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Existence and uniqueness of positive solutions for nonlinear Caputo-Hadamard fractional differential equations

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Abstract:

We prove the existence and uniqueness of a positive solution of nonlinear Caputo-Hadamard fractional differential equations. In the process we employ the Schauder and Banach fixed point theorems and the method of upper and lower solutions to show the existence and uniqueness of a positive solution. Finally, an example is given to illustrate our results.

Keywords: Fractional differential equations; Positive solutions; Upper and lower solutions; Existence; Uniqueness; Fixed point theorems.

MSC (2020): 26A33, 34A12, 34G20.

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1. Introduction

Fractional differential equations arise from a variety of applications including in various fields of science and engineering. In particular, problems concerning qualitative analysis of the positivity of such solutions for fractional differential equations (FDE) have received the attention of many authors, see [1, 2, 3, 4, 5, 6, 8, 9, 13, 14, 15, 16] and the references therein.

Recently, Zhang in [16] investigated the existence and uniqueness of positive solutions for the nonlinear fractional differential equation

$$\begin{cases} D^{\alpha}x\,(t) = f(t, x(t)),\ 0 < t \leq 1, \\ x\,(0) = 0, \end{cases}$$

where D^{α} is the standard Riemann Liouville fractional derivative of order $0 < \alpha < 1$, and $f : [0, 1] \times [0, \infty) \to [0, \infty)$ is a given continuous function. By using the method of the upper and lower solution and cone fixed-point theorem, the author obtained the existence and uniqueness of a positive solution.

The nonlinear fractional differential equation boundary value problem

$$\begin{cases} D^{\alpha}x\,(t) + f(t, x(t)) = 0,\ 0 < t < 1, \\ x\,(0) = x\,(1) = 0, \end{cases}$$

has been investigated in [2], where $1 < \alpha \leq 2$, and $f : [0, 1]\times[0, \infty) \to [0, \infty)$ is a given continuous function. By means of some fixed-point theorems on cone, some existence and multiplicity results of positive solutions have been established.

In [9], Matar discussed the existence and uniqueness of the positive solution of the following nonlinear fractional differential equation

$$\begin{cases} {}^{C}D^{\alpha}x\,(t) = f(t, x(t)),\ 0 < t \leq 1, \\ x\,(0) = 0,\ x'\,(0) = \theta > 0, \end{cases}$$

where $\{{}^{c}D^{\alpha}x\,(t) =$ is the standard Caputo fractional derivative of order $1 < \alpha \leq 2$, and $f : [0, 1] \times [0, \infty) \to [0, \infty)$ is a given continuous function. By employing the method of the upper and lower solutions and Schauder and Banach fixed point theorems, the author obtained positivity results.

The nonlinear fractional differential equation

$$\begin{cases} {}^{C}D^{\alpha}x\,(t) = x\,(t) = f(t, x(t)) +^{C} D^{\alpha-1}g\,(t, x\,(t))\,,\ 0 < t \leq T, \\ x\,(0) = \theta_1 > 0,\ x'\,(0) = \theta_2 > 0, \end{cases}$$

has been investigated in [4], where $1 < \alpha \leq 2$, $g, f : [0,T] \times [0,\infty) \to [0,\infty)$ are given continuous functions, g is non-decreasing on x and $\theta_2 \geq g(0,\theta_1)$. By employing the method of the upper and lower solutions and Schauder and Banach fixed point theorems, the authors obtained positivity results.

Ahmad and Ntouyas in [1] studied the existence and uniqueness of solutions to the following boundary value problem

$$\begin{cases} \mathcal{D}_1^{\alpha}\left(\mathcal{D}_1^{\beta}u(t) - g(t,u_t)\right) = f(t,u_t),\ t \in [1,b], \\ u(t) = \phi(t),\ t \in [1-r,1], \\ \mathcal{D}_1^{\beta}u(1) = \eta \in \mathbf{R}, \end{cases}$$

where $\mathcal{D}_1^{\alpha}$ and $\mathcal{D}_1^{\beta}$ are the Caputo-Hadamard fractional derivatives, $0 < \alpha, \beta < 1$. By employing the fixed point theorems, the authors obtained existence and uniqueness results.

