Edge even graceful labeling of torus grid graph

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Received: February 2020 | Accepted: April 2020

Abstract:

We study the family of torus grid graphs. We also obtain necessary and sufficient conditions to be edge even graceful labeling for all of the cases of every member of this family.

Keywords: Graceful labeling; Edge even graceful labeling; Torus grid graph.

MSC (2020): 05C78.

Cite this article as (IEEE citation style):

S. N. Daoud and W. Saleh, "Edge even graceful labeling of torus grid graph", *Proyecciones (Antofagasta, On line)*, vol. 39, no. 4, pp. 1033-1082, Aug. 2020, doi: 10.22199/issn.0717-6279-2020-04-0065.

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1. Introduction

The scope of Graph Theory plays a very significant base in many areas of applied and pure sciences. One amidst the necessary areas in Graph Theory is Graph Labeling of a graph that is associate of integers either to the vertices or edges or each subject to assured conditions. Graph labeling began nearly fifty years ago. Over these decades, over two hundred ways of labeling techniques were given and over 2500 papers were revealed. In spite of this vast literature, simply a number of general results were discovered. Today, graph labeling has a lot of attention from completely different good researches in Graph Theory that has rigorous applications in several disciplines such as communication networks, writing theory, graph decomposition problems and so on. Additional, fascinating applications of graph labeling are given in [1,3,4,5,6,18,19,26,27,29].

A function g is named a graceful labeling of a graph G with vertices n and edges m if $g : V(G) \longrightarrow \{0, 1, 2, \dots, m\}$ is (1-1) and $g^* : E(G) \longrightarrow \{1, 2, \dots, m\}$ is given as $g^*(e = uv) = |g(u) - g(v)|$ is correspondence. This kind of graph labeling first is obtained in [24] as a β -valuation, lately is called graceful labeling in [17]. A function g is called an odd graceful labeling of a graph G with vertices n and edges m if $g : v(G) \longrightarrow \{0, 1, 2, \dots, 2m-1\}$ is injective and the induced function $g^* : v(G) \longrightarrow \{1, 3, \dots, 2m-1\}$ is defined as $g^*(e = uv) = |g(u) - g(v)|$ is one-to-one correspondence and we will use o.g.l referring to this type of labeling. Also, the readers can see [15,16,25] for a lot of results. A function g is named an edge graceful labeling of a graph G with vertices n and edges m if $g : E(G) \longrightarrow \{1, 2, \dots, m\}$ is one-to-one correspondence and $g^* : V(G) \longrightarrow \{0, 1, 2, \dots, n-1\}$ that is known as $g^*(u) = \sum_{e-uv \in E(G)} g(e) \pmod{n}$ is one-to-one correspondence and we will use e.g.l referring to this type of labeling. Also, the readers can see [20,21,23] for a lot of results. A function g is named an edge odd graceful labeling of a graph G with vertices n and edges m if $g : E(G) \longrightarrow \{1, 3, \dots, 2m-1\}$ is one-to-one correspondence and $g^* : V(G) \longrightarrow \{0, 1, 2, \dots, 2m-1\}$ is given as $g^*(u) = \sum_{e-uv \in E(G)} g(e) \pmod{2m}$ is injective, will use e.o.g.l referring to this type of labeling. This type of graph labeling can be found in initial introduced by Solairaju and Chithra in 2009 [28]. For more information on this labeling see [7,8,10]. A function g is named an edge even graceful labeling of a graph G with vertices n and edges m if $g : E(G) \longrightarrow \{2, 4, \dots, 2m-2\}$ is one-to-one correspondence and

$g^* : V(G) \longrightarrow \{0, 2, 4, \dots, 2m - 2\}$ given as $g^*(u) = \sum_{e-uv \in E(G)} g(e) \pmod{2r}$, where $r = \max\{n, m\}$ is (1-1), we will use e.e.g.l referring to this type of labeling. This type of graph labeling can be found initial introduced by Elsonbaty and Daoud [11]. For more information on this labling see [9] and [14]. For more information about other types of graph labelings, see [2,12,13,22,30]

2. Labeling of Torus grid Graph

$T_{p,q}$ which is torus grid graph is a graph constructed from the Cartesian product $C_p \times C_q$ where C_p and C_q are the cycle graphs. That is the torus grid graph consists of p copies of C_q represented by circles and can be numbered from the inner most circle to the outer circle as: $C_q^{(1)}, C_q^{(2)}, C_q^{(3)}, \dots, C_q^{(p-1)}, C_q^{(p)}$ and we decision them circles, also q copies of C_p represented by ovals or elliptic shapes transverse the p circles and can be numbered clockwise as $C_p^{(1)}, C_p^{(2)}, C_p^{(3)}, \dots, C_p^{(q-1)}, C_p^{(q)}$ and that we call them ovals. See Figure 2.1.

Figure 2.1: Tous grid graph $T_{p,q}$

Theorem 2.0.1. Let q be an even number and it belongs to $\mathbf{Z}^+$ and $2 \leq p$ is greater than or equal 2. Then $T_{p,q}$, is an e. e. g. graph.

Proof. By taking $n = |V(T_{p,q})| = pq$, $m = |E(T_{p,q})| = 2pq$ and $r = \max m, n = 2pq$ and $g : E(T_{p,q}) \longrightarrow \{2, 4, \dots, 2pq - 2\}$. Assume that $T_{p,q}$ is the torus grid graph is that is given in Figure 2.2. First we label the ovals $C_p^{(j)}$, $1 \leq j \leq q$ as follows: turn clockwise to label the edges

$$u_{(p-1)q+1}u_{(p-2)q+1}, u_{(p-1)q+2}u_{(p-2)q+2}, \\ u_{(p-1)q+3}u_{(p-2)q+3}, \dots, u_{pq-1}u_{(p-1)q-1}, u_{pq}u_{(p-1)q}$$

by $2, 4, 6, \dots, 2q - 2, 2q$, then turn counter-clockwise to label the edges

$$u_{(p-2)q+1}u_{(p-4)q+1}, u_{(p-1)q}u_{(p-3)q}, u_{(p-1)q-1}u_{(p-3)q-1}, \\ \dots, u_{(p-2)q+3}u_{(p-4)q+3}, u_{(p-2)q+2}u_{(p-4)q+2}$$

by $2q + 2, 2q + 4, 2q + 6, \dots, 4q - 2, 4q$ and so on.

