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On additive maps of MA-semirings with involution

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Abstract:

*We extend the concept of *-derivations of rings to a certain class of semirings called MA-semirings and establish some results on commutativity forced by the *-derivations satisfying different criteria. We specially focus on the results on certain conditions under which additive mappings become Jordan *-derivations. -code, which, when self-dual, produces an unimodular lattice by Construction A.*

Keywords: MA-semirings; *-semirings ; *-derivations; Jordan *- derivations.

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1. Introduction and Preliminaries

The concept of involution is studied by many algebraists for algebras, groups, rings and other structures [5, 7, 11, 12, 13, 14, 16]. Another aspect which carries much importance in ring theory is derivation. M. Bresar and J. Vukman [6] studied the concept of *-derivation and Jordan *-derivation for rings. We can roughly say that a *-derivation is a derivation with involution. In the present paper, we canonically extend the concept of *-derivation for a class of semirings called MA-semirings introduced by Javed et al [8]. For more on MA-semirings one can see [1, 2, 3, 4, 9, 15]. We generalize some results for *-derivations of MA-semirings established in [10] for rings.

Now we include some definitions and preliminaries necessary for completion. An additive inverse semiring S with absorbing zero '0' is called an MA-Semiring if $r + r^{'} \in Z, \forall r \in S$, where Z is the center of S and $r^{'}$ is the pseudo inverse of r. Obviously every ring is an MA-semiring but the following example shows that converse may not be true.

Example 1.1. *[8] The set $S = \{0, 1, 2, 3, 4, ...\}$ with addition $\oplus$ and multiplication $\odot$ respectively defined by $a \oplus b = sup\{a, b\}$ and $a \odot b = inf\{a, b\}$ is an MA-semiring. In fact S is a commutative prime MA-semiring.*

Such examples motivate us to generalize the results of ring theory for MA-semirings. Throughout the paper, by a *-semiring S, we mean a *-MA-semiring unless stated otherwise. S is prime if $aRb = 0$ implies that $a = 0$ or $b = 0$. S is semiprime if $aRa = 0$ implies that $a = 0$. An additive mapping $* : S \longrightarrow S$ is involution if $\forall u, v \in S$, $(u^*)^* = u$ and $(uv)^* = v^*u^*$. By a *-semiring we simply mean a semiring S with involution $*$. Following example describes a *-MA-semiring.

Example 1.2. *If $(R, +, \cdot)$ is an MA-Semiring, then the set R with addition '+' and multiplication $\bullet$ defined as $a \bullet b = b.a$ forms an MA-semiring called the opposite MA-Semiring of R. We usually notate it as R^o.*
Let $(R, +, \cdot)$ be an MA-semiring and R^o its opposite MA-semiring. Consider $S = R \times R^o$ with $(a, b) \oplus (c, d) = (a+c, b+d)$ and $(a, b) \odot (c, d) = (a.c, b \bullet d) = (ac, db)$. Then $(S, \oplus, \odot)$ forms an MA-semiring. Define $* : S \to S$ by $(x, y)^* = (y, x)$. Therefore $*$ defines an involution on S and $(S, \oplus, \odot)$ forms a $*$-MA-semiring or MA-semiring with involution.

S is 2-torsion free if for $u \in S$, $2u = 0$ implies $u = 0$ and 3-torsion if $3u = 0$ implies $u = 0$. An additive mapping $d : S \longrightarrow S$ is said to be a derivation if $d(uv) = d(u)v + ud(v)$. A *-derivation is an additive mapping $d : S \longrightarrow S$ such that $d(uv) = d(u)v^* + ud(v)$. By Jordan *-derivation, we mean an additive mapping $d : S \longrightarrow S$ satisfying $d(u^2) = d(u)u^* + ud(u)$. An additive mapping $F : S \to S$ is generalized derivation associated with a derivation d if $F(xy) = F(x)y + xd(y)$. We define Commutator as $[u, v] = uv + v^{'}u$. By Jordan product we mean $u \circ v = uv + yu$ for all $u, v \in S$. Following identities will be used frequently: $[u, uv] = u[u, v]$, $[uv, w] = u[v, w] + [u, w]v$, $[u, yw] = [u, v]w + v[u, w]$, $[u, v] + [v, u] = v(u + u^{'}) = u(v+v^{'})$, $(uv)^{'} = u^{'}v = uv^{'}$, $[u, v]^{'} = [u, v^{'}] = [u^{'}, v]$, $u\circ(v+w) = u\circ v+u\circ w$ (see [8],[15]).

