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# Fixed point theorems in the study of positive strict set-contractions

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**Abstract**

*The author uses fixed point index properties and Inspired by the work in Benmezai and Boucheneb (see Theorem* 3.8 *in [3]) to prove new fixed point theorems for strict set-contraction defined on a Banach space and leaving invariant a cone.*

**Keywords:** *Cones, fixed point theory, strict set-contractions, positive solution, general minorant principle, boundary value problem.*

**2010 Mathematics Subject Classifications:** *47H10, 47H11, 47H30.*

## 1. Introduction

In the study non-linear operators in ordered Banach spaces having an invariant cone it is often convenient to make use of minorants, majorants and the special concept of the derivatives in order to establish the existence of non-zero fixed points. Krasnoselskii has provided in [11] many interesting fixed point theorems stating that if such an operator is approximatively linear at 0 and $+\infty$, and the spectral radii of the linear approximations are oppositely located with respect to 1, then it has a fixed point. Amann in [2] has generalized these results for monotones operators which are strict set-contractions.

The main goal of this paper is to study strict-set contraction in ordered Banach spaces having an invariant cone and to give sufficient conditions on minorants and majorants which yield the existence of at least one non-zero fixed point ( see [4], [3], [1] and [5]). We will assume that the mapping $T$ has an asymptotically linear majorant $h$ and has a minorant $g$ which is right differentiable at zero and existence of the fixed point is obtained under additional conditions about the positive spectra of the derivatives. The proofs are based on the fixed point index theory, developed in [13] (see also the monographs [7] and [8]). In order to be more precise, let $X$ be a Banach space, $C$ be a cone in $X$, and let $T : C \longrightarrow C$ be a completely continuous mapping. Recently, Mechrouk have proved in [12] that if $T$ has a positive right differentiable at zero minorant $h : K \longrightarrow K$ and an asymptotically linear positive majorant $g : P \longrightarrow P$ satisfying $\theta_P^{g'(\infty)} < 1 < \lambda_P^{h'(0)}$, then $T$ has at least one positive nontrivial fixed point, where the constants $\lambda_P^{h'(0)}$ and $\theta_P^{g'(\infty)}$ play an important role in the statement of the obtained existence and nonexistence results and sometimes they replace the positive spectral radius. Motivated by the above work, we consider in this paper the case where the operator $T$ is a strict set-contraction.

The paper is organized as follows. Section 2 gives some preliminaries. Section 3 is devoted to prove new fixed point theorems for positive maps having approximative minorant and majorant at 0 and $\infty$ in specific classes of operators. Applications to the existence of solutions to a third order boundary value problem with mixed boundary conditions are presented in the last section.

## 2. Abstract Background

We will use extensively in this work cones and the fixed point index theory, so let us recall some facts related to these two tools. Let $X$ be a real Banach space endowed with norm $\| \ . \ \|$, and let $L(X) = L(X, X)$ be the set of all linear continuous mapping from $X$ into $X$. A nonempty closed convex subset $C$ of $X$ is said to be a cone if $(tC) \subset C$ for all $t \geq 0$ and $C \cap (-C) = \{0_X\}$. It is well known that a cone $C$ induces a partial order in the Banach space $X$. We write for all $x, y \in X : x \preceq y$ if $y - x \in C$, $x \prec y$ if $y - x \in C$, $y \neq x$ and $xy$ if $y - x \notin C$. Notations $\succeq$, $\succ$ and  denote respectively the reverse situations. We say that the cone $C$ is normal with a constant $n_C > 0$ if for all $u, v$ in $C$, $u \preceq v$ implies $\|u\| \leq n_C \|v\|$.
Let $C$ be a cone in $X$ and let $L : X \to X$.

**Definition 2.1.** *The mapping $L$ is said to be positive if $L(C) \subset C$. In this case, a nonnegative constant $\mu$ is said to be a positive eigenvalue of $L$ if there exists $u \in C \{0_X\}$ such that $Lu = \mu u$.*

**Definition 2.2.** *Let $A$ be a nonempty set and let $B$ be an ordered set. A map $g : A \longrightarrow B$ is said to be a majorant of the map $f : A \longrightarrow B$ if $f(x) \leq g(x)$ for all $x \in X$. Minorant is defined by reversing the above inequality sign.*

**Definition 2.3.** *Let $C$ be a cone in $X$ and $L : X \to X$ a continuous map. $L$ is said to be*
**a)** *positive, if $L(C) \subset C$,*
**b)** *strongly positive, if $C$ has a nonempty interior ($intC \neq \emptyset$) and $L(C \{0_X\}) \subset intC$,*
**c)** *increasing, if for all $u, v \in X$, $u \preceq v$ implies $Lu \preceq Lv$.*