When $p \equiv 0 \pmod{4}$, we turn counter-clockwise to label the edges

$$u_{2q+1}u_1, u_qu_{3q}, u_{q-1}u_{3q-1}, \dots, u_{2q+3}u_3, u_{2q+2}u_2$$

by $(p-2)q + 2, (p-2)q + 4, (p-2)q + 6, \dots, pq - 2, pq$, we turn clockwise to label the edges

$$u_1u_{q+1}, u_2u_{q+2}, u_3u_{q+3}, \dots, u_{q-1}u_{2q-1}, u_qu_{2q}$$

by $pq + 2, pq + 4, pq + 6, \dots, (p+2)q - 2, (p+2)q$, then turn counter-clockwise to label the edges

$$u_{q+1}u_{3q+1}, u_{2q}u_{4q}, u_{2q-1}u_{4q-1}, \dots, u_{q+3}u_{3q+3}, u_{q+2}u_{3q+2}$$

by $(p+2)q + 2, (p+2)q, (p+2)q + 6, \dots, (p+4)q - 2, (p+4)q$ and so on. When $p \equiv 2 \pmod{4}$, turn clockwise to label the edges

$$u_{2q+1}u_1, u_{2q+1}u_2, u_{2q+3}u_3, \dots, u_{3q-1}u_{q-1}, u_{3q}u_q$$

by $(p-2)q+2, (p-2)q+4, (p-2)q+6, \dots, pq-2, pq$, then turn counter-clockwise anticlockwise to label the edges

$$u_1u_{q+1}, u_qu_{2q}, u_{q-1}u_{2q-1}, \dots, u_3u_{q+3}, u_2u_{q+2}$$

by $pq+2, pq+4, pq+6, \dots, (p+2)q-2, (p+2)q$, then turn clockwise to label the edges

$$u_{q+1}u_{3q+1}, u_{q+2}u_{3q+2}, u_{q+3}u_{3q+3}, \dots, u_{2q-1}u_{4q-1}, u_{2q}u_{4q}$$

by $(p+2)q+2, (p+2)q+4, (p+2)q+6, \dots, (p+4)q-2, (p+4)q$ and so on.

Now, turn counter-clockwise to label the edges

$$u_{(p-1)q-1}u_{(p-3)q-1}, u_{pq}u_{(p-2)q}, u_{pq-1}u_{(p-2)q-1}, \\ \dots, u_{(p-1)q+3}u_{(p-3)q+3}, u_{(p-1)q+2}u_{(p-3)q+2}$$

by $2(p-1)q+2, 2(p-1)q, 2(p-1)q+6, \dots, 2pq-2, 2pq$.

Second we label the edges of the circles $C_q^{(j)}$, $1 \leq j \leq p$ begin with the edges of inner most circle $C_q^{(1)}$ then the edges of outer circle $C_q^{(p)}$, then the edges of the circles $C_q^{(2)}, C_q^{(4)}, \dots, C_q^{(p-2)}$.

Finally the edges of the circles $C_q^{(p-1)}, C_q^{(p-3)}, \dots, C_q^{(3)}$, respectively as follows: $g(u_i u_{i+1}) = 2pq + 2i, 1 \leq i \leq q-1$,

$$g(u_q u_1) = 2(p+1)q, g(u_{(p-1)q+i} u_{(p-1)q+i+1}) = 2q(p+1) + 2i, 1 \leq i \leq q-1,$$

$$g(u_{pq} u_{(p-1)q+1}) = 2q(p+2), g(u_{(j-1)q+i} u_{(j-1)q+i+1}) = 2q \left(p + \frac{j+2}{2} \right) + 2i, \\ g(u_{jq} u_{(j-1)q+1}) = 2q \left(p + \frac{j+4}{2} \right), 2 \leq j \leq p-2, j \text{ is an even and } 1 \leq i \leq q-1, \\ g(u_{(j-1)q+i} u_{(j-1)q+i+1}) = (4p-j+1)q + 2i, g(u_{jq} u_{(j-1)q+i+1}) = (4p-(j-3))q, \\ 3 \leq j \leq p-1, k \text{ is odd and } 1 \leq i \leq q-1.$$

Thus the labels of corresponding vertices mod $4pq$ in case of $p \equiv 0 \pmod{4}$ become :

$$g^*(u_i) \equiv 2pq + 4i + 2, g^*(u_{2q+i}) \equiv 2pq - 8q + 4i + 2, g^*(u_{4q+i}) \equiv 2pq - 16q + 4i + 2, \dots, g^*\left(u_{\left(\frac{p}{2}+3\right)q+i}\right) \equiv 20q + 4i + 2, \dots, g^*\left(u_{(p-5)q+i}\right) \equiv 2pq - 12q + 4i + 2, \\ g^*\left(u_{(p-3)q+i}\right) \equiv 2pq - 4q + 4i + 2, g^*\left(u_{(p-1)q+i}\right) \equiv 2pq + 4q + 4i + 2, g^*(u_{n+i}) \equiv 2pn + 12n + 4i + 2, g^*(u_{3q+i}) \equiv 2pq + 20q + 4i + 2, g^*(u_{5q+i}) \equiv 2pq + 28q + 4i + 2, \\ \dots, g^*\left(u_{\left(\frac{p}{2}-7\right)q+i}\right) \equiv 4pq - 12q + 4i + 2, g^*\left(u_{\left(\frac{p}{2}-3\right)q+i}\right) \equiv 4pq - 4q + 4i + 2, \dots, g^*\left(u_{\left(\frac{p}{2}\right)q+i}\right) \equiv 2pq + 8q + 4i + 2, g^*\left(u_{(p-4)q+i}\right) \equiv 2pq + 16q + 4i + 2 \text{ and } g^*\left(u_{\left(\frac{p}{2}+2\right)q+i}\right) \equiv 4pq + 8q + 4i + 2, 1 \leq i \leq q.$$

In case of $p \equiv 2 \pmod{4}$, the labels of corresponding vertices mod $4pq$ can be:

$$g^*(u_i) \equiv 2pq + 4i + 2, g^*(u_{2q+i}) \equiv 2pq - 8q + 4i + 2, g^*(u_{4q+i}) \equiv 2pq - 16q + 4i + 2, \dots, g^*\left(u_{\frac{p}{2}-5}q+i\right) \equiv 20q + 4i + 2, g^*\left(u_{\frac{p}{2}-3}q+i\right) \equiv 12q + 4i + 2, \\ g^*\left(u_{\frac{p}{2}-1}q+i\right) \equiv 4pq - 24q + 4i + 2, g^*\left(u_{\frac{p}{2}-6}q+i\right) \equiv 4pq - 16q + 4i + 2, g^*\left(u_{\frac{p}{2}-4}q+i\right) \equiv 4pq - 8q + 4i + 2, g^*\left(u_{\frac{p}{2}-2}q+i\right) \equiv 4i + 2, g^*\left(u_{\frac{p}{2}}q+i\right) \equiv 8q + 4i + 2, \\ g^*\left(u_{\frac{p}{2}+2}q+i\right) \equiv 16q + 4i + 2, \dots, g^*\left(u_{(p-5)q+i}\right) \equiv 2pq - 12q + 4i - 2, g^*\left(u_{(p-3)q+i}\right) \equiv 2pq - 4q + 4i - 2, g^*\left(u_{(p-1)q+i}\right) \equiv 2pq - 20q + 4i - 2, \\ g^*\left(u_{\left(\frac{p}{2}+1\right)q+i}\right) \equiv 4pq - 4q + 4i - 2, g^*\left(u_{\left(\frac{p}{2}+3\right)q+i}\right) \equiv 4pq - 12q + 4i - 2, g^*\left(u_{\left(\frac{p}{2}+5\right)q+i}\right) \equiv 4pq - 20q + 4i - 2, \dots, g^*\left(u_{(p-6)q+i}\right) \equiv 2pq + 24q + 4i + 2, \\ g^*\left(u_{(p-4)q+i}\right) \equiv 2pq + 16q + 4i + 2, g^*\left(u_{(p-2)q+i}\right) \equiv 2pq + 8q + 4i + 22, 1 \leq i \leq q. \quad \square$$

Illustration: The e. e. g. l. $T_{8,11}$, $T_{8,12}$, $T_{10,8}$ and $T_{10,9}$, respectively are displayed in Figure 2.3.