In this paper, we are interested in the analysis of qualitative theory of the problems of the positive solutions to fractional differential equations. Inspired and motivated by the works mentioned above and the papers [1]-[6], [8], [9], [13]-[16] and the references therein, we concentrate on the positivity of solutions for the nonlinear Caputo-Hadamard fractional differential equation

$$\begin{cases} \mathcal{D}_1^{\alpha}x(t) = f(t,x(t)) + \mathcal{D}_1^{\alpha-1}g(t,x(t)),\ 1 < t \leq T, \\ x(1) = \theta_1 > 0,\ x'(1) = \theta_2 > 0, \end{cases} \tag{1.1}$$

where $1 < \alpha \leq 2$, $g, f : [1,T] \times [0,\infty) \to [0,\infty)$ are given continuous functions, g is non-decreasing on x and $\theta_2 \geq g(1,\theta_1)$. To show the existence and uniqueness of the positive solution, we transform (1.1) into an integral equation and then by the method of upper and lower solutions and use Schauder and Banach fixed point theorems.

This paper is organized as follows. In section 2, we introduce some notations and lemmas, and state some preliminaries results needed in later section. Also, we present the inversion of (1.1) and the Banach and Schauder fixed point theorems. For details on Banach and Schauder theorems we refer the reader to [12]. In Section 3, we give and prove our main results on positivity, and we provide an example to illustrate our results.

2. Preliminaries

Let $X = C([1,T])$ be the Banach space of all real-valued continuous functions defined on the compact interval $[1,T]$, endowed with the maximum norm. Define the subspace $\mathcal{A}=\{x\in X : x(t)\geq 0,\ t\in[1,T]\}$ of X. By a positive solution $x\in X$, we mean a function $x(t)>0$, $1\leq t\leq T$.

Let $a,b\in\mathbf{R}^{+}$ such that $b>a$. For any $x\in[a,b]$, we define the upper-control function $U(t,x)=\sup\{f(t,\lambda) : a\leq\lambda\leq x\}$, and lower-control function $L(t,x)=\inf\{f(t,\lambda) : x\leq\lambda\leq b\}$. Obviously, $U(t,x)$ and $L(t,x)$ are monotonous non-decreasing on the argument x and $L(t,x)\leq f(t,x)\leq U(t,x)$.

We introduce some necessary definitions, lemmas and theorems which will be used in this paper. For more details, see [7, 10, 11].

Definition 2.1 ([7]). *The Hadamard fractional integral of order $\alpha>0$ for a continuous function $x:[1,+\infty)\to\mathbf{R}$ is defined as*

$$\mathcal{I}_1^{\alpha}x(t)=\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}x(s)\frac{ds}{s},\ \alpha>0.$$

Definition 2.2 ([7]). *The Caputo-Hadamard fractional derivative of order $\alpha>0$ for a continuous function $x:[1,+\infty)\to\mathbf{R}$ is defined as*

$$\mathcal{D}_1^{\alpha}x(t)=\frac{1}{\Gamma(n-\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{n-\alpha-1}\delta^n(x)(s)\frac{ds}{s}, n-1<\alpha<n,$$

where $\delta^n=\left(t\frac{d}{dt}\right)^n$, $n\in\mathbf{N}$.

Lemma 2.3 ([7]). *Let $n-1<\alpha\leq n$, $n\in\mathbf{N}$ and $x\in C^n([1,T])$. Then*
$(\mathcal{D}_1^{\alpha}\mathcal{I}_1^{\alpha}x)(t)=x(t)$,
$(\mathcal{I}_1^{\alpha}\mathcal{D}_1^{\alpha}x)(t)=x(t)-\sum_{k=0}^{n-1}\frac{x^{(k)}(1)}{\Gamma(k+1)}(\log t)^k$.

Lemma 2.4 ([7]). *For all $\mu>0$ and $\nu>-1$,*

$$\frac{1}{\Gamma(\mu)}\int_1^t\left(\log\frac{t}{s}\right)^{\mu-1}(\log s)^{\nu}\frac{ds}{s}=\frac{\Gamma(\nu+1)}{\Gamma(\mu+\nu+1)}(\log t)^{\mu+\nu}.$$

The following lemma is fundamental to our results.