**Definition 2.4.** *Let $L_1, L_2 : X \to X$ be positive maps. We write $L_1 \preceq L_2$ if for all $x \in C$, $L_1 x \preceq L_2 x$.*

**Definition 2.5.** *Let $B(X)$ be the set of all bounded subsets of $X$ and $\psi : B(X) \longrightarrow \mathbf{R}^+$ be a measure of non-compactness on $X$; that is $\psi$ satisfies for $A$, $B \in B(X)$*

1. *$\psi(A) = 0 \Longleftrightarrow A$ is relatively compact on $X$.*
2. *$A \subseteq B$ imply $\psi(A) \leq \psi(B)$.*
3. *$\psi(\bar{co}A) = \psi(\overline{A}) = \psi(A)$.*

4. $\psi(A\bigcup B)=\max\{\psi(A),\psi(B)\}$.

5. *for all* $t\in[0,1], \psi(tA+(1-t)B)\leq t\psi(A)+(1-t)\psi(B)$

6. *if* $(A_n)_n\subset B(X)$ *is a decreasing sequence of closed nonempty sets with* $\lim\psi(A_n)=0$, *then* $\cap_{n\geq 1}A_n$ *is a nonempty compact set.*

**Definition 2.6.** *A function* $f:\Omega\subset X\to X$ *is said to be a strict-set contraction if it is continuous, bounded, and there exists a constant* $k\in[0,1)$ *such that* $\psi(f(S))\leq k\psi(S)$ *for all bounded sets* $S\subset\Omega$.

**Definition 2.7 ([14]).** *A map* $g:C\longrightarrow X$ *is said to be differentiable at* $x_0\in C$ *along* $C$ *if there exists* $g'(x_0)\in L(X)$ *such that*

$$\lim_{h\in C,h\longrightarrow 0}\frac{\parallel g(x_0+h)-g(x_0)-g'(x_0)h\parallel}{\parallel h\parallel}=0.$$

*We say that* $g'(x_0)$ *is the derivative of* $g$ *at* $x_0$ *along* $C$, *is uniquely determined.*

*The map* $g$ *is said to be asymptotically linear along* $C$ *if there exists* $g'(\infty)\in L(X)$ *such that*

$$\lim_{x\in C,\|x\|\longrightarrow+\infty}\frac{\parallel g(x)-g'(\infty)x\parallel}{\parallel x\parallel}=0.$$

*Again,* $g'(\infty)$ *is uniquely determined and called the derivative at infinity along* $C$.

**Lemma 2.8 ([11]).** *The derivative* $g'(\nu)$, $(\nu=+\infty$, *or* $x_0\in C)$, *with respect to a cone of the positive operator* $g$ *is a linear positive operator.*

Detailed presentation of the differentiability with respect to a cone can be found in [11] and [14].

The main results of this paper are proved by means of the fixed point index theory for strict-set contraction mappings develloped in [13].

Let us recall some lemmas providing fixed point index computations. Let $C$ be a cone in $X$. Let for $R>0$, $C_R=C\cap B(0_X,R)$ where $B(0_X,R)$ is the open ball of radius $R$ centred at $0_X$, and let $\partial C_R$ be its boundary and consider a strict-set contraction mapping, $f:\overline{C_R}\to C$.

**Lemma 2.9 ([7]).** *If $fx \neq \lambda x$ for all $x \in \partial C_R$ and $\lambda \geq 1$ then $i(f, C_R, C) = 1$.*

**Lemma 2.10 ([7]).** *If there exists $e \succ 0_X$ such that $x \neq fx + te$ for all $t \geq 0$ and all $u \in \partial C_R$ then $i(f, C_R, C) = 0$.*

From the two Lemma above, we conclude the following Lemma.

**Lemma 2.11.** *If $fxx$ for all $x \in \partial C_R$ then $i(f, C_R, C) = 1$.*

**Lemma 2.12.** *If $fxx$ for all $x \in \partial C_R$ then $i(f, C_R, C) = 0$.*

A detailed presentation of the fixed point index theory for strict-set contraction mappings can be found in [13].