(a) $T_{8,11}$

(b) $T_{8,12}$

(c) $T_{10,8}$

(d) $T_{10,9}$ Figure 2.3: The e. e. g. l. of $T_{8,11}$, $T_{8,12}$, $T_{10,8}$ and $T_{10,9}$

Lemma 2.0.2. *If $q = 2$ and $p > 1$ is an odd number and belong to $\mathbf{Z}^+$. Then $T_{p,2}$ is an e. e. g. graph.*

Proof. Consider that $T_{p,2}$ which is in Figure 2.4.

Now, let us label the edges of the left oval as follows:

$$g(u_{2p-1}u_{2p-3}) = 2; g(u_{2p-(4j-1)}u_{2p-(4j+3)}) = 4j+4, 1 \leq j \leq \frac{p-3}{2}, g(u_1u_3) = 2p+2; g(u_{4j-3}u_{4j+1}) = 2p+4j+2, 1 \leq j \leq \frac{p-1}{2}.$$

Hence, label the edges of the right oval as follows:

$$g(u_{4j-2}u_{4j+2}) = 2p+4j+4; g(u_{2p-4j+2}u_{2p-4j-2}) = 4j+6, 1 \leq j \leq \frac{p-3}{2}, g(u_{2p-4}u_{2q}) = 4; g(u_{2p}u_{2q-2}) = 6, g(u_2u_4) = 2p+4.$$

Moreover, we label the edges of the upper half circles as follows:

$$g(u_1u_2) = 4p+2; g(u_{2p-1}u_{2p}) \equiv 4j+6; g(u_{4j-1}u_{4j}) = 4p+8j+2; g(u_{2p-4j-1}u_{2p-4j}) = 4p+8j+6, 1 \leq j \leq \frac{p-3}{2} \text{ and } g(u_{2p-3}u_{2p-2}) = 8p-2.$$

Finally, we label the edges of the lower half circles as follows:

$$g(u_1u_2) = 4p+4; g(u_{2p-1}u_{2p}) = 4j+8; g(u_{4j-1}u_{4j}) = 4p+8j+4; g(u_{2p-4j-1}u_{2p-4j}) = 4p+8j+8, 1 \leq j \leq \frac{p-3}{2} \text{ and } g(u_{2p-3}u_{2p-2}) = 8m.$$

Thus, the labels of the corresponding vertices mod $8p$ become:

$$g^*(u_{4j+1}) \equiv 4p-8j+14, 0 \leq j \leq \frac{p-3}{2}; g^*(u_{4j+2}) \equiv 4p-8j+18; g^*(u_{4j+3}) \equiv 4p-8j+22; g^*(u_{4j+4}) \equiv 4p-8j+26, 1 \leq j \leq \frac{p-5}{2}; g^*(u_{2p-3}) \equiv 8; g^*(u_{2p-2}) \equiv 14; g^*(u_{2p-1}) \equiv 4q+16; g^*(u_{2p}) \equiv 24 \quad \square$$

Figure 2.4: $T_{p,2}$ $p > 1$ where p is odd

Illustration: The e. e. g. l. of $T_{11,2}$ is presented in Figure 2.5.

Figure 2.5: e. e. g. l. of $T_{11,2}$

Lemma 2.0.3. *If $q = 3$ and p is an odd positive integer greater than 1. Then the torus grid graph $T_{p,3}$ is an edge even graceful graph.*

Proof. Assume that $T_{p,3}$ which is in Figure 2.6. We label the edges of the ovals begin with the edges of the oval $C_p^{(1)}$ to the edges of the oval $C_p^{(3)}$ such as: First, turn clockwise to label the edges $u_{3m-2}u_{3m-5}, u_{3m-1}u_{3m-4}, u_{3m}u_{3m-3}$ by 4, 6, 8, then turn counter-clockwise to label the edges $u_{3m-5}u_{3m-11}, u_{3m-3}u_{3m-9}, u_{3m-4}u_{3m-11}$ by 10, 12, 14 and so on. Next, turn clockwise to label the edges u_4u_1, u_5u_2, u_6u_3 by $3m+1, 3m+3, 3m+5$, then turn counter-clockwise anticlockwise to label the edges u_1u_7, u_3u_9, u_2u_8 by $3p+7, 3p+9, 3p+11$, then turn clockwise to label the edges $u_7u_{13}, u_8u_{14}, u_9u_{15}$ by $3p+13, 3p+15, 3p+17$ and so on. Now, turn clockwise to label the edges $u_{3p-8}u_{3p-2}, u_{3p-7}u_{3p-1}, u_{3p}u_{3p-3}$ by $6p-2, 6p, 2$. In this step, we label the edges of the circles $C_3^{(j)}, 1 \leq j \leq p$ begin with the edges of the inner most circle $C_3^{(1)}$, then the edges of the circle $C_3^{(p)}$ then the edges of the circles

$$C_3^{(2)}, C_3^{(p-2)}, C_3^{(4)}, C_3^{(p-4)}, C_3^{(6)}, C_3^{(p-6)}, \dots, C_3^{(3)}, C_3^{(p-1)}$$

as follows:

$$\begin{aligned} g(u_1u_2) &= 6p+2; g(u_2u_3) = 6p+6; g(u_3u_1) = 6p+4; g(u_{3p-2}u_{3p-1}) = \\ &= 6p+8, g(u_{3p-1}u_{3p}) = 6p+12, g(u_{3p}u_{3p-2}) = 6p+10, g(u_{3j-2}u_{3j-1}) = 6p+ \\ &= 6j+2, g(u_{3j-1}u_{3j}) = 6p+8j+4, g(u_{3j}u_{3j-2}) = 6p+6j+6, 2 \leq j \leq p-1 \\ &\text{where } j \text{ is even and} \\ g(u_{3j-2}u_{3j-1}) &= 12p-6j+8, g(u_{3j-1}u_{3j}) = 12p-6j+10, g(u_{3j}u_{3j-2}) = \\ &= 12p-6j+12, 3 \leq j \leq p-2, \text{ where } j \text{ is odd.} \end{aligned}$$

Thus, the labels of the corresponding vertices mod $12m$ become:

$$\begin{aligned} g^*(u_1) &\equiv 6p+14, g^*(u_2) \equiv 6p+12, g^*(u_3) \equiv 6p+24, g^*(u_{3p-2}) \equiv 6p+ \\ &= 20, g^*(u_{3p-1}) \equiv 6p+26, g^*(u_{3p}) \equiv 32, g^*(u_{3j+i-3}) \equiv 6p+6j+4i+12, 2 \leq i \leq \\ &= 3, 2 \leq j \leq p-1 \text{ where } j \text{ is even; } g^*(u_{3j+i-3}) \equiv 6p-6j+4i+18, 1 \leq i \leq 3, 3 \leq \\ &= j \leq p-4 \text{ where } j \text{ is odd; } g^*(u_{3p-8}) \equiv 34, g^*(u_{3p-7}) \equiv 38, g^*(u_{3p}) \equiv 6p+42 \end{aligned}$$

□

Figure 2.6: $T_{p,3}, p > 1, p$ is odd

Illustration: The e. e. g. l. of $T_{11,3}$ is presented in Figure 2.7.