2. Main Results

Theorem 2.1. *Let S be a semiprime *-semiring. If T is an additive mapping satisfying*

$$T(uv) + T(u^{'})v^* = 0, \forall u, v \in S \tag{2.1}$$

Then $[T(S), S] = 0$ and hence $T(S) \subseteq Z(S)$.

Proof. In (2.1) writing uw for u, we get $\forall u, v \in S$

$$T(uwv) + T(u^{'}w)v^* = 0 \tag{2.2}$$

and wv for v in (2.2), we get $T(uwv) + T(u^{'}(wv)^* = 0$, which implies

$$T(uwv) + T(u^{'})v^*w^* = 0 \tag{2.3}$$

From (2.3), put $T(uwv) = T(uw)v^*$, we get

$$T(uw)v^* + T(u^{'})v^*w^* = 0 \tag{2.4}$$

From (2.1) using $T(uv) = T(u)v^*$ into (4), we obtain $T(u)w^*v^* + T(u^{'})v^*w^* = 0$, which implies $T(u)w^*v^* + T(u)^{'}v^*w^* = 0$ and therefore $T(u)(w^*v^* + v^{*'}w^*) = 0$, which further gives

$$T(u)[w^*, v^*] = 0 \tag{2.5}$$

In (2.5) replacing w by w^* and v by v^*, we obtain

$$T(u)[w, v] = 0 \tag{2.6}$$

In (2.6), replacing w by $wT(u)$, we obtain $T(u)[wT(u),v]=0$, which implies $T(u)w[T(u),v]+T(u)[w,v]T(u)=0$. Using (2.6) again, we obtain

$$T(u)w[T(u),v]=0 \tag{2.7}$$

Replacing w by vw in (2.7), we obtain

$$T(u)vw[T(u),v]=0 \tag{2.8}$$

Multiplying (2.7) by $v^{'}$ from the left, we obtain

$$v^{'}T(u)w[T(u),v]=0 \tag{2.9}$$

Adding (2.8) and (2.9), we obtain $[T(u),v]w[T(u),v]=0$ which implies $[T(u),v]S[T(u),v]=0$. Since S is semiprime, therefore the last equation yields $[T(u),v]=0$, which gives $[T(S),S]=0$ and hence $T(S)\subseteq Z(S)$. □

Theorem 2.2. *Let S be prime *-semiring. If S admits a nontrivial *-derivation d such that $d(uv)+d(u^{'})d(v)=0, \forall u,v\in S$, then $d=0$.*

Proof. By hypothesis for all $u,v\in S$, we have

$$d(uv)+d(u^{'})d(v)=0 \tag{2.10}$$

By definition of *-derivation,from (2.10), we obtain

$$d(u)v^{*}+ud(v)+d(u^{'})d(v)=0 \tag{2.11}$$

In (2.11) replacing u by uw, we obtain $d(uw)v^{*}+uwd(v)+d(u^{'}w)d(v)=0$ and again using (2.11), we obtain $d(u)d(w)v^{*}+xwd(v)+d(u^{'}w)d(v)=0$, which after simplification implies $(d(u)d(w)+d(u^{'})d(w))v^{*}+(u+d(u^{'}))wd(v)=0$. Using (10), we obtain $(u+d(u^{'}))wd(v)=0$ and therefore $(u+d(u^{'}))Sd(v)=0$. As S is prime, either $(u+d(u^{'}))=0$ or $d(v)=0$. If $(u+d(u^{'}))=0$, then $u=d(u)$, a contradiction, which shows that $d(v)=0$ and therefore $d=0$. □

Theorem 2.3. *Let S be prime *-semiring. If S admits a *-derivation d such that $d\neq I^{*}$ and $d(uv)+d(v)d(u^{'})=0, \forall u,v\in S$, then $d=0$ (where $I^{*}(u)=u^{*}$)).*