Lemma 2.5. *Let $x\in C^1([1,T]), x^{(2)}$ and $\frac{\partial g}{\partial t}$ exist, then x is a solution of (1.1) if and only if*

$$x(t)=\theta_1+(\theta_2-g(1,\theta_1))\log t+\int_1^t g(s,x(s))\frac{ds}{s}$$
$$+\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}f(s,x(s))\frac{ds}{s}. \tag{2.1}$$

Proof. Let x be a solution of (1.1). First we write this equation as

$$\mathcal{I}_1^{\alpha}\mathcal{D}_1^{\alpha}x\left(t\right)=\mathcal{I}_1^{\alpha}\left(f(t,x(t))+\mathcal{D}_1^{\alpha-1}g\left(t,x\left(t\right)\right)\right),\ 1<t\leq T.$$

From Lemma 2.3, we have

$$\begin{aligned}
&x(t)-x(1)-x'(1)\log t\\
&=\mathcal{I}_1^{\alpha}\mathcal{D}_1^{\alpha-1}g\left(t,x\left(t\right)\right)+\mathcal{I}_1^{\alpha}f(t,x(t))\\
&=\mathcal{I}_1^{1}\mathcal{I}_1^{\alpha-1}\mathcal{D}_1^{\alpha-1}g\left(t,x\left(t\right)\right)+\mathcal{I}_1^{\alpha}f(t,x(t))\\
&=\mathcal{I}_1^{1}\left(g\left(t,x\left(t\right)\right)-g\left(1,x\left(1\right)\right)\right)+\mathcal{I}_1^{\alpha}f(t,x(t))\\
&=\textstyle\int_1^t g\left(s,x\left(s\right)\right)\frac{ds}{s}-g\left(1,x\left(1\right)\right)\log t+\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}f(s,x(s))\frac{ds}{s},
\end{aligned}$$

then, by using the initial conditions, we obtain (2.1).

Conversely, suppose x satisfies (2.1). By Lemma 2.3, we observe that

$$\begin{aligned}
\mathcal{D}_1^{\alpha}x(t)&=\mathcal{D}_1^{\alpha}\Big(\theta_1+\left(\theta_2-g\left(1,\theta_1\right)\right)\log t+\textstyle\int_1^t g\left(s,x\left(s\right)\right)\frac{ds}{s}\\
&\quad+\textstyle\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}f(s,x(s))\frac{ds}{s}\Big)\\
&=\mathcal{D}_1^{\alpha}\left(\theta_1+\left(\theta_2-g\left(1,\theta_1\right)\right)\log t\right)\\
&\quad+\mathcal{D}_1^{\alpha}\mathcal{I}_1^{1}g\left(t,x\left(t\right)\right)+\mathcal{D}_1^{\alpha}\mathcal{I}_1^{\alpha}f(t,x(t))\\
&=\mathcal{D}_1^{\alpha-1}g\left(t,x\left(t\right)\right)+f(t,x(t)).
\end{aligned}$$

Moreover, the initial conditions $x\left(1\right)=\theta_1$ and $x'(1)=\theta_2$ hold. This completes the proof. □

Definition 2.6. *Let $x^*,x_*\in\mathcal{A}$, $a\leq x_*\leq x^*\leq b$, satisfy*

$$\begin{cases}\mathcal{D}_1^{\alpha}x^*\left(t\right)-\mathcal{D}_1^{\alpha-1}g\left(t,x^*\left(t\right)\right)\geq U(t,x^*(t)),\ 1<t\leq T,\\ x^*\left(1\right)\geq\theta_1,\ x^{*\prime}\left(1\right)\geq\theta_2,\end{cases}$$

or

$$\begin{aligned}
x^*\left(t\right)&\geq\theta_1+\left(\theta_2-g\left(1,\theta_1\right)\right)\log t+\int_1^t g\left(s,x^*\left(s\right)\right)\frac{ds}{s}\\
&+\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}U(s,x^*(s))\frac{ds}{s},\quad 1\leq t\leq T,
\end{aligned}$$

and

$$\begin{cases}\mathcal{D}_1^{\alpha}x_*\left(t\right)-\mathcal{D}_1^{\alpha-1}g\left(t,x_*\left(t\right)\right)\leq L(t,x_*(t)),\ 1<t\leq T,\\ x_*\left(1\right)\leq\theta_1,\ x_*'\left(1\right)\leq\theta_2,\end{cases}$$

or

$$x_*(t) \leq \theta_1 + (\theta_2 - g(1,\theta_1))\log t + \int_1^t g(s, x_*(s))\frac{ds}{s}$$
$$+\frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} L(s, x_*(s))\frac{ds}{s}, \quad 1 \leq t \leq T.$$

Then the functions x^* and x_* are called a pair of upper and lower solutions for the equation (1.1).

Lastly in this section, we state the fixed point theorems which enable us to prove the existence and uniqueness of a positive solution of (1.1).