Figure 2.7: The e. e. g. l. of $T_{11,3}$

Lemma 2.0.4. Let $q = 5$ and $p > 1$ be an odd number that belongs to $\mathbf{Z}^+$. Then $T_{p,5}$ is an e. e. g. graph.

Proof. Consider that $T_{p,5}$, is given in Figure 2.8. First we label the edges of the ovals begin with the edges of the oval $C_p^{(1)}$ to the edges of the oval $C_p^{(5)}$ as follows:

Turn clockwise to label the edges

$u_{5p-4}u_{5p-9}, u_{5p-3}u_{5p-8}, u_{5p-2}u_{5p-7}, u_{5p-1}u_{5p-6}, u_{5p}u_{5p-5}$ by 2, 4, 8, 10, 12, then turn counter-clockwise to label the edges

$u_{5p-9}u_{5p-19}, u_{5p-5}u_{5p-15}, u_{5p-6}u_{5p-16}, u_{5p-7}u_{5p-17}, u_{5p}u_{5p-18}$ by 14, 16, 18, 20, 22 and so on.

Turn clockwise to label the edges

$u_6u_1, u_7u_2, u_8u_3, u_9u_4, u_{10}u_5$ by $5(p-3)+14, 5(p-3)+16, 5(p-3)+18, 5(p-3)+20, 5(p-3)+22$,

then move clockwise to label the edges, then turn counter-clockwise to label the edge

$u_1u_{11}, u_5u_{15}, u_4u_{14}, u_3u_{13}, u_2u_{12}$ by $5(p-3)+24, 5(p-3)+26, 5(p-3)+28, 5(p-3)+30, 5(p-3)+32$ and so on. Finally turn clockwise to label the edges $u_{5q-14}u_{5p-4}, u_{5p-13}u_{5p-3}, u_{5p-12}u_{5p-2}, u_{5p-11}u_{5p-1}, u_{5p-10}u_{5p}$ by $10(p-1)+4, 10(p-1)+6, 10(p-1)+8, 10p, 6$.

Next, label the edges of circles $C_5^{(p)}, 1 \leq j \leq p$ begin with the inner most circle $C_5^{(1)}$, then the edges of the circle $C_5^{(p)}$, then the edges of the circle $C_5^{(2)}$, then the edges of the circles $C_5^{(4)}, C_5^{(6)}, \dots, C_5^{(p-1)}$, then the edges of the circles $C_5^{(3)}, C_5^{(5)}, \dots, C_5^{(p-2)}$ as follows:

move clockwise to label the edges

$g(u_1u_2) = 10p+2; g(u_2u_3) = 10p+4; g(u_3u_4) = 10p+8; g(u_4u_5) = 10p+6, g(u_5u_1) = 10(p+1), g(u_{5p+i-5}u_{5p+i-4}) = 10(p+1)+2i, 1 \leq i \leq 4, g(u_{5p}u_{5q-4}) = 10(p+2), g(u_6u_7) = 10p+22, g(u_7u_8) = 10p+26, g(u_8u_9) = 10p+24, g(u_9u_{10}) = 10p+28, g(u_{10}u_6) = 10(p+3), g(u_{5j-5+i}u_{5j-4+i}) = 10(p+j)+2i, g(u_{5j}u_{5j-4}) = 10(p+j+1), 1 \leq i \leq 4, 4 \leq j \leq p-1$ where j is even and

$g(u_{5j-5+i}u_{5j-4+i}) = 10(2p-j+1)+2i, g(u_{5j}u_{5j-4}) = 10(2p-j+2), 1 \leq i \leq 4, 3 \leq j \leq p-2$, where j is odd.

Thus, the labels of the corresponding vertices mod $12p$ become:

$g^*(u_1) \equiv 10p+20, g^*(u_2) \equiv 10p+24, g^*(u_3) \equiv 10p+30, g^*(u_4) \equiv 10p+32, g^*(u_5) \equiv 10p+34, g^*(u_{5p-4}) \equiv 10p+28, g^*(u_{5p-3}) \equiv 10p+26, g^*(u_{5p-2}) \equiv$

$10p + 36, g^*(u_{5p-1}) \equiv 10p + 26, g^*(u_{5p}) \equiv 56, g^*(u_6) \equiv 10p + 40, g^*(u_7) \equiv$
 $10p + 46, g^*(u_8) \equiv 10p + 48, g^*(u_9) \equiv 10p + 50, g^*(u_{10}) \equiv 10p + 56, g^*(u_{5j+i-5}) \equiv$
 $10(p + j) + 4i + 16, 1 \leq i \leq 5, 4 \leq j \leq p - 3, j \text{ is even}; g^*(u_{5j+i-5}) \equiv 10(p -$
 $j) + 4i + 26, 1 \leq i \leq 5, 3 \leq j \leq p - 4, j \text{ is odd}; g^*(u_{5p+4}) \equiv 50, g^*(u_{5p-13}) \equiv$
 $54, g^*(u_{5p-12}) \equiv 58, g^*(u_{5p-11}) \equiv 62, g^*(u_{5p-10}) \equiv 10p + 70, g^*(u_{5p-9}) \equiv$
 $8, g^*(u_{5p-8}) \equiv 12, g^*(u_{5p-7}) \equiv 18, g^*(u_{5p-6}) \equiv 22, g^*(u_{5p-5}) \equiv 26. \quad \square$

Figure 2.8: $T_{p,5}, p > 1, p$ is odd

Illustration: The e. e. g. l. of $T_{11,5}$ is presented in Figure 2.9.

Figure 2.9: The e. e. g. l. of $T_{11,5}$

Lemma 2.0.5. Assume that $q > 1$ is an odd number that belongs to $\mathbf{Z}^+$ and $p = 3$. Then $T_{3,q}$ is an e. e. g. graph.