Proof. By the hypothesis for all $u, v \in S$

$$d(u,v) + d(v)d(u^{'}) = 0 \tag{2.12}$$

In (2.12) writing uv for v, we obtain $d(uuv) + d(uv)d(u^{'}) = 0$ which further gives on simplification $d(u)v^*(u^* + d(u^{'})) + u(d(uv) + d(v)d(u^{'})) = 0$. Using (2.12) again, we obtain $d(u)v^*(u^* + d(u^{'})) = 0$, which implies $d(u)S(u^* + d(u^{'})) = 0$. By the primeness of S, we have either $u^* + d(u^{'}) = 0$ or $d(u) = 0$. If $u^* + d(u^{'}) = 0$, then $d(u^{'}) = u^* = I^*(u)$, which implies that $d = I^*$, a contradiction. Therefore we obtain $d(u) = 0$ and $d = 0$ as required. □

Theorem 2.4. *Let S be prime *-semiring and $a \in S$. If S admits a *-derivation d such that $[d(u), a] = 0$ $\forall u \in S$, then $a \in Z(S)$ or $d(a) = 0$.*

Proof. We have forall $u \in S$

$$[d(u), a] = 0 \tag{2.13}$$

In (13) replacing u by uv, we obtain $[d(uv), a] = 0$. On simplification, we obtain $d(u)[v^*, a] + [d(u), a]v^* + u[d(v), a] + [u, a]d(v) = 0$. Using (13), again, we obtain

$$d(u)[v^*, a] + [u, a]d(v) = 0 \tag{2.14}$$

Replacing u by a in (2.14), we obtain $d(a)[v^*, a] + [a, a]d(v) = 0$ and therefore

$$d(a)[v^*, a] + a(d(v)a + d(v)a^{'}) = 0 \tag{2.15}$$

From (2.13), replacing u by v, we obtain $d(v)a = ad(v)$, and hence using it in (2.15), we have $d(a)[v^*, a] + a[d(v), a] = 0$. Using (2.13) again, we have

$$d(a)[v^*, a] = 0 \tag{2.16}$$

Replacing v by v^*, we obtain

$$d(a)[v, a] = 0 \tag{2.17}$$

In (2.17), replacing v by vu and using it again, we obtain $d(a)S[u, a] = 0$. By the primeness of S, we have $d(a) = 0$ or $[u, a] = 0$ and therefore $d(a) = 0$ or $a \in Z(S)$. □

Theorem 2.5. *Let S be semiprime *-semiring. If S admits a *-derivation d such that $d[u, v] = 0$, then $d = 0$ or S is commutative.*

Proof. We have for all $u, v \in S$

$$d[u,v] = 0 \tag{2.18}$$

Replacing u by uv in (2.18), we obtain $d[uv, v] = 0$ and therefore $d[u,v]v^* + [u,v]d(v) = 0$. Using (2.18) again, we obtain

$$[u,v]d(v) = 0 \tag{2.19}$$

Replacing u by su in (2.19), we obtain $[su, v]d(v) = 0$ which implies $s[u,v]d(v) + [s,v]ud(v) = 0$. Using (2.19) again, we obtain $[s,v]ud(v) = 0$ and therefore

$$[s,v]Rd(v) = 0 \tag{2.20}$$

By primeness of S, (2.20) yields either $[s,v] = 0$ or $d(v) = 0$. Now take $K = \{v \in S : d(v) = 0\}$ and $L = \{v \in S : [s,v] = 0, \forall s \in S\}$. Clearly $S = K \bigcup L$. We claim that either $S = K$ or $S = L$. For this we can show that either $L \subseteq K$ or $K \subseteq L$. Suppose that $u \in K \setminus L$ and $v \in L \setminus K$. Clearly $u + v \in K + L \subseteq S = K \bigcup L$. Therefore $u + v \in K$ or $u + v \in L$. Firstly, If $u + v \in K$, then $d(u + v) = 0$ which implies $d(u) + d(v) = 0$ and therefore $d(v) = 0$ which means $v \in K$, a contradiction. Secondly, if $u + v \in L$ $[u + v, r] = [u, r] + [v, r] = [u, r] = 0, \forall r \in S$, which implies $u \in L$, a contradiction. Therefore, we have either $L \subseteq K$ or $K \subseteq L$ and hence either $S = K$ or $S = L$. This proves that that either $d = 0$ or S is commutative. $\square$