Definition 2.7. *Let $(X, \|.\|)$ be a Banach space and $\Phi : X \to X$. The operator Φ is a contraction operator if there is an $\lambda \in (0,1)$ such that $x, y \in X$ imply*

$$\|\Phi x - \Phi y\| \leq \lambda \|x - y\|.$$

Theorem 2.8 (Banach [12]). *Let $\mathcal{C}$ be a nonempty closed convex subset of a Banach space X and $\Phi : \mathcal{C} \to \mathcal{C}$ be a contraction operator. Then there is a unique $x \in \mathcal{C}$ with $\Phi x = x$.*

Theorem 2.9 (Schauder [12]). *Let $\mathcal{C}$ be a nonempty closed convex subset of a Banach space X and $\Phi : \mathcal{C} \to \mathcal{C}$ be a continuous compact operator. Then Φ has a fixed point in $\mathcal{C}$.*

3. Main results

In this section, we consider the results of existence problem for many cases of the FDE (1.1). Moreover, we introduce the sufficient conditions of the uniqueness problem of (1.1).

To transform equation (2.1) to be applicable to Schauder fixed point, we define an operator $\Phi : \mathcal{A} \longrightarrow X$ by

$$(\Phi x)(t) = \theta_1 + (\theta_2 - g(1,\theta_1))\log t + \int_1^t g(s, x(s))\frac{ds}{s}$$
$$+\frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} f(s, x(s))\frac{ds}{s},\ t \in [1, T], \tag{3.1}$$

where the figured fixed point must satisfy the identity operator equation $\Phi x = x$.

Theorem 3.1. *Assume that x^* and x_* are upper and lower solutions of (1.1). Then the FDE (1.1) has at least one solution $x \in X$ satisfying $x_*(t) \leq x(t) \leq x^*(t)$, $t \in [1,T]$.*

Proof. Let $\mathcal{C} = \{x \in \mathcal{A} : x_*(t) \leq x(t) \leq x^*(t),\ t \in [1,T]\}$, endowed with the norm $\|x\| = \max_{t\in[1,T]} |x(t)|$, then we have $\|x\| \leq b$. Hence, $\mathcal{C}$ is a convex, bounded and closed subset of the Banach space X. Moreover, the continuity of g and f imply the continuity of the operator Φ on $\mathcal{C}$ defined by (3.1). Now, the continuity of g and f on $[1,T]\times\mathcal{C}$ also imply that there exist positive constants c_f and c_g such that

$$\max\{f(t,x(t)) : t \in [1,T],\ x \in \mathcal{C}\} < c_f,$$

and

$$\max\{g(t,x(t)) : t \in [1,T],\ x \in \mathcal{C}\} < c_g.$$

Then

$$\begin{aligned} |(\Phi x)(t)| &\leq |\theta_1 + (\theta_2 - g(1,\theta_1))\log t| + \textstyle\int_1^t |g(s,x(s))| \frac{ds}{s} \\ &+ \tfrac{1}{\Gamma(\alpha)} \textstyle\int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} |f(s,x(s))| \frac{ds}{s} \\ &\leq \theta_1 + (\theta_2 + c_0 + c_g)\log T + \tfrac{c_f(\log T)^\alpha}{\Gamma(\alpha+1)}, \end{aligned}$$

where $c_0 = |g(1,\theta_1)|$. Thus,

$$\|\Phi x\| \leq \theta_1 + (\theta_2 + c_0 + c_g)\log T + \frac{c_f(\log T)^\alpha}{\Gamma(\alpha+1)}.$$

Hence, $\Phi(\mathcal{C})$ is uniformly bounded. Next, we prove the equicontinuity of $\Phi(\mathcal{C})$. Let $x \in \mathcal{C}$, then for any $t_1, t_2 \in [1,T]$, $t_2 > t_1$, we have