Proof. Consider that $T_{3,q}$ is given in Figure 2.10. As first step, we label the edges of the oval $C_3^{(j)}$, $1 \leq j \leq q$ clockwise starting with the edges of the oval $C_3^{(1)}$ to the edges of the oval $C_3^{(q)}$ as follows:

$$g(u_{2q+i}u_{q+i}) = 2i, g(u_{q+i}u_i) = 2q + 2i, g(u_{2q+i}u_i) = 4q + 2i, 1 \leq i \leq n.$$

Secondly, we label the edges of the circles $C_q^{(j)}$, $1 \leq j \leq q$, respectively as follows: $g(u_iu_{i+1}) = 6q + 2i, g(u_qu_1) = 8q, g(u_{q+i}u_{q+i+1}) = 10q + 2i, g(u_qu_1) = 8q, g(u_{2q+i}u_{2q+i+1}) = 8q + 2i, g(u_qu_1) = 8q, 1 \leq i \leq q - 1$.

Thus the labels of the corresponding vertices mod $12q$ become: $g^*(u_1) = 8q + 6, g^*(u_i) = 6q + 8i - 2, g^*(u_{q+1}) = 6, g^*(u_{q+i}) = 10q + 8i - 2, g^*(u_{2q+1}) = 10q + 6, g^*(u_{2q+i}) = 8q + 8i - 2, 2 \leq i \leq q$. $\square$

Figure 2.10: $T_{2,q}$, $q > 1$ is odd positive.

Illustration: The e. e. g. l. of $T_{3,15}$ is given in Figure 2.11.

Theorem 2.0.6. Assume that $p > 2$ is an even number that belongs to $\mathbf{Z}^+$ and $q > 1$ is an odd number that belongs to $\mathbf{Z}^+$. Then $T_{p,q}$ is an e. e. g. graph.

Proof. By using $n = |V(T_{p,q})| = pq$ and $m = |E(T_{p,q})| = 2pq$ and $r = \max\{m, n\} = 2pq$ and let $g : E(T_{p,q}) \longrightarrow \{2, 4, \dots, 4pq - 2\}$, hence there are four cases:

Case 1 $q \equiv 0 \pmod{8}$. Assume that $T_{p,q}$ is given as in Figure 2.12. As first step, we label the edges of the ovals $C_p^{(j)}$, $1 \leq j \leq q$ begin with the edges of the oval $C_p^{(1)}$ as follows:
we turn clockwise to label the edges

$$u_{(p-1)q+1}u_{(p-2)q+1}, u_{(p-1)q+2}u_{(p-2)q+2}, u_{(p-1)q+3}u_{(p-2)q+3}, \\ \dots, u_{pq+1}u_{(p-1)q-1}, u_{pq}u_{(p-1)q}$$

by $2, 4, 6, \dots, 2q - 2, 2q$, then we turn counter-clockwise to label the edges

$$u_{(p-2)q+1}u_{(p-4)q+1}, u_{(p-1)q}u_{(p-3)q}, u_{(p-1)q-1}u_{(p-3)q-1}, \\ \dots, u_{(p-2)q+3}u_{(p-4)q+3}, u_{(p-2)q+2}u_{(p-4)q+2}$$

by $2q + 2, 2q + 4, 2q + 6, \dots, 4q - 2, 4q$ and so on. If $p \equiv 1 \pmod{4}$, we turn clockwise to label the edges $u_{q+1}u_1, u_{q+2}u_2, u_{q+3}u_3, \dots, u_{2q-1}u_{q-1}, u_{2q}u_q$ by $(p-1)q + 2, (p-1)q + 4, (p-1)q + 6, \dots, (p+1)q - 2, (p+1)q$, then we turn counter-clockwise to label the edges

$u_1u_{2q+1}, u_qu_{3q}, u_{q-1}u_{3q-1}, \dots, u_3u_{2q+3}, u_2u_{2n+2}$ by $(p+1)n + 2, (p+1)q + 4, (p+1)q + 6, \dots, (p+3)q - 2, (p+3)q$, next we turn clockwise to label the edges

$u_{2q+1}u_{4q+1}, u_{2q+2}u_{4q+2}, u_{2q+3}u_{4q+3}, \dots, u_{3q-1}u_{5q-1}, u_{3q}u_{5q}$ by $(p+3)q + 2, (p+3)q + 4, (p+3)q + 6, \dots, (p+5)q - 2, (p+5)q$ and so on.

Figure 2.12: $T_{p,q}$, p is even and $q \equiv 0 \pmod{8}$

If $p \equiv 1 \pmod{4}$, we turn counterterm-clockwise to label the edges

$$u_{q+1}u_1, u_{2q}u_q, u_{2q-1}u_{q-1}, \dots, u_{q+3}u_3, u_{q+2}u_2$$

by $(p-1)q+2, (p-1)q+4, (p-1)q+6, \dots, (p+1)q-2, (p+1)q$, then we turn clockwise to label the edges $u_1u_{2q+1}, u_2u_{2q+2}, u_3u_{2q+3}, \dots, u_{q-1}u_{3q-1}, u_qu_{3q}$ by $(p+1)q+2, (p+1)q+4, (p+1)q+6, \dots, (p+3)q-2, (p+3)q$, then we turn counter-clockwise anticlockwise to label the edges

$$u_{2q+1}u_{4q+1}, u_{3q}u_{5q}, u_{3q-1}u_{5q-1}, \dots, u_{2q+3}u_{4q+3}, u_{2q+2}u_{4q+2}$$

by $(p+3)q+2, (p+3)q+4, (p+3)q+6, \dots, (p+5)q-2, (p+5)q$ and so on.

Finally, we turn clockwise to label the edges

Case 1

$$u_{(p-3)q+1}u_{(p-1)q+1}, u_{(p-3)q+2}u_{(p-1)q+2}, u_{(p-3)q+3}u_{(p-1)q+3}, \\ \dots, u_{(p-2)q-1}u_{pq-1}, u_{(p-2)q}u_{pq}$$