In all this section $E$ is a real Banach space, $K$ is a nontrivial cone in $E$ and $L(E)$ denote the set of all linear continuous self mapping on $E$ endowed with the norm, $\|L\| = \sup\limits_{\|u\|=1} \|Lu\|$. Let $C^+(E)$ denote the subset of $L(E)$ consisting of all strict set-contraction positive operators. Hereafter $\preceq$ is the order induced by the cone $K$ on $E$ and we set,

$$L_K(E) = \{L \in L(E),\ L \text{ is increasing }\}$$

and

$$C_K(E) = \{L \in L_K(E) : L \text{ is a strict-set contraction}\}.$$

Now, for $L \in L_K(E)$ we define the subset

$$\Theta_P^L = \{\theta \geq 0 : \text{ there exists } u \in P\,\{0_E\} \text{ such that } Lu \succeq \theta u\}.$$

**Remark 2.13.** *Note that*
**i)** $0 \in \Theta_P^L$ *and if* $\theta \in \Theta_P^L$*, then* $[0, \theta] \subset \Theta_P^L$.
**ii)** $\Lambda_P^L \subset \Lambda_K^L$ *and* $\Theta_P^L \subset \Theta_K^L$.
**iii)** *If* $\mu$ *is positive eigenvalue of* $L$*, then* $\mu \in \Theta_P^L \cap [0, \| L \|]$.
**iv)** *If* $L^{-1}(0_E) \cap K = \{0_E\}$ *and* $P \subset K$ *then* $\Theta_P^L = \Theta_K^L$.

In all this paper, we set for $L \in L_K^P(E)$,

$$\theta_P^L = \sup \Theta_P^L$$

The constant $\theta_P^L$ replaces the spectral radius of $L$ which in our case is not necessarily an eigenvalue of $L$ having an eigenvector in $K$. So, it is

natural to ask what represents this constant with respect to the operator $L$.
If $L : E \longrightarrow E$ is a bounded linear operator, then we define, $r(L)$, its spectral radius by

$$r(L) = \lim_{n \longrightarrow +\infty} \| L^n \|^{\frac{1}{n}} .$$

Lemma 2.14 gives sufficient conditions for the existence of $\theta_P^L$.

**Lemma 2.14 ([3]).** *Assume $L \in L_K(E)$. Then the subset $\Theta_P^L$ is bounded from above by $r(L)$.*

**Lemma 2.15 ([3]).** *Assume that the cone $K$ is solid, and let $L \in C_K(E)$ be strongly positive and increasing. Then $\theta_K^L$ is the unique positive eigenvalue of $L$.*

**Lemma 2.16 ([10]).** *Let $D \subset E$, $D$ be a bounded set and $f$ is uniformly continuous and bounded from $J \times S$ into $E$, then*

$$\psi(f(J \times S)) = \max_{t \in J} \psi(f(t, S), \forall S \subset D.$$

## 3. Main results

**Lemma 3.1.** *Suppose that $T$ has a right differentiable at zero majorant $g : K \longrightarrow K$ such that $g(0) = 0$, $g'(0) \in C_K(E)$ satisfying $r\left(g'(0)\right) < 1$ and $K$ is a normal cone. Then $T$ has at least one positive fixed point.*

**Proof.** Let us prove existence of $r > 0$ small enough, such that for all $t \in [0, 1]$ equation $tTu = u$ has no solution in $\partial K_r = K \cap B\left(0_E, r\right)$ where $B\left(0_E, r\right)$ is the open ball of radius $r$ centred at $0_E$, and let $\partial K_r$ be its boundary.
By the contrary suppose that for each integer $n \geq 1$ there exist $t_n \in [0, 1]$ and $u_n \in \partial K_{\frac{1}{n}}$, such that

$$u_n = t_n T u_n.$$

Note that $v_n = u_n / \|u_n\| \in \partial K_1$ and satisfies

$$v_n \preceq \frac{g(u_n)}{\|u_n\|}. \tag{3.1}$$

Thus, we have:

$$(3.2)\qquad \frac{g(u_n)}{\|u_n\|} = \frac{g(u_n) - g'(0)(u_n)}{\|u_n\|} + \frac{g'(0)(u_n)}{\|u_n\|}.$$

We set

$$G_n(u_n) = \frac{g(u_n) - g'(0)(u_n)}{\|u_n\|}.$$

Clearly

$$(3.3)\qquad v_n \preceq G_n(u_n) + g'(0)(v_n).$$

We obtain from (3.3), that is

$$\begin{aligned} v_n &\preceq G_n(u_n) + g'(0)(v_n) \\ &\preceq G_n(u_n) + g'(0)(G_n(u_n)) + \left[g'(0)\right]^2 (v_n) \\ &\preceq G_n(u_n) + g'(0)(G_n(u_n)) + \left[g'(0)\right]^2 (G_n(u_n)) + \left[g'(0)\right]^3 (v_n) \\ &\vdots \\ &\preceq I_{n,k} + \left[g'(0)\right]^k (v_n) \end{aligned}$$

where $I_{n,k} = \sum_{i=0}^{k} \left[g'(0)\right]^i (G_n(u_n))$ and we have from the normality of the cone $K$