$$\begin{aligned} &|(\Phi x)(t_2) - (\Phi x)(t_1)| \\ &\leq (\theta_2 + c_0)(\log t_2 - \log t_1) + \left|\textstyle\int_1^{t_2} g(s,x(s))\frac{ds}{s} - \int_1^{t_1} g(s,x(s))\frac{ds}{s}\right| \\ &+ \left|\tfrac{1}{\Gamma(\alpha)}\textstyle\int_1^{t_2}\left(\log\frac{t_2}{s}\right)^{\alpha-1} f(s,x(s))\frac{ds}{s} - \tfrac{1}{\Gamma(\alpha)}\int_1^{t_1}\left(\log\frac{t_1}{s}\right)^{\alpha-1} f(s,x(s))\frac{ds}{s}\right| \\ &\leq (\theta_2 + c_0)\left(\log\tfrac{t_2}{t_1}\right) + \textstyle\int_{t_1}^{t_2}|g(s,x(s))|\frac{ds}{s} \\ &+ \tfrac{1}{\Gamma(\alpha)}\textstyle\int_1^{t_1}\left(\left(\log\frac{t_2}{s}\right)^{\alpha-1} - \left(\log\frac{t_1}{s}\right)^{\alpha-1}\right)|f(s,x(s))|\frac{ds}{s} \\ &+ \tfrac{1}{\Gamma(\alpha)}\textstyle\int_{t_1}^{t_2}\left(\log\frac{t_2}{s}\right)^{\alpha-1}|f(s,x(s))|\frac{ds}{s} \\ &\leq (\theta_2 + c_0 + c_g)\left(\log\tfrac{t_2}{t_1}\right) + \tfrac{c_f}{\Gamma(\alpha+1)}\left((\log t_2)^\alpha - (\log t_1)^\alpha\right). \end{aligned}$$

As $t_1 \to t_2$ the right-hand side of the previous inequality is independent of x and tends to zero. Therefore, $\Phi(\mathcal{C})$ is equicontinuous. The Arzelè-Ascoli Theorem implies that $\Phi : \mathcal{C} \longrightarrow X$ is compact. The only thing to apply Schauder fixed point is to prove that $\Phi(\mathcal{C}) \subseteq \mathcal{C}$. Let $x \in \mathcal{C}$, then by hypotheses, we have

$$\begin{aligned}(\Phi x)(t) &= \theta_1 + (\theta_2 - g(1,\theta_1))\log t + \textstyle\int_1^t g(s, x(s))\frac{ds}{s} \\ &+ \tfrac{1}{\Gamma(\alpha)} \textstyle\int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} f(s, x(s))\frac{ds}{s} \\ &\leq \theta_1 + (\theta_2 - g(1,\theta_1))\log t + \textstyle\int_1^t g(s, x^*(s))\frac{ds}{s} \\ &+ \tfrac{1}{\Gamma(\alpha)} \textstyle\int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} U(s, x(s))\frac{ds}{s} \\ &\leq \theta_1 + (\theta_2 - g(1,\theta_1))\log t + \textstyle\int_1^t g(s, x^*(s))\frac{ds}{s} \\ &+ \tfrac{1}{\Gamma(\alpha)} \textstyle\int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} U(s, x^*(s))\frac{ds}{s} \\ &\leq x^*(t),\end{aligned}$$

and

$$\begin{aligned}(\Phi x)(t) &= \theta_1 + (\theta_2 - g(1,\theta_1))\log t + \textstyle\int_1^t g(s, x(s))\frac{ds}{s} \\ &+ \tfrac{1}{\Gamma(\alpha)} \textstyle\int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} f(s, x(s))\frac{ds}{s} \\ &\geq \theta_1 + (\theta_2 - g(1,\theta_1))\log t + \textstyle\int_1^t g(s, x_*(s))\frac{ds}{s} \\ &+ \tfrac{1}{\Gamma(\alpha)} \textstyle\int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} L(t, x(s))\frac{ds}{s} \\ &\geq \theta_1 + (\theta_2 - g(1,\theta_1))\log t + \textstyle\int_1^t g(s, x_*(s))\frac{ds}{s} \\ &+ \tfrac{1}{\Gamma(\alpha)} \textstyle\int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} L(t, x_*(s))\frac{ds}{s} \\ &\geq x_*(t).\end{aligned}$$

Hence, $x_*(t) \leq (\Phi x)(t) \leq x^*(t)$, $t \in [1,T]$, that is, $\Phi(\mathcal{C}) \subseteq \mathcal{C}$. According to Schauder fixed point theorem, the operator Φ has at least one fixed point $x \in \mathcal{C}$. Therefore, the FDE (1.1) has at least one positive solution $x \in X$ and $x_*(t) \leq x(t) \leq x^*(t),\ t \in [1,T]$. □

Next, we consider many particular cases of the previous theorem.