by $2q(p-1)+2, 2q(p-1)+4, 2q(p-1)+6, \dots, 2pq-2, 2pq$. Second, we label the edges of the circles $C_q^{(j)}, 1 \leq j \leq p$ starting with the edges of the inner most circle $C_q^{(1)}$ as follows: $g(u_1u_2) = 2pq+4, g(u_2u_3) = 3pq+2, g(u_3u_4) = 2pq+6, g(u_4u_5) = 2pq+8, g(u_5u_6) = 2pq+12, g(u_6u_7) = 2pq+10, g(u_7u_8) = 2pq+14, g(u_8u_9) = 2pq+16, g(u_9u_{10}) = 2pq+20, g(u_{10}u_{11}) = 2pq+18, g(u_{11}u_{12}) = 2pq+22, \dots, g(u_{q-9}u_{q-8}) = 2q(p+1)-18, g(u_{q-8}u_{q-7}) = 2q(p+1)-16, g(u_{q-7}u_{q-6}) = 2q(p+1)-12, g(u_{q-6}u_{q-5}) = 2q(p+1)-14, g(u_{q-5}u_{q-4}) = 2q(p+1)-10, g(u_{q-4}u_{q-3}) = 2q(p+1)-8, g(u_{q-3}u_{q-2}) = 2q(p+1)-4, g(u_{q-2}u_{q-1}) = 2q(p+1)-6, g(u_{q-1}u_q) = 2q(p+1)-2$ and $g(u_qu_1) = 2q(p+1)$, then label the edges of outer circle $C_q^{(p)}$ as follows: $g(u_{(p-1)q+1}u_{(p-1)q+i-1}) = 2q(p+1)+2i, 1 \leq i \leq q-2, g(u_{q-1}u_{pq}) = 2q(p+2)$ and $g(u_{pq}u_{q(p-1)-1}) = 2q(p+1)-2$ then we label the edges the circle $C_q^{(2)}$ as follows: $g(u_{q+1}u_{q+2}) = 2q(p+2)+2, g(u_{q+2}u_{q+3}) = 2q(p+2)+4, g(u_{q+3}u_{q+4}) = 2q(p+2)+8, g(u_{q+4}u_{q+5}) = 2q(p+2)+6, g(u_{q+5}u_{q+6}) = 2q(p+2)+10, g(u_{q+6}u_{q+7}) = 2q(p+2)+12, g(u_{q+7}u_{q+8}) = 2q(p+2)+16, g(u_{q+8}u_{q+9}) = 2q(p+2)+14, g(u_{q+9}u_{q+10}) = 2q(p+2)+18, g(u_{q+10}u_{q+11}) = 2q(p+2)+20, g(u_{q+11}u_{q+12}) = 2q(p+2)+24, \dots, g(u_{2q+9}u_{2q+8}) = 2q(p+3)-16, g(u_{2q+8}u_{2q+7}) = 2q(p+3)-18, g(u_{2q+7}u_{2q+6}) = 2q(p+3)-14, g(u_{2q+6}u_{2q+5}) = 2q(p+3)-12, g(u_{2q+5}u_{2q+4}) = 2q(p+3)-8, g(u_{2q+4}u_{2q+3}) = 2q(p+3)-10, g(u_{2q+3}u_{2q+2}) = 2q(p+3)-6, g(u_{2q+2}u_{2q+1}) =$

$2q(p+3)-4, g(u_{2q-1}u_{2q}) = 2q(p+3)$ and $g(u_{2q}u_{q+1}) = 2q(p+3)-2$. Next, we label the edges of circles $C_q^{(3)}, C_q^{(5)}, \dots, C_q^{(p-2)}$ as follows: $g(u_{(j-1)q+i}u_{(j-1)q+i+1}) = 4q\left(p - \frac{j-1}{2}\right) + 2i$ and $g(u_{jq}u_{(j-1)q+1}) = 4q(p - (2j-4)) - 2, 3 \leq j \leq p-2, j$ is odd and $1 \leq i \leq q-1$. Finally, we label the edges of the circles $C_q^{(4)}, C_q^{(6)}, \dots, C_q^{(p-1)}$ as follows: $g(u_{(j-1)q+i}u_{(j-1)q+i+1}) = 2q(p+j)+2i$ and $g(u_{jq}u_{(j-1)q+1}) = 2q(p+j+1), 4 \leq j \leq p-1, j$ is even and $1 \leq i \leq q-1$.

Thus the labels of the vertices of the inner post circle $C_q^{(1)} \bmod 4pq$ are $g^*(u_1) \equiv 2pq+2q+8, g^*(u_2) \equiv 2pq+2q+10, g^*(u_3) \equiv 2pq+2q+12, g^*(u_4) \equiv 2pq+2q+18, g^*(u_5) \equiv 2pq+2q+24, g^*(u_6) \equiv 2pq+2q+26, g^*(u_7) \equiv 2pq+2q+28, g^*(u_8) \equiv 2pq+2q+34, g^*(u_9) \equiv 2pq+2q+40, g^*(u_{10}) \equiv 2pq+2q+42, g^*(u_{11}) \equiv 2pq+2q+44, g^*(u_{12}) \equiv 2pq+2q+50, \dots, g^*(u_{q-8}) \equiv 2pq+6q-30, g^*(u_{q-7}) \equiv 2pq+6q-24, g^*(u_{q-6}) \equiv 2pq+6q-22, g^*(u_{q-5}) \equiv 2pq+6q-20, g^*(u_{q-4}) \equiv 2pq+6q-14, g^*(u_{q-3}) \equiv 2pq+6q-8, g^*(u_{q-2}) \equiv 2pq+6q-6, g^*(u_{q-1}) \equiv 2pq+6q-4$ and $g^*(u_q) \equiv 2pq+6q+2$.

The labels of the vertices of the outer circle $C_q^{(p)} \bmod 4pq$ are: $g^*(u_{(p-1)q+1}) \equiv 2pq+4q+4, g^*(u_{(p-1)q+i}) \equiv 2pq+2q+8i-2, 2 \leq i \leq q-2, g^*(u_{pq-1}) \equiv 2pq+10q-8$ and $g^*(u_{pq}) \equiv 2pq+10q-2$. The labels of the vertices of the outer circle $C_q^{(2)} \bmod 4pq$ become: $g^*(u_{q+1}) \equiv 2pq+6q+4, g^*(u_{q+2}) \equiv 2pq+6q+10, g^*(u_{q+3}) \equiv 2pq+6q+16, g^*(u_{q+4}) \equiv 2pq+6q+18, g^*(u_{q+5}) \equiv 2pq+6q+20, g^*(u_{q+6}) \equiv 2pq+6q+26, g^*(u_{q+7}) \equiv 2pq+6q+32, g^*(u_{q+8}) \equiv 2pq+6q+34, g^*(u_{q+9}) \equiv 2pq+6q+36, g^*(u_{q+10}) \equiv 2pq+6q+42, \dots, g^*(u_{2q-8}) \equiv 2pq+10q-30, g^*(u_{2q-7}) \equiv 2pq+10q-28, g^*(u_{2q-6}) \equiv 2pq+10q-22, g^*(u_{2q-5}) \equiv 2pq+10q-16, g^*(u_{2q-4}) \equiv 2pq+10q-14, g^*(u_{2q-3}) \equiv 2pq+10q-12, g^*(u_{2q-2}) \equiv 2pq+10q-6, g^*(u_{2q-1}) \equiv 2pq+10q$ and $g^*(u_{2q}) \equiv 2pq+10q+2$.

The labels of the vertices of circles $C_q^{(3)}, C_q^{(5)}, \dots, C_q^{(p-2)} \bmod 4pq$ become: $g^*(u_{jq+i}) \equiv 2pq+2(j-1)q+4i+2, 2 \leq j \leq p-3, j$ is even and $1 \leq i \leq q$. The labels of the vertices of circles $C_q^{(4)}, C_q^{(6)}, \dots, C_q^{(p-1)} \bmod 4pq$ become: $g^*(u_{jq+i}) \equiv 2pq+2(j+2)q+4i+2, 3 \leq j \leq p-2, j$ is odd and $1 \leq i \leq q$.

Case 2 $q \equiv 2 \bmod 8$. Let $T_{p,q}$ be the torus grid graph that is given in Figure12. As first step, we label the edges of the ovals $C_p^{(j)}, 1 \leq j \leq n$ as the same in case (1).