Theorem 2.6. *Let S be prime *-semiring. If S admits a *-derivation d such that $d(u \circ v) = 0$, $\forall u, v \in S$, then $d = 0$ or S is commutative.*

Proof. For any $u, v \in S$, We have

$$d(u \circ v) = 0 \tag{2.21}$$

In (2.21) replacing u by uv, we obtain $d((uv) \circ v) = 0$. But $d((uv) \circ v) = d(u \circ v)v)$. Therefore $d(u \circ v)v) = 0$ and hence $d(u \circ v)v^* + (u \circ v)d(v) = 0$. Using (2.21) again, we obtain

$$(u \circ v)d(v) = 0 \tag{2.22}$$

In (2.22) replacing u by sv, we obtain $((sv) \circ v)d(v) = 0$, which implies

$$(s \circ v)Sd(v) = 0 \tag{2.23}$$

Since S is prime, therefore (2.23) yields either $(s \circ v) = 0$ or $d(v) = 0$. Let $K = \{v \in S : d(v) = 0\}$ and $L = \{v \in S : s \circ v = 0, \forall s \in S\}$. Clearly $S = K \bigcup L$. Our claim is that either $S = K$ or $S = L$. For this we show that either $K \subseteq L$ or $L \subseteq K$. Suppose that $u \in K \setminus L$ and $v \in L \setminus K$. Clearly $u+v \in K+L \subseteq S = K \bigcup L$, which implies $u+v \in K$ or $u+v \in L$. Firstly, If $u + v \in K$, then $d(u + v) = d(u) + d(v) = d(v) = 0$ which means $v \in K$, a contradiction. Secondly, if $u+v \in L$, then $r \circ (u+v) = r \circ u + r \circ v = r \circ u = 0, \forall r \in S$. which means $u \in L$, a contradiction. Therefore we obtain either $L \subseteq K$ or $K \subseteq L$, which implies that either $S = K$ or $S = L$. If $S = K$, then $d = 0$. On the other hand, if $S = L$, then for any $s, v \in S$

$$s \circ v = 0 \tag{2.24}$$

In (2.24) replacing s by sw, we obtain $(sw) \circ v = 0$, which implies $swv + vsw = 0$. Since $s = s + s^{'} + s$ and $s + s^{'} \in Z(S)$ therefore last equation becomes $swv + v(s + s^{'} + s)w = 0$ which gives on simplification that $s(w \circ v) + [v, s]w = 0$. Using (2.24) again, we obtain $[v, s]w = 0$. Replacing w by wu, we obtain $[v, s]Su = 0$,. By the primeness of S, since $S \neq 0$, we obtain $[v, s] = 0$. This proves that S is commutative. □

Theorem 2.7. *Let S be prime *-semiring. If S admits a *-derivation d such that $d(u) \circ v = 0$, $\forall u, v \in S$, then $d = 0$ or S is commutative.*

Proof. We have for any $u, v \in S$

$$d(u) \circ v = 0 \tag{2.25}$$

In (2.25) replacing u by uw, we obtain $(d(u)w^* + ud(w)) \circ v = 0$. Since $v + v^{'} \in Z$, $v + v^{'} + v = v$ and $v^{'} + v + v^{'} = v^{'}$, after simplification we obtain

$$(d(u) \circ v)w^* + d(u)[w^*, v] + u(v \circ d(w)) + [v, u]d(w) = 0 \tag{2.26}$$

Using (26), we obtain

$$d(u)[w^*, v] + [v, u]d(w) = 0 \tag{2.27}$$

Replacing u by v, (2.27), we obtain $d(v)[w^*, v] + [v, v]d(w) = 0$. Using the definition of S and simplifying we obtain

$$d(v)[w^*, v] + v(vd(w) + v^{'}d(w)) = 0 \tag{2.28}$$

From (2.25), we have $d(w)v = v^{'}d(w)$. Hence (2.28) becomes $d(v)[w^*, v]+$ $v(d(w) \circ v) = 0$. Using (2.25) again, we obtain

$$d(v)[w^*, v] = 0 \tag{2.29}$$

In (2.29) replacing w by w^*, we obtain

$$d(v)[w, v] = 0 \tag{2.30}$$

Replacing w by uw in (2.30), we obtain $d(v)[uw, v] = 0$ which further implies $d(v)u[w, v] + [d(u), v]w = 0$. Using (2.30) again, we obtain

$$d(v)S[w, v] = 0 \tag{2.31}$$

Since S is prime, therefore from (2.31), we have $d(v) = 0$ or $[w, v] = 0$. The remaining part is same as that of Theorem 2.5. □