Corollary 3.2. *Assume that there exist continuous functions k_1, k_2,k_3 and k_4 such that*

$$0 < k_1(t) \leq g(t, x(t)) \leq k_2(t) < \infty,\ (t, x(t)) \in [1,T] \times [0, +\infty),$$

$$\theta_2 \geq k_1(1),\ \theta_2 \geq k_2(1), \tag{3.2}$$

and

$$0 < k_3(t) \leq f(t, x(t)) \leq k_4(t) < \infty,\ (t, x(t)) \in [1,T] \times [0, +\infty). \tag{3.3}$$

Then, the FDE (1.1) has at least one positive solution $x \in X$. Moreover,

$$\theta_1 + (\theta_2 - k_1(1))\log t + \int_1^t k_1(s)\frac{ds}{s} + \mathcal{I}_1^{\alpha} k_3(t)$$

$$\leq x(t)$$
$$\leq \theta_1 + (\theta_2 - k_2(1)) \log t + \int_1^t k_2(s) \frac{ds}{s} + \mathcal{I}_1^\alpha k_4(t). \tag{3.4}$$

Proof. By the given assumption (3.3) and the definition of control function, we have $k_3(t) \leq L(t,x) \leq U(t,x) \leq k_4(t)$, $(t, x(t)) \in [1,T] \times [a,b]$. Now, we consider the equations

$$\begin{cases} \mathcal{D}_1^\alpha x(t) = k_3(t) + \mathcal{D}_1^{\alpha-1} k_1(t), \ x(1) = \theta_1, \ x'(1) = \theta_2, \\ \mathcal{D}_1^\alpha x(t) = k_4(t) + \mathcal{D}_1^{\alpha-1} k_2(t), \ x(1) = \theta_1, \ x'(1) = \theta_2. \end{cases} \tag{3.5}$$

Obviously, Equations (3.5) are equivalent to

$$\begin{aligned} x(t) &= \theta_1 + (\theta_2 - k_1(1)) \log t + \textstyle\int_1^t k_1(s) \frac{ds}{s} + \mathcal{I}_1^\alpha k_3(t), \\ x(t) &= \theta_1 + (\theta_2 - k_2(1)) \log t + \textstyle\int_1^t k_2(s) \frac{ds}{s} + \mathcal{I}_1^\alpha k_4(t). \end{aligned}$$

Hence, the first implies

$$x(t) - \theta_1 - (\theta_2 - k_1(1)) \log t - \int_1^t k_1(s) \frac{ds}{s} = \mathcal{I}_1^\alpha k_3(t) \leq \mathcal{I}_1^\alpha (L(t, x(t))),$$

and the second implies

$$x(t) - \theta_1 - (\theta_2 - k_2(1)) \log t - \int_1^t k_2(s) \frac{ds}{s} = \mathcal{I}_1^\alpha k_4(t) \geq \mathcal{I}_1^\alpha (U(t, x(t))),$$

which are the upper and lower solutions of Equations (3.5), respectively. An application of Theorem 3.1 yields that the FDE (1.1) has at least one solution $x \in X$ and satisfies Equation (3.4). □

Corollary 3.3. *Assume that (3.2) holds and* $0 < \sigma < k(t) = \lim_{x \to \infty} f(t,x) < \infty$ *for* $t \in [1,T]$. *Then the FDE (1.1) has at least a positive solution* $x \in X$.

Proof. By assumption, if $x > \rho > 0$, then $0 \leq |f(t,x) - k(t)| < \sigma$ for any $t \in [1,T]$. Hence, $0 < k(t) - \sigma \leq f(t,x) \leq k(t) + \sigma$ for $t \in [1,T]$ and $\rho < x < +\infty$. Now if $\max\{f(t,x) : t \in [1,T], \ x \leq \rho\} \leq \nu$, then $k(t) - \sigma \leq f(t,x) \leq k(t) + \sigma + \nu$ for $t \in [1,T]$, and $0 < x < +\infty$. By Corollary 3.3, the FDE (1.1) has at least one positive solution $x \in X$ satisfying

$$\begin{aligned} &\theta_1 + (\theta_2 - k_1(1)) \log t + \textstyle\int_1^t k_1(s) \frac{ds}{s} + \mathcal{I}_1^\alpha k(t) - \frac{\sigma (\log t)^\alpha}{\Gamma(\alpha+1)} \\ &\leq x(t) \\ &\leq \theta_1 + (\theta_2 - k_2(1)) \log t + \textstyle\int_1^t k_2(s) \frac{ds}{s} + \mathcal{I}_1^\alpha k(t) + \frac{(\sigma+\nu)(\log t)^\alpha}{\Gamma(\alpha+1)}. \end{aligned}$$

□

Corollary 3.4. *Assume that* $0 < \sigma < f(t, x(t)) \leq \gamma x(t) + \eta < \infty$ *for* $t \in [1,T]$, *and* σ, η *and* γ *are positive constants. Then, the FDE (1.1) has at least one positive solution* $x \in C([1,\delta])$, *where* $1 < \delta < T$.