$$\begin{aligned} 1 &\leq c_K \parallel I_{n,k} \parallel + c_K \parallel \left[g'(0)\right]^k (v_n) \parallel \\ &\leq c_K \parallel I_{n,k} \parallel + c_K \parallel \left[g'(0)\right]^k \parallel \end{aligned}$$

in which by letting $n \longrightarrow \infty$, yields $1 \leq c_K \parallel [g'(0)]^k \parallel$ . Then $1 \leq c_K^{\frac{1}{k}} \parallel [g'(0)]^k \parallel^{\frac{1}{k}}$ . leading to the contradiction by letting $k \longrightarrow \infty$, $1 \leq r\,(g'(0)) < 1$, and proves existence of $r > 0$ small enough such that for all $t \in [0, 1]$ equation $tTu = u$ has no solution in $\partial K_r$. For a such $r > 0$, we deduce from Lemma 2.9 that

$$i(T, K_r, K) = 1$$

and $T$ has a positive fixed point $u$ with $\|u\| < r$. This completes the proof. □

Arguing as in the proof of Theorem 3.1, we obtain the following result.

**Lemma 3.2.** *Suppose that $T$ has an asymptotically linear majorant $g : K \longrightarrow K$ such that $g'(\infty) \in C_K(E)$ satisfying $r(g'(\infty)) < 1$ and $K$ is normal. Then $T$ has at least one positive fixed point.*

**Theorem 3.3.** *Suppose that the cone $K$ is a normal cone and $T$ has an asymptotically linear majorant $g : K \longrightarrow K$ such that $g'(\infty) \in C_K(E)$. Suppose that $T$ has a right differentiable at zero minorant $h : K \longrightarrow K$ such that $h(0) = 0$ and $h'(0) \in C_K(E)$ satisfying $r(g'(\infty)) < 1 < \theta_P^{h'(0)}$. Then $T$ has at least one positive nontrivial fixed point.*

**Proof.** We have to prove existence of $0 < r < R$ such that

$$i(T, K_r, K) = 0 \text{ and } i(T, K_R, K) = 1.$$

In such a situation, additivity and solution properties of the fixed point index imply that

$$i(T, K_R\overline{K_r}, K) = i(T, K_R, K) - i(T, K_r, K) = 1$$

and $T$ has a positive fixed point $u$ with $r < \|u\| < R$.

Let us prove existence of $R > 0$ Big enough, such that for all $t \in [0, 1]$ equation $tTu = u$ has no solution in $\partial K_R$. By the contrary suppose that for each integer $n \geq 1$ there exist $t_n \in [0, 1]$ and $u_n \in \partial K_n$ such that

$$u_n = t_n T u_n.$$

Note that $w_n = u_n / \|u_n\| \in \partial K_1$ and satisfies

$$w_n \leq \frac{g(u_n)}{\|u_n\|}. \tag{3.4}$$

Thus, we have:

$$\frac{g(u_n)}{\|u_n\|} = \frac{g(u_n) - g'(\infty)(u_n)}{\|u_n\|} + \frac{g'(\infty)(u_n)}{\|u_n\|}. \tag{3.5}$$

We set

$$G_n(u_n) = \frac{g(u_n) - g'(\infty)(u_n)}{\|u_n\|}.$$

Clearly

$$w_n \preceq G_n(u_n) + g'(\infty)(w_n).$$

By the same argument used in Lemma 3.2, we obtain that:

$$w_n \preceq J_{n,k} + [g'(\infty)]^k (w_n).$$

where $J_{n,k} = \displaystyle\sum_{i=0}^{k} [g'(\infty)]^i (G_n(u_n))$ and we have from the normality of the cone $K$

$$\begin{aligned} 1 &\leq c_K \parallel J_{n,k} \parallel + c_K \parallel \left[g'(\infty)\right]^k (v_n) \parallel \\ &\leq c_K \parallel J_{n,k} \parallel + c_K \parallel \left[g'(\infty)\right]^k \parallel \end{aligned}$$

in which by letting $n \longrightarrow \infty$, yields $1 \leq c_K \parallel [g'(\infty)]^k \parallel$ . Then $1 \leq c_K^{\frac{1}{k}} \parallel [g'(\infty)]^k \parallel^{\frac{1}{k}}$ . leading to the contradiction by letting $k \longrightarrow \infty$, $1 \leq r(g'(\infty)) < 1$, and proves existence of $R > 0$ big enough such that for all $t \in [0, 1]$ equation $tTu = u$ has no solution in $\partial K_R$. For a such $R > 0$, we deduce from Lemma 2.9 that