Then, we label the edges of the circles $C_q^{(j)}, 1 \leq j \leq p$ begin with the edges of the inner most circle $C_q^{(1)}$ such as: $g(u_1u_2) = 2pq+2, g(u_2u_3) = 2pq+4, g(u_3u_4) = 2pq+8, g(u_4u_5) = 2pq+6, g(u_5u_6) = 2pq+10, g(u_6u_7) =$

$2pq+12, g(u_7u_8) = 2pq+16, g(u_8u_9) = 2pq+14, g(u_9u_{10}) = 2pq+18, g(u_{10}u_{11}) = 2pq+20, g(u_{11}u_{12}) = 2pq+24, \dots, g(u_{q-9}u_{q-8}) = 2q(p+1)-18, g(u_{q-8}u_{q-7}) = 2q(p+1)-16, g(u_{q-7}u_{q-6}) = 2q(p+1)-12, g(u_{q-6}u_{q-5}) = 2q(p+1)-14, g(u_{q-5}u_{q-4}) = 2q(p+1)-10, g(u_{q-4}u_{q-3}) = 2q(p+1)-8, g(u_{q-3}u_{q-2}) = 2q(p+1)-4, g(u_{q-2}u_{q-1}) = 2q(p+1)-6, g(u_{q-1}u_q) = 2q(p+1)-2$ and $g(u_qu_1) = 2q(p+1)$.

Then we label the edges of outer circle $C_q^{(p)}$ as follows: $g(u_{(p-1)q+i}u_{q+3}) = 2q(p+1) + 2i, 1 \leq i \leq q-2, g(u_{q-1}u_{pq}) = 2q(p+2)$ and $g(u_{pq}u_{q(p-1)+1}) = 2q(p+2)-2$. Then, we label the edges the circle $C_q^{(2)}$ as follows: $g(u_{q+1}u_{q+2}) = 2q(p+2) + 2, g(u_{q+2}u_{q+3}) = 2q(p+2) + 4, g(u_{q+3}u_{q+4}) = 2q(p+2) + 8, g(u_{q+4}u_{q+5}) = 2q(p+2) + 6, g(u_{q+5}u_{q+6}) = 2q(p+2) + 10, g(u_{q+6}u_{q+7}) = 2q(p+2) + 12, g(u_{q+7}u_{q+8}) = 2q(p+2) + 16, g(u_{q+8}u_{q+9}) = 2q(p+2) + 14, g(u_{q+9}u_{q+10}) = 2q(p+2) + 18, g(u_{q+10}u_{q+11}) = 2q(p+2) + 20, g(u_{q+11}u_{q+12}) = 2q(p+2) + 24, \dots, g(u_{2q-9}u_{2q-8}) = 2q(p+3) - 18, g(u_{2q-8}u_{2q-7}) = 2q(p+3) - 16, g(u_{2q-7}u_{2q-6}) = 2q(p+3) - 12, g(u_{2q-6}u_{2q-5}) = 2q(p+3) - 14, g(u_{2q-5}u_{2q-4}) = 2q(p+3) - 10, g(u_{2q-4}u_{2q-3}) = 2q(p+3) - 8, g(u_{2q-3}u_{2q-2}) = 2q(p+3) - 6, g(u_{2q-2}u_{2q-1}) = 2q(p+3) - 4, g(u_{2q-1}u_{2q}) = 2n(p+3)$ and $g(u_{2q}u_{q+1}) = 2q(p+3)-2$. Then we label the edges of circles $C_q^{(3)}, C_q^{(5)}, \dots, C_q^{(p-2)}$ and $C_q^{(4)}, C_q^{(6)}, \dots, C_q^{(p-1)} \bmod 4pq$ as the same in case (1).

Thus, the labels of the vertices of the inner most circle $C_q^{(1)} \bmod 4pq$ are: $g^*(u_1) \equiv 2pq+2q+6, g^*(u_2) \equiv 2pq+2q+10, g^*(u_3) \equiv 2pq+2q+16, g^*(u_4) \equiv 2pq+2q+18, g^*(u_5) \equiv 2pq+2q+20, g^*(u_6) \equiv 2pq+2q+26, g^*(u_7) \equiv 2pq+2q+32, g^*(u_8) \equiv 2pq+2q+34, g^*(u_9) \equiv 2pq+2q+36, g^*(u_{10}) \equiv 2pq+2q+42, \dots, g^*(u_{q-8}) \equiv 2pq+6q-30, g^*(u_{q-7}) \equiv 2pq+6q-24, g^*(u_{q-6}) \equiv 2pq+6q-22, g^*(u_{q-5}) \equiv 2pq+6q-20, g^*(u_{q-4}) \equiv 2pq+6q-14, g^*(u_{q-3}) \equiv 2pq+6q-8, g^*(u_{q-2}) \equiv 2pq+6q-6, g^*(u_{q-1}) \equiv 2pq+6q-4$ and $g^*(u_q) \equiv 2pq+6q+2$.

The labels of the vertices of the outer circle $C_q^{(p)} \bmod 4pq$ are as follows: $g^*(u_{(q-1)q+1}) \equiv 2pq+4q+4, g^*(u_{(p-1)q+i}) \equiv 2pq+2q+8i-2, 2 \leq i \leq q-2, g^*(u_{pq-1}) \equiv 2pq+10q-8$ and $g^*(u_{pq}) \equiv 2pq+10q-2$.

The labels of the vertices of the circle $C_q^{(2)} \bmod 4pq$ are as follows: $g^*(u_{q+1}) \equiv 2pq+6q+4, g^*(u_{q+2}) \equiv 2pq+6q+10, g^*(u_{q+3}) \equiv 2pq+6q+16, g^*(u_{q+4}) \equiv 2pq+6q+18, g^*(u_{q+5}) \equiv 2pq+6q+20, g^*(u_{q+6}) \equiv 2pq+6q+26, g^*(u_{q+7}) \equiv 2pq+6q+32, g^*(u_{q+8}) \equiv 2pq+6q+34, g^*(u_{q+9}) \equiv 2pq+6q+36, g^*(u_{q+10}) \equiv 2pq+6q+42, \dots, g^*(u_{2q-8}) \equiv 2pq+10q-30, g^*(u_{2q-7}) \equiv 2pq+10q-24, g^*(u_{2q-6}) \equiv 2pq+10q-22, g^*(u_{2q-5}) \equiv 2pq+10q-20, g^*(u_{2q-4}) \equiv 2pq+10q-14, g^*(u_{2q-3}) \equiv 2pq+10q-10, g^*(u_{2q-2}) \equiv 2pq+10q-6, g^*(u_{2q-1}) \equiv 2pq+10q$ and $g^*(u_{2p}) \equiv 2pq+10q+2$.

The labels of the vertices of circles $C_q^{(3)}, C_q^{(5)}, \dots, C_q^{(p-2)}$ and $C_q^{(4)}, C_q^{(6)}, \dots, C_q^{(p-1)} \bmod 4pq$ are the same as in case (1).