Theorem 2.8. *Let S be a 2-torsion free semiprime *-semiring. Suppose that*
$au^*b^* + bua = 0, \forall u \in S$, for some $a, b \in S$. Then $ab = 0 = ba$. Moreover if S is prime, then either $a = 0$ or $b = 0$.

Proof. By the hypothesis

$$au^*b^* + bua = 0 \tag{2.32}$$

In (2.32) replacing u by vbu, we obtain $a(vbu)^*b^* + bua = 0$

$$au^*b^*v^*b^* + bvbua = 0 \tag{2.33}$$

From (2.32), using $au^*b^* = bua^*$ into (2.33), we obtain $bua^{'}v^*b^* +$ $bvbua = 0$ which further implies

$$bubva + bvbua = 0 \tag{2.34}$$

In particular for $v = u$, we obtain $2bubua = 0, \forall u \in S$ and 2-torsion freeness of S further yields

$$bubua = 0 \tag{2.35}$$

Again from (2.32), using $au^*b^* = bua^*$ into (2.35), we obtain

$$buau^*b^* = 0 \tag{2.36}$$

In (2.35), replacing v by uav, we obtain $bub(uav)a + b(uav)bua = 0$. Using (2.32), we obtain $(buau^*b^*)'va + buavbua = 0$. Using (2.36) again, we obtain $buaSbua = 0$ and therefore by the semiprimeness, we obtain

$$bua = 0 \tag{2.37}$$

This implies $abuab = 0$. By the semiprimeness of S, we have $ab = 0$. Again from (2.37), we have $bauba = 0$, which implies $ba = 0$. Hence we conclude that $ab = 0 = ba$. Moreover if S is prime then (2.37) yields either $a = 0$ or $b = 0$. □

Theorem 2.9. *Let S be a 2-torsion free semiprime *-semiring and $F : S \longrightarrow S$ be an additive mapping satisfying*

$$F(uv'u) + F(u)v^*u^* + uf(v)u^* + uvf(u) = 0, \forall u, v \in S \tag{2.38}$$

*associated with the Jordan *-derivation f. Then F is a Jordan *-derivation.*

Proof. Replacing u by $u + w$ by in (2.38), we obtain

$$F((u+w)v'(u+w))+F(u+w)v^*(u+w)^*+(u+w)f(v)(u+w)^*+(u+w)vf(u+w) = 0$$

which further implies
$F(uv'u) + F(wv'u) + F(uv'w) + F(wv'w) + F(u)v^*u^*$
$+F(u)v^*w^* + F(w)v^*w^* + F(w)v^*w^* + (uf(v)u^*$
$+wf(v)u^*)+uf(v)w^*+wf(v)w^*+uvf(u)+wvf(u)+uvf(w)+wvf(w) = 0.$
Using (2.38) again we obtain
$F(wv'u) + F(uv'w) + F(u)v^*w^*$

$$+ F(w)v^*w^* + wf(v)u^* + uf(v)w^* + wvf(u) + uvf(w) = 0 \tag{2.39}$$

In (2.39), replacing w by u^2, we obtain
$F(u^2v'u) + F(uv'u^2) + F(u)v^*u^{*2} + F(u^2)v^*u^{2*}$

$$+ u^2f(v)u^* + uf(v)u^{2*} + u^2vf(u) + uvf(u^2) = 0 \tag{2.40}$$

Replacing v by $uv + vu$ in (2.38), we obtain
$F(u(uv+vu)'u)+F(u)(uv+vu)^*u^*+uf(uv+vu)u^*+u((uv+vu))f(u) = 0,$

which further implies
$F(u^2v^{'}u + uv^{'}u^2) + F(u)u^*v^*u^* + F(u)v^*u^{*2}$

$$+uf(u)v^*u^* + u^2f(v)u^* + uf(v)u^{*2} + uvf(u)u^* + u^2vf(u) + uvuf(u) = 0 \tag{2.41}$$