$$i(T, K_R, K) = 1$$

Let $e > 0$ such that $h'(0)(e) \succeq \theta_K^{h'(0)} e$ and let us prove existence of $r > 0$ small enough, such that for all $t > 0$ equation $Tu + te = u$ has no solution in $\partial K_r$. By the contrary suppose that for each integer $n \geq 1$ there exist $t_n \in \mathbf{R}^+$ and $u_n \in \partial K_{\frac{1}{n}}$ such that

$$u_n = Tu_n + t_n e.$$

Note that $v_n = u_n / \|u_n\| \in \partial K_1$ and satisfies the inequality:

$$v_n \succeq (Tu_n / \|u_n\|) \succeq \frac{h(u_n)}{\|u_n\|}. \tag{3.6}$$

Thus, we have

$$\frac{h(u_n)}{\|u_n\|} = \frac{h(u_n) - h'(0)(u_n)}{\|u_n\|} + \frac{h'(0)(u_n)}{\|u_n\|}. \tag{3.7}$$

We set

$$H_n(u_n) = \frac{h(u_n) - h'(0)(u_n)}{\|u_n\|}.$$

Then, one has

$$v_n \succeq H_n(u_n) + h'(0)(v_n). \tag{3.8}$$

and

$$v_n \succeq \frac{t_n e}{\|u_n\|} \succeq t_n e \tag{3.9}$$

We obtain

$$\begin{aligned} v_n &\succeq H_n(u_n) + h'(0)(v_n) \\ &\succeq H_n(u_n) + h'(0)(H_n(u_n)) + \left[h'(0)\right]^2(v_n) \\ &\succeq H_n(u_n) + h'(0)(H_n(u_n)) + \left[h'(0)\right]^2(H_n(u_n)) + \left[h'(0)\right]^3(v_n) \\ &\vdots \\ &\succeq H_n(u_n) + h'(0)(H_n(u_n)) + \left[h'(0)\right]^2(H_n(u_n)) + \cdots + \left[h'(0)\right]^k(H_n(u_n)) + \left[h'(0)\right]^k(v_n) \\ &= I_{n,k} + \left[h'(0)\right]^k(v_n). \end{aligned}$$

We have from (3.9)

$$v_n \succeq I_{n,k} + t_n \left(\theta_K^{h'(0)}\right)^k e$$

where $I_{n,k} = \sum_{i=0}^{k} \left[h'(0)\right]^i (H_n(u_n))$ and the normality of $K$ leads to

$$c_K \parallel v_n \parallel \geq t_n \left(\theta_K^{h'(0)}\right)^k \parallel e \parallel - \parallel I_{n,k} \parallel \tag{3.10}$$

letting $n \to \infty$, we obtain

$$c_K \geq t \left(\theta_K^{h'(0)}\right)^k \parallel e \parallel$$

Taking in account, $\theta_K^{h'(0)} > 1$, we obtain the contradiction

$$0_E \prec t \parallel e \parallel \preceq c_K / \left(\theta_K^{h'(0)}\right)^k \to 0_E \text{ as} k \to \infty.$$

Thus, we have from Lemma 2.10, $i(T, K_r, K) = 0$. This completes the proof. □

### 3.1. Application to second order bvp

Throughout the remainder of this paper, we apply the above results to a second-order differential equation in Banach spaces:

$$\begin{cases} u''(t) + f(t, u(t)) = \theta & 0 < t < 1 \\ u(0) = \int_0^1 s\, u(s) ds \; u(1) = \theta, \end{cases} \tag{3.11}$$

where $f \in C[[0,1] \times P, P$, $\theta$ is the zero element of $E$.
We consider problem (3.11) in $C(J,E)$, with $J = [0,1]$. Clearly that $(C(J,E), \| \, . \, \|_c)$ is a Banach space with the norm $\| \, u \, \|_c = \max_{t \in J} \| \, u(t) \, \|$ for $u \in C(J,E)$.
We suppose that:
$(H)$ $f \in C[J \times P, P]$, and let $l > 0$, $f(t,x)$ is uniformly continuous on $J \times (P \cap S_l)$ and there exists a constant $L_l$ with $0 \leq L_l < \frac{5}{2}$ such that

$$\psi(f(t,S)) \leq L_l \psi(S), \, \forall t \in J, \, S \subset P \cap S_l,$$

where $S_l = \{u \in E, \, \| \, u \, \| < r\}$ and here $\psi$ denotes the Kuratowski measure of non-compactness on $S$.