Proof. Consider the equation

$$\left\{\begin{array}{l}\mathcal{D}_1^{\alpha}x(t)-\mathcal{D}_1^{\alpha-1}g\left(t,x(t)\right)=\gamma x(t)+\eta,\ 1<t\leq T,\\ x(1)=\theta_1>0,\ x'(1)=\theta_2>0.\end{array}\right. \tag{3.6}$$

Equation (3.6) is equivalent to integral equation

$$\begin{array}{rcl} x(t) & = & \theta_1+\left(\theta_2-g\left(0,\theta_1\right)\right)\log t+\int_1^t g\left(s,x\left(s\right)\right)\frac{ds}{s} \\ & & +\frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\left(\gamma x(s)+\eta\right)\frac{ds}{s} \\ & & =\theta_1+\left(\theta_2-g\left(1,\theta_1\right)\right)\log t+\int_1^t g\left(s,x\left(s\right)\right)\frac{ds}{s} \\ & & +\frac{\eta(\log t)^{\alpha}}{\Gamma(\alpha+1)}+\frac{\gamma}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}x(s)\frac{ds}{s}. \end{array}$$

Let ω and ϕ be positive real numbers. Choose an appropriate $\delta\in(1,T)$ such that $0<\frac{\gamma(\log\delta)^{\alpha}}{\Gamma(\alpha+1)}<\phi<1$ and

$$\omega>(1-\phi)^{-1}\left(\theta_1+\left(\theta_2+c_0+c_g\right)\log\delta+\frac{\eta\left(\log\delta\right)^{\alpha}}{\Gamma(\alpha+1)}\right).$$

Then if $1\leq t\leq\delta$, the set $B_\omega=\{x\in X:|x(t)|\leq\omega,\ 1\leq t\leq\delta\}$ is convex, closed, and bounded subset of $C\left([1,\delta]\right)$. The operator $F:B_\omega\longrightarrow B_\omega$ given by

$$\begin{array}{rcl} \left(Fx\right)(t) & = & \theta_1+\left(\theta_2-g\left(1,\theta_1\right)\right)\log t+\int_1^t g\left(s,x\left(s\right)\right)\frac{ds}{s} \\ & & +\frac{\eta(\log t)^{\alpha}}{\Gamma(\alpha+1)}+\frac{\gamma}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}x(s)\frac{ds}{s}, \end{array}$$

is compact as in the proof of Theorem 3.1. Moreover,

$$\left|\left(Fx\right)(t)\right|\leq\theta_1+\left(\theta_2+c_0+c_g\right)\log\delta+\frac{\eta\left(\log\delta\right)^{\alpha}}{\Gamma(\alpha+1)}+\frac{\gamma\left(\log\delta\right)^{\alpha}}{\Gamma(\alpha+1)}\left\|x\right\|.$$

If $x\in B_\omega$, then

$$\left|\left(Fx\right)(t)\right|\leq\left(1-\phi\right)\omega+\phi\omega=\omega,$$

that is $\|Fx\|\leq\omega$. Hence, the Schauder fixed theorem ensures that the operator F has at least one fixed point in B_ω, and then Equation (3.6) has at least one positive solution $x^*(t)$, where $1<t<\delta$. Therefore, if $t\in[1,\delta]$ one can asserts that

$$x^*(t)=\theta_1+\left(\theta_2-g\left(1,\theta_1\right)\right)\log t+\int_1^t g\left(s,x^*(s)\right)\frac{ds}{s}$$
$$+\frac{\eta\left(\log t\right)^{\alpha}}{\Gamma(\alpha+1)}+\frac{\gamma}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}x^*(s)\frac{ds}{s}.$$

The definition of control function implies

$$U(t,x^*(t)) \le \gamma x^*(t)+\eta = \mathcal{D}_1^{\alpha} x^*(t) - \mathcal{D}_1^{\alpha-1} g\left(t,x^*(t)\right),$$

then x^* is an upper positive solution of the FDE (1.1). Moreover, one can consider

$$x_*(t) = \theta_1 + (\theta_2 - g\left(1,\theta_1\right))\log t + \int_1^t g\left(s,x_*(s)\right)\frac{ds}{s} + \frac{\sigma\left(\log t\right)^{\alpha}}{\Gamma(\alpha+1)}$$

as a lower positive solution of Equation (1.1). By Theorem 3.1, the FDE (1.1) has at least one positive solution $x \in C\left([1,\delta]\right)$, where $1<\delta<T$ and $x_*(t) \le x(t) \le x^*(t)$. □

The last result is the uniqueness of the positive solution of (1.1) using Banach contraction principle.