From (2.40), we have
$(F(u^2)v^*u^{2*})^{'} = F(u^2v^{'}u) + F(uv^{'}u^2)$

$$+ F(u)v^*u^{*2} + u^2f(v)u^* + uf(v)u^{2*} + u^2vf(u) + uvf(u^2) \tag{2.42}$$

Using (2.42) into (2.41), we obtain $F(u)u^*v^*u^* + (F(u^2)v^*u^*)^{'} + uf(u)v^*u^* = 0$ and therefore $F(u)u^{*'}v^*u^* + F(u^2)v^*u^* + u^{'}f(u)v^*u^* = 0$, which further implies

$$(F(u^2) + (F(u))'u^{*'} + u^{'}f(u))v^*u^* = 0S \tag{2.43}$$

Setting $F(u^2) + (F(u))'u^* + u^{'}f(u) = A(u)$ in (2.43), we obtain

$$A(u)v^*u^* = 0 \tag{2.44}$$

Replacing v by v^* in (2.44), we obtain

$$A(u)vu^* = 0 \tag{2.45}$$

which implies that$u^*A(u)Ru^*A(u) = 0$. By the semiprimeness of S, we obtain

$$u^*A(u) = 0 \tag{2.46}$$

Replacing v by $u^*vA(u)$, we obtain $A(u)u^*RA(u)u^* = 0$ and by the semiprimeness, we get

$$A(u)u^* = 0 \tag{2.47}$$

In (2.47) replacing u by $u + v$, we obtain $A(u + v)(u + v)^* = 0$, which further implies $(A(u) + B(u, v) + A(v))(u^* + v^*) = 0$, where $B(u, v) = F(uv + vu) + (F(u)v^*)^{'} + (F(v)u^*)^{'} + u^{'}f(v) + v^{'}f(u)$. Hence we have $A(u)u^* + B(u, v)u^* + A(v)u^* + A(u)v^* + B(u, v)v^* + A(v)v^* = 0$. Using (2.47) again, we obtain

$$B(u, v)u^* + A(v)u^* + A(u)v^* + B(u, v)v^* = 0 \tag{2.48}$$

In (2.48) replacing u by $u^{'}$, we obtain $B(u^{'},v)u^{*'} + A(v)u^{*'} + A(u^{'})v^{*} + B(u^{'},v)v^{*} = 0$, which further implies $B(u,v)u^{*} + (A(v)u^{*})^{'} + A(u)v^{*} + (B(u,v)v^{*})^{'} = 0$ and hence

$$B(u,v)u^{*} + A(u)v^{*} = A(v)u^{*} + B(u,v)v^{*} \tag{2.49}$$

Using (2.49) into (2.48), we obtain $2(B(u,v)u^{*} + A(u)v^{*}) = 0$ and by 2-torsion freeness of S, we obtain

$$B(u,v)u^{*} + A(u)v^{*} = 0 \tag{2.50}$$

Multiplying (2.50) by $A(u)$ from the right, we obtain $B(u,v)u^{*}A(u) + A(u)v^{*}A(u) = 0$. Using (2.46), we obtain $A(u)v^{*}A(u) = 0$. Replacing v by v^{*}, we obtain $A(u)RA(u) = 0$. By the semiprimeness of S, we obtain $A(u) = 0$. Therefore $F(u^{2}) + (F(u))'u^{*} + u^{'}F(u) = 0$, which further implies $F(u^{2}) = F(u)u^{*} + uF(u)$ and this completes the proof. □

Theorem 2.10. *Let S be a 2-torsion and 3-torsion free semiprime *-semiring and $D : S \longrightarrow S$ be an additive mapping satisfying*

$$D(uv^{'}u) + D(u)v^{*}u^{*} + uD(v)u^{*} + uvD(u) = 0, \forall u, v \in S \tag{2.51}$$

*Then D is a Jordan *-derivation.*

Proof. In (2.50), replacing u by u^{2}, we obtain

$$D(u^{2}v^{'}u^{2}) + D(u^{2})v^{*}u^{*2} + u^{2}D(v)u^{*2} + u^{2}vD(u^{2}) = 0 \tag{2.52}$$