Tt is easy to see that the problem (3.11) has a solution $u = u(t)$ if and only if $u$ is a solution of the operator equation

$$Tu(x) = \int_0^1 H(t,s)\, f(s,u(s))\, ds, \tag{3.12}$$

where $H(t,s)$ is the Green's function associated with (3.11) given by

$$H(t,s) = G(t,s) + \frac{6}{5}(1-t) \int_0^1 \tau \, G(s,\tau) d\tau, \tag{3.13}$$

where

$$G(t,s) \quad = \quad \begin{cases} t\,(1-s) & t \leq \; s, \\ s\,(1-t) & s \leq \; t. \end{cases} \tag{3.14}$$

Let $e(x) = x(1-x)$, $\forall x \in [0,1]$. We may prove the following Lemma.

**Lemma 3.4.** *The Green's function $G(t,s)$ possesses the following properties:*

*1* $G(t,s) \geq 0$, $\forall t, s \in [0,1]$.

*2* $e(t)\, e(s) \leq G(t,s) \leq e(t) \leq \bar{e} = \max_{[0,1]} e(t) = \frac{1}{4}$.

*3* $G(t,s) \leq e(s)$, $\forall t, s \in [0,1]$.

*4 Let* $\delta \in (0, \frac{1}{2})$, $J_\delta = [\delta, 1-\delta]$, *then*

$$G(t,s) \geq \delta G(\tau,s), \, \forall t \in J_\delta \, \forall \tau, s \in [0,1].$$

**Proposition 3.5.** *Assume that $(H)$ hold. Then for $t, s \in [0,1]$, we have:*

$$\frac{1}{10} e(t)\, e(s) \leq H(t,s) \leq \frac{8}{5} e(s) \leq \frac{2}{5} = B_H, \, \forall t, s \in [0,1].$$

**Proof.**

$$\begin{aligned} H(t,s) &\geq \frac{6}{5}(1-t)\int_0^1 \tau\, G(s,\tau)\, dz \\ &\geq \frac{6}{5}(1-t)\int_0^1 \tau\, e(s)\, e(\tau)\, dz \\ &= \frac{6}{5}t(1-t)e(s)\int_0^1 \tau\ e(\tau)\, dz \\ &= \frac{6}{5}\frac{1}{12}e(t)e(s) \\ &= \frac{1}{10}e(t)e(s). \end{aligned}$$

In other hand, since $G(t,s) \leq e(s)$, we have:

$$\begin{aligned} H(t,s) &\leq G(t,s) + \frac{6}{5}(1-t)\int_0^1 \tau\, e(s)\, dz \\ &\leq e(s) + \frac{6}{5}e(s)\int_0^1 \tau\, e(s)\, e(\tau)\, dz \\ &= \frac{8}{5}\, e(s). \end{aligned}$$

□

**Proposition 3.6.** *Assume that $(H)$ hold. Then for $t \in J_\delta$, $\tau, s \in [0,1]$, we have:*

$$H(t,s) \leq \delta\, H(\tau,s).$$

**Proof.**

$$\begin{aligned} H(t,s) &\geq \delta\, G(\tau,s) + \frac{6}{5}\int_0^1 G(s,\tau)\,\tau\, dz \\ &\geq \delta\, G(\tau,s) + \frac{6\delta}{5}\int_0^1 G(s,\tau)\,\tau\, dz \\ &\geq \delta\, G(\tau,s) + \frac{6\delta}{5}(1-\tau)\int_0^1 G(s,v)\, v\, dv \\ &\geq \delta\, H(\tau,s),\ \forall\, \tau, s \in [0,1]. \end{aligned}$$

□

Let $Q$ and $K$ denote two cones such that: Let the set

$$Q = \{\ u\ \in C(J,\ E),\ \ u(t) =\geq \theta,\ t \in J\}$$

and

$$K = \{ u \in Q,\ u(t) \geq \delta u(s),\ t \in J_\delta, s \in [0,1]\}.$$

It is easy to see that $K$ is a cone of $C(J,\, E)$. We will make use of the following Lemma:

**Lemma 3.7.** *Suppose that* $(H)$ *holds and* $L_l < (B_H)^{-1}$. *Then for each* $l > 0$, $T$ *is a strict-set contraction on* $Q \cap B_l$, *that is there exists a constant* $\gamma_l = B_H\, L_l$, *such that*

$$\psi(T(S)) \leq \gamma_l \psi(S),\ \forall S \subset Q \cap B_l,$$

*where* $B_l = \{u \in C(J,\, E),\ \| \ u \ \|_c \leq l\}$.