Theorem 3.5. *Assume that x^* and x_* are upper and lower solutions of (1.1) and there exist positive constants β_1 and β_2 such that*

$$\begin{aligned} |g(t,y\left(t\right)) - g(t,x\left(t\right)) &\le \beta_1 \left\|y-x\right\|, \\ |f(t,y\left(t\right)) - f(t,x\left(t\right))| &\le \beta_2 \left\|y-x\right\|, \end{aligned}$$

for $t\in[1,T]$ and $x,y \in X$. If

$$\beta_1 \log T + \frac{\beta_2\left(\log T\right)^{\alpha}}{\Gamma(\alpha+1)} < 1, \tag{3.7}$$

then the FDE (1.1) has a unique positive solution $x \in \mathcal{C}$.

Proof. From Theorem 3.1, it follows that the FDE (1.1) has at least one positive solution in $\mathcal{C}$. Hence, we need only to prove that the operator defined in (3.1) is a contraction on $\mathcal{C}$. In fact, for any $x,y\in\mathcal{C}$, we have

$$\begin{aligned} \left|\left(\Phi x\right)(t) - \left(\Phi y\right)(t)\right| \;\le\; & \textstyle\int_1^t \left|g\left(s,x\left(s\right)\right) - g\left(s,y\left(s\right)\right)\right| \frac{ds}{s} \\ & +\tfrac{1}{\Gamma(\alpha)} \textstyle\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} \left|f(s,x(s)) - f(s,y(s))\right| \frac{ds}{s} \\ \le\; & \left(\beta_1 \log T + \tfrac{\beta_2(\log T)^{\alpha}}{\Gamma(\alpha+1)}\right)\left\|x-y\right\|. \end{aligned}$$

Hence, the operator Φ is a contraction mapping by (3.7). Therefore, the FDE (1.1) has a unique positive solution $x\in\mathcal{C}$. □

Finally, we give an example to illustrate our results.

Example 3.6. *We consider the nonlinear fractional differential equation*

$$(3.8)\qquad \begin{cases} \mathcal{D}_1^{\frac{6}{5}}x(t) - \mathcal{D}_1^{\frac{1}{5}}\frac{1+x(t)}{3+x(t)} = \frac{1}{e+t}\left(e + \frac{tx(t)}{2+x(t)}\right), \ 1 < t \leq e, \\ x(1) = 1, \ x'(1) = 1, \end{cases}$$

where $\theta_1 = \theta_2 = 1$, $T = e$, $g(t,x) = \frac{1+x}{3+x}$ *and* $f(t,x) = \frac{1}{e+t}\left(e + \frac{tx}{2+x}\right)$. *Since* g *is non-decreasing on* x,

$$\lim_{x \longrightarrow \infty} \frac{1+x}{3+x} = \lim_{x \longrightarrow \infty} \frac{1}{e+t}\left(e + \frac{tx}{2+x}\right) = 1,$$

and

$$\frac{1}{3} \leq g(t,x) \leq 1, \ \frac{1}{2} \leq f(t,x) \leq 1,$$

for $(t,x) \in [1,T] \times [0,+\infty)$*, hence by any of the above Corollaries, the equation (3.8) has a positive solution which verifies* $x_*(t) \leq x(t) \leq x^*(t)$ *where*

$$x^*(t) = 1 + \log t + \frac{(\log t)^{\frac{6}{5}}}{\Gamma(11/5)} \text{and } x_*(t) = 1 + \log t + \frac{1}{2}\frac{(\log t)^{\frac{6}{5}}}{\Gamma(11/5)},$$

are respectively the upper and lower solutions of (3.8). Also, we have

$$\beta_1 \log T + \frac{\beta_2 (\log T)^{\alpha}}{\Gamma(\alpha+1)} \simeq 0.344 < 1,$$

then by Theorem 3.5, the equation (3.8) has a unique positive solution which is bounded by x_* *and* x^*.

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