In (2.51) replacing v by uvu, we obtain $D(u^{2}v^{'}u^{2}) + D(u)u^{*}v^{*}u^{*2} + uD(uvu)u^{*} + u(uvu)D(u) = 0$. Using (2.51) into the last equation, we obtain $D(u^{2}v^{'}u^{2}) + D(u)u^{*}v^{*}u^{*2} + u(D(u)v^{*}u^{*} + uD(v)u^{*} + uvD(u))u^{*} + u(uvu)D(u) = 0$. Therefore

$$D(u^{2}v^{'}u^{2}) + D(u)u^{*}v^{*}u^{*2} + uD(u)v^{*}u^{*2} + u^{2}D(v)u^{*2} + u^{2}vD(u)u^{*} + u^{2}vuD(u) = 0 \tag{2.53}$$

Since $v + v^{'} \in Z$, $v + v^{'} + v = v$, $v^{'} + v + v^{'} = v^{'}$, therefore from (2.52) , we have

$$D(u^{2}v^{'}u^{2}) + u^{2}D(v)u^{*2} = D(u^{2})v^{*'}u^{*2} + u^{2'}vD(u^{2} \tag{2.54}$$

Using (2.54) into (2.53), we obtain $D(u)u^*v^*u^{*2}+uD(u)v^*u^{*2}+u^2vD(u)u^*+u^2vuD(u)+D(u^2)v^{*'}u^{*2}+u^{2'}vD(u^2=0$. This further implies $u^2v^{'}(D(u^2+D(u)u^{*'}+u^{'}D(u))+(D(u^2)+D(u)u^{*'}+u^{'}D(u))v^{*'}u^{*2}=0$ and therefore $u^2v(D(u^2+D(u)u^{*'}+u^{'}D(u))+(D(u^2)+D(u)u^{*'}+u^{'}D(u))v^*u^{*2}=0$. Setting $A(u)=D(u^2+D(u)u^{*'}+u^{'}D(u)$ into the last equation, we obtain

$$u^2vA(u)+A(u)v^*u^{*2}=0 \tag{2.55}$$

In view of Theorem 2.8, we can write

$$A(u)u^2=0 \tag{2.56}$$

$$u^2A(u)=0 \tag{2.57}$$

linearizing (2.56), we obtain

$$A(u+v)(u+v)^2=0 \tag{2.58}$$

We can easily see that $A(u+v)=A(u)+B(u,v)+A(v)$, where $B(u,v)=D(uv+vu)+(D(u))^{'}v^*+(D(v))^{'}u^*+u^{'}D(v)+v^{'}D(u)$. Hence (59) becomes $A(u)u^2+B(u,v)u^2+A(v)u^2+A(u)v^2+B(u,v)v^2+A(v)v^2+A(u)(uv+vu)$ $+B(u,v)(uv+vu)+A(v)(uv+vu)=0$. Using (2.56) again in the last equation, we obtain

$$\begin{aligned} &B(u,v)u^2+A(v)u^2+A(u)v^2+B(u,v)v^2+A(u)(uv+vu)\\ &+B(u,v)(uv+vu)+A(v)(uv+vu)=0 \end{aligned} \tag{2.59}$$

We can easily observe that $A(u^{'})=A(u)$ and $B(u^{'},v)=(B(u,v))^{'}$. Replacing u by $u^{'}$ in (2.59), we obtain

$$\begin{aligned} &(B(u,v))^{'}u^2+A(v)u^2+A(u)v^2+(B(u,v))^{'}v^2+A(u)(uv+vu)^{'}\\ &+B(u,v)(uv+vu)+A(v)(uv+vu)^{'}=0 \end{aligned} \tag{2.60}$$

From (2.60), we have

$$A(v)u^2+A(u)v^2+B(u,v)(uv+vu)$$

$$(2.61)=(B(u,v))u^2+(B(u,v))v^2+A(u)(uv+vu)+A(v)(uv+vu)$$

Using (2.61) into (2.59), we obtain $2((B(u,v))u^2+(B(u,v))v^2+A(u)(uv+vu)+A(v)(uv+vu))=0$. Since S is 2-torsion free, therefore