**Proof.** By $(H)$, $f$ is uniformly continuous on $J \times (P \cap S_l)$. From Lemma 2.16, we obtain

$$\psi(f(J \times S)) = \max_{t \in J} \psi(f(t, S) \leq L_l \psi(S).$$

Since $f$ is uniformly continuous and bounded on $S \subset Q \cap B_l$, then $T$ is continuous and bounded from $Q \cap B_l$ into $Q$. □

**Lemma 3.8.** *Suppose that* $(H)$ *holds. Then* $T(K) \subset K$ *and* $T : K \longrightarrow K$ *is a strict-set contraction.*

**Proof.** We have from Lemma 3.6 that

$$\begin{aligned} \min_{t \in J_\delta} Tu(t) &= \min_{t \in J_\delta} \int_0^1 H(t,s)\, f(s, u(s))\, ds \\ &\geq \delta \int_0^1 H(\tau, s)\, f(s, u(s))\, ds \\ &= Tu(\tau),\ \forall \tau \in J. \end{aligned}$$

Therefore, $Tu \in K$ and $T(K) \subset K$.
Furthermore, we have

$$\begin{aligned} \psi(T(S(t))) &= \psi\left(\int_0^1 H(t,\tau)\, f(\tau, S(\tau))\, d\tau\right) \\ &\leq \psi\left(\int_0^1 \frac{2}{5}\, f(\tau, S(\tau))\, d\tau\right) \end{aligned}$$

$$\begin{aligned}
&\leq \frac{2}{5}\psi\left(\int_0^1 f(\tau, S(\tau))\,d\tau\right)\\
&\leq \frac{2}{5}\psi\left(\int_0^1 \max_{t\in J} f(\tau, S(\tau))\,d\tau\right)\\
&= \frac{2}{5}\psi\left(f(J,S)\right)\\
&\leq \frac{2}{5}L_l\psi\left(S\right)\\
&= \gamma_l\psi\left(S\right).
\end{aligned}$$

This prove that $T : K \longrightarrow K$ is a strict-set contraction. □

We also consider the associated linear eigenvalue problem

$$\begin{cases} u''(t)\ + u(t)\ = \mu\, u(t) & 0 < t < 1 \\ u(0) = \displaystyle\int_0^1 s\, u(s)ds =\ u(1)\ =\ \theta, \end{cases} \tag{3.15}$$

**Lemma 3.9.** *The linear eigenvalue problem* (3.15) *has a unique positive eigenvalue* $\mu_\star > 0$.

**Proof.** Let the set

$$X = \{\ u\ \in C^1(J,E),\ u(0) = \int_0^1 s\,u(s)ds\ u(1) = \theta\}$$

be equipped with the norm defined for $u \in X$ by $\parallel u \parallel_X = \sup_{t\in J} \parallel u'(t) \parallel$ and $K_X = K \cap X\ isaconeon$X.
$Considertheoperator$L:X$\rightarrow X$ defined for $u \in X$ by

$$Lu(x) = \int_0^1 H(t,s)\,u(s)\,ds, \tag{3.16}$$

where $H$ is the green's function defined in (3.13).
Clearly that $\mu_\star > 0$ is a positive eigenvalue of (3.15) if and only if $\mu_\star^{-1} > 0$ is a positive eigenvalue of $L$. In view of Lemma 2.15, let us prove that $L(K_X{}^\star) \subset int(K_X)$. To this end, consider the set

$$O\ = \{u\ \in K_X,\ u > \theta \text{ in } (0,\ 1),\ u'(0) > \theta,\ u'(1) < \theta\}.$$

We have $O \subset K_X$ and the complement of a set $O_X$ is $O^c$, such that

$$O^c = F_1 \cup F_2 \cup F_3,$$

where

$$F_1 = \{u \in X,\ \exists x_0 \in [0,\ 1]\ with\ u(x_0) \leq \theta\},$$

$$F_2 = \{u \in X,\ u'(0) \leq \theta\},$$

$$F_3 = \{u \in X,\ u'(1) \geq \theta\}.$$

In the fact $F_1,\ F_2$ and $F_3$ are closed sets on X. To this aim let $(u_n)_n \subset F_1$, tending to $u$ in $X$ and $(x_n)_n \subset [0,\ 1]$ tending to $\overline{x}$ in $[0,\ 1]$, with $u'(x_n) \leq \theta$. We distinguish the following cases

- $\overline{x} \in ]0,\ 1[$; in such situation $u(\overline{x}) = \lim_{n\to+\infty} u_n(x_n) \leq \theta$ and $u \in F_1$.
- $\overline{x} = 0$; in this case we obtain

$$u'(0) = \lim_{n\to+\infty} \frac{u_n(x_n)}{x_n} \leq \theta, \text{ and } u \in F_2.$$

- $\overline{x} = 1$; in this case we obtain

$$u'(1) = \lim_{n\to+\infty} \frac{u_n'(x_n)}{x_n - 1} \geq \theta \text{ and } u \in F_3.$$

