_{B_k} \left(\frac{osc_{B_k}(f(\cdot - y) - f(\cdot))}{r_k}\right)^p r_k^n y = I + II.
\end{aligned}$$

By Lemma 2.1 and by the Minkowski inequality we obtain

$$\begin{aligned}
I \quad \leq \quad & \int_{|y|>\delta} |K_w(y)| V_p^p\left(f(\cdot - y) - f(\cdot)\right) y \\
& \int_{|y|>\delta} |K_w(y)| \nabla (f(\cdot - y) - f(\cdot))^p_{Lp} y \\
& \int_{|y|>\delta} |K_w(y)| \nabla f^p_{L_p} y.
\end{aligned}$$

We estimate II as follows

$$\begin{aligned}
II \quad & \int_{|y|\leq\delta} |K_w(y)| V_p^p\left(f(\cdot - y) - f(\cdot)\right) y \\
& \qquad\qquad\qquad\qquad \omega_p(f,\delta)^p \int_{|y|\leq\delta} |K_w(y)|y.
\end{aligned}$$

from the previous estimates it follows

$$V_p(T_w f - f, \mathbf{R})^p \nabla f^p_{L_p} \int_{|y|>\delta} |K_w(y)|y + \omega_p(f,\delta)^p \int_{|y|\leq\delta} |K_w(y)|y,$$

due to Proposition 3.4 and eq:Aprox-Def3 the theorem follows. $\square$

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