(2.62) $B(u,v)u^2 + B(u,v)v^2 + A(u)(uv+vu) + A(v)(uv+vu)) = 0$

We can easily see that $A(2u) = 4A(u)$ and $B(2u,v) = 2B(u,v)$. Replacing u by $2u$ in (2.62), we obtain $8B(u,v)u^2 + 2B(u,v)v^2 + 8A(u)(uv+vu) + 2A(v)(uv+vu) = 0$, which can also be written as $2(4B(u,v)u^2 + B(u,v)v^2 + 4A(u)(uv+vu) + A(v)(uv+vu)) = 0$. By the 2-torsion freeness of S, we obtain

(2.63)$4B(u,v)u^2 + B(u,v)v^2 + 4A(u)(uv+vu) + A(v)(uv+vu) = 0$

Since Since $v + v^{'} \in Z$, $v + v^{'} + v = v$, $v^{'} + v + v^{'} = v^{'}$, therefore from (2.62), we have

(2.64) $B(u,v)v^2 + A(v)(uv+vu)) = (B(u,v))^{'}u^2 + (A(u))^{'}(uv+vu)$

Using (2.64) into (2.63), we obtain $4B(u,v)u^2 + (B(u,v))^{'}u^2 + 4A(u)(uv+vu) + (A(u))^{'}(uv+vu) = 0$. Since $u + u^{'} + u = u, u^{'} + u + u^{'} = u^{'}$, therefore $3B(u,v)u^2 + 3A(u)(uv+vu) = 0$ and hence by 3-torsion freeness of S, we have

(2.65) $$B(u,v)u^2 + A(u)(uv+vu) = 0$$

Multiplying (2.65) by $A(u)u$ from the right and using (2.57), we obtain

(2.66) $$A(u)uvA(u)u + A(u)vuA(u)u = 0$$

In (2.66), replacing v by vu, we obtain $A(u)uvuA(u)u + A(u)vu^2A(u)u = 0$. Using (2.56), we obtain $A(u)uvuA(u)u = 0$, which further implies $uA(u)uRuA(u)u = 0$. By the semiprimeness of S, we obtain

(2.67) $$A(u)u = 0$$

Hence (2.65) becomes

(2.68) $$B(u,v)u^2 + A(u)vu = 0$$

Multiplying (2.68) by $A(u)$ from the right and using (2.57), we obtain $A(u)vuA(u) = 0$, which implies $uA(u)vuA(u) = 0$ and hence$uA(u)RuA(u) = 0$ and by the semiprimeness, we have

$$uA(u) = 0 \tag{2.69}$$

From (2.68), we have $(B(u,v)u+A(u)v)u = 0$, which implies $(B(u,v)u+A(u)v)u(B(u,v)u + A(u)v) = 0$ and therefore

$$B(u,v)u + A(u)v = 0 \tag{2.70}$$

Multiplying (2.70) by $A(u)$ from the right, we obtain $B(u,v)uA(u) + A(u)vA(u) = 0$ and using (2.70) again, we obtain $A(u)vA(u) = 0$. Since S is semiprime, therefore $A(u) = 0$. This means $D(u^2) + D(u)u^{*'} + u^{'}D(u) = 0$ and hence $D(u^2) = D(u)u^* + uD(u)$, which shows that D is Jordan *-derivation. □

Concluding Remarks

This article presents some criteria for *-derivations which induce commutativity in additive inverse semirings with involution. Secondly we present some additive mappings satisfying certain conditions under which they become Jordan *-derivations. Therefore ideas presented in this article are useful. We propose some open problems as follows:

1. Let S be a semiprime *-semiring and d a nonzero *-derivation of S satisfying $d(u) \circ v = 0, \forall u, v \in S$. Is S commutative?
2. Let S be a semiprime *-semiring and d a nonzero *-derivation of S satisfying $d(u \circ v) = 0, \forall u, v \in S$. Is S commutative?
3. Let S be a prime *-semiring, d a nonzero *-derivation of S and F an additive mapping defined by $F(xy) = F(x)y + xd(y)$. If F satisfies $F(u \circ v) = 0, \forall u, v \in S$. Is S commutative?

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