Therefore $O$ is an open subset of X.
On the other hand, let $u \in K_X\backslash\{\theta\}$ and $v = Lu$, clearly $v > \theta$ on $(0,1)$, and we have

$$v'(t) = \int_0^1 \frac{\partial G}{\partial t}(t,\ s)\, u(s)\, ds = \int_0^1 (\frac{s^3}{5} - \frac{6}{5}s)\, u(s)\, ds + \int_t^1 u(s) ds,\ \forall t \in [0,\ 1].$$

Then

1. $v'(0) = \int_0^1 (\frac{s^3}{5} - \frac{6}{5}s + 1)\, u(s)\, ds > \theta$
2. $v'(1) = \int_0^1 (\frac{s^3}{5} - \frac{6}{5}s)\, u(s)\, ds < \theta.$

That is,

$$L({K_X}^\star) \subset O \subset int(K_X).$$

Finally, lemma 2.15 guarantees existence of a unique positive eigenvalue of $L$ and we have $\mu_\star^{-1} = \theta_{K_X}$. □

Let introduce the following notations

$$f^0 = \limsup_{u\to 0}\left(\max_{0\le t\le 1}\frac{f(t,u)}{u}\right) \quad f^\infty = \limsup_{u\to \infty}\left(\max_{0\le t\le 1}\frac{f(t,u)}{u}\right)$$

**Theorem 3.10.** *The problem* (3.11) *admits a positive solution whenever one of the following conditions:*

$$f^\infty < \mu_\star < f^0 \tag{3.17}$$

**Proof.** Let $L : E \longrightarrow E$ be the operator defined by (3.16). It is easy to see that $L$ is an increasing and strict set contraction operator.

Since $L(K) \subset K_X$, it follows from **iv)** Remark 2.13 that

$$\mu_\star^{-1} = \theta_K^L$$

where $K_X$ is the cone defined in the proof of Lemma 3.9.

Let $F : K \to K$, the Nemytskii operator defined for $u \in K$ by $Fu(x) = f(x, u(x))$. We present the proof of Theorem 3.10 in the case where (3.17) holds. Hypothesis (3.17) implies that there exists $\epsilon > 0$, small enough such that

$$(\mu_\star + \epsilon)\, u - Hu \le Fu \le (\mu_\star - \epsilon)\, u + c \ \text{ for all } u \in K$$

where $Hu(x) = \max\left\{f(x, u(x)) - f^0 u(x), 0\right\}$. So, we get:

$$\alpha\, Lu - LHu \le Tu \le \beta\, Lu + B_H c \ \text{ for all } u \in K,$$

where $B_H = \frac{2}{5}$, $\alpha = (\mu_\star + \epsilon)$ and $\beta = (\mu_\star - \epsilon)$.

We introduce the following notation:

$$h(u) = \alpha\, Lu - LHu, \ g(u) = \beta\, Lu + cM$$

So we have

$$h(u) \le Tu \le g(u) \ \text{ for all } u \in K$$

Using the fact that $H(u) = \circ\left(\|u\|\right)$ near 0, we may show that $h'(0)$ is the derivative of $h$ along $K$ at zero, and $g'(0)$ is the derivative of $g$ along $K$ at $\infty$ such that

$$\frac{1}{\beta}\, g'(\infty)[u] = \frac{1}{\alpha}\, h'(0)[u] = Lu, \forall u \in K.$$

With this aim, let $u \in K$, there exists $\omega > 0$ such that

$$\|h(u) - h'(0)[u]\| = \|LHu\| \leq \omega \|Hu\| . \tag{3.18}$$

Therefore, we have:
$$\lim_{u\in K,\, \|u\| \longrightarrow 0} \frac{\|h(u) - h'(\infty)[u]\|}{\|u\|} \leq \omega \lim_{\|u\| \longrightarrow 0} \frac{\|Hu\|}{\|u\|}.$$
This means that

$$\lim_{u\in K,\, \|u\| \longrightarrow 0} \frac{\|h(u) - h'(0)[u]\|}{\|u\|} = 0.$$

Therefore, we have:
$$\lim_{u\in K,\, \|u\| \longrightarrow +\infty} \frac{\|g(u) - g'(\infty)[u]\|}{\|u\|} = \lim_{\|u\| \longrightarrow +\infty} \frac{cM}{\|u\|}.$$
Clearly that $h'(0), g'(\infty) \in C_K(E)$ and $\theta_P^{h'(0)} = \alpha\mu_\star^{-1}$ and $r(g'(\infty)) = \beta\mu_\star^{-1}$. So from the choice of $\alpha$, and $\beta$, we have

$$r(g'(\infty)) < 1 < \theta_P^{h'(0)}.$$

We deduce from the Theorem 3.3 existence results for positive solution to the bvp (3.11) . □

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