Case 3 $q \equiv 4 \bmod 8$. Assume that $T_{p,q}$ is given in Figure 2.12. First, we label the edges of the ovals $C_p^{(j)}, 1 \leq j \leq q$ as the same in case (1). Secondly, we label the edges of the circles $C_q^{(j)}, 1 \leq j \leq p$ begin with the edges of the inner most circle $C_q^{(1)}$ as follows: $g(u_1u_2) = 2pq + 4, g(u_2u_3) = 2pq + 2, g(u_3u_4) = 2pq + 6, g(u_4u_5) = 2pq + 8, g(u_5u_6) = 2pq + 12, g(u_6u_7) = 2pq + 10, g(u_7u_8) = 2pq + 14, g(u_8u_9) = 2pq + 16, g(u_9u_{10}) = 2pq + 20, g(u_{10}u_{11}) = 2pq + 18, g(u_{11}u_{12}) = 2pq + 22, \dots, g(u_{q-9}u_{q-8}) = 2q(p+1) - 18, g(u_{q-8}u_{q-7}) = 2q(p+1) - 16, g(u_{q-7}u_{q-6}) = 2q(p+1) - 12, g(u_{q-6}u_{q-5}) = 2q(p+1) - 14, g(u_{q-5}u_{q-4}) = 2q(p+1) - 10, g(u_{q-4}u_{q-3}) = 2q(p+1) - 8, g(u_{q-3}u_{q-2}) = 2q(p+1) - 4, g(u_{q-2}u_{q-1}) = 2q(p+1) - 6, g(u_{q-1}u_q) = 2q(p+1) - 2$ and $g(u_qu_1) = 2q(p+1)$. Hence the edges of outer circle $C_q^{(p)}$ are labeled as: $g(u_{(p-1)q+1}u_{(p-1)q+2}) = 4, g(u_{(p-1)q+2}u_{(p-1)q+3}) = 2q(p+1) + 2, g(u_{(p-1)q+i}u_{(p-1)q+i+1}) = 2q(p+1) + 2i, 3 \leq i \leq q-2$ and $g(u_{pq}u_{q(p-1)+1}) = 2q(p+2)$. Then we label the edges the circle $C_q^{(2)}$ as follows: $g(u_{q+1}u_{q+2}) = 2q(p+2) + 2, g(u_{q+2}u_{q+3}) = 2q(p+2) + 4, g(u_{q+3}u_{q+4}) = 2q(p+2) + 8, g(u_{q+4}u_{q+5}) = 2q(p+2) + 6, g(u_{q+5}u_{q+6}) = 2q(p+2) + 10, g(u_{q+6}u_{q+7}) = 2q(p+2) + 12, g(u_{q+7}u_{q+8}) = 2q(p+2) + 16, g(u_{q+8}u_{q+9}) = 2q(p+2) + 14, g(u_{q+9}u_{q+10}) = 2q(p+2) + 18, g(u_{q+10}u_{q+11}) = 2q(p+2) + 20, \dots, g(u_{2q-9}u_{2q-8}) = 2q(p+3) - 16, g(u_{2q-8}u_{2q-7}) = 2q(p+3) - 18, g(u_{2q-7}u_{2q-6}) = 2q(p+3) - 14, g(u_{2q-6}u_{2q-5}) = 2q(p+3) - 12, g(u_{2q-5}u_{2q-4}) = 2q(p+3) - 8, g(u_{2q-4}u_{2q-3}) = 2q(p+3) - 10, g(u_{2q-3}u_{2q-2}) = 2q(p+3) - 6, g(u_{2q-2}u_{2q-1}) = 2q(p+3) - 4, g(u_{2q-1}u_{2q}) = 2q(p+3)$ and $g(u_{2q}u_{q+1}) = 2q(p+3) - 2$. Then we label the edges of circles $C_q^{(3)}, C_q^{(5)}, \dots, C_q^{(p-2)}$ and $C_q^{(4)}, C_q^{(6)}, \dots, C_q^{(p-1)} \bmod 4pq$ as the same in case (1).

Thus, the labels of the vertices of the inner most circle $C_q^{(1)} \bmod 4pq$ are: $g^*(u_1) \equiv 2pq + 2q + 8, g^*(u_2) \equiv 2pq + 2q + 10, g^*(u_3) \equiv 2pq + 2q + 12, g^*(u_4) \equiv 2pq + 2q + 18, g^*(u_5) \equiv 2pq + 2q + 24, g^*(u_6) \equiv 2pq + 2q + 26, g^*(u_7) \equiv 2pq + 2q + 28, g^*(u_8) \equiv 2pq + 2q + 34, g^*(u_9) \equiv 2pq + 2q + 40, g^*(u_{10}) \equiv 2pq + 2q + 42, g^*(u_{11}) \equiv 2pq + 2q + 44, g^*(u_{12}) \equiv 2pq + 2q + 50, \dots, g^*(u_{q-8}) \equiv 2pq + 6q - 30, g^*(u_{q-7}) \equiv 2pq + 6q - 24, g^*(u_{q-6}) \equiv 2pq + 6q - 22, g^*(u_{q-5}) \equiv 2pq + 6q - 20, g^*(u_{q-4}) \equiv 2pq + 6q - 14, g^*(u_{q-3}) \equiv 2pq + 6q - 8, g^*(u_{q-2}) \equiv 2pq + 6q - 6, g^*(u_{q-1}) \equiv 2pq + 6q - 4$ and $g^*(u_q) \equiv 2pq + 6q + 2$.

The labels of the vertices of the outer circle $C_q^{(2)} \bmod 4pq$ are as follows: $g^*(u_{q+1}) \equiv 2pq + 6q + 4, g^*(u_{q+2}) \equiv 2pq + 6q + 10, g^*(u_{q+3}) \equiv 2pq + 6q + 16, g^*(u_{q+4}) \equiv 2pq + 6q + 18, g^*(u_{q+5}) \equiv 2pq + 6q + 20, g^*(u_{q+6}) \equiv$

$$2pq+6q+26, g^*(u_{q+7}) \equiv 2pq+6q+32, g^*(u_{q+8}) \equiv 2pq+6q+34, g^*(u_{q+9}) \equiv 2pq+6q+36, g^*(u_{q+10}) \equiv 2pq+6q+42, \dots, g^*(u_{2q-8}) \equiv 2pq+10q-30, g^*(u_{2q-7}) \equiv 2pq+10q-28, g^*(u_{2q-6}) \equiv 2pq+10q-22, g^*(u_{2q-5}) \equiv 2pq+10q-16, g^*(u_{2q-4}) \equiv 2pq+10q-14, g^*(u_{2q-3}) \equiv 2pq+10q-12, g^*(u_{2q-2}) \equiv 2pq+10q-6, g^*(u_{2q-1}) \equiv 2pq+10q, g^*(u_{2q}) \equiv 2pq+10q+2.$$

The labels of the vertices of circles $C_q^{(3)}, C_q^{(5)}, \dots, C_q^{(p-2)}$ and $C_q^{(4)}, C_q^{(6)}, \dots, C_q^{(p-1)} \bmod 4pq$ as the same in case (1).

Case 4 $q \equiv 6 \pmod{8}$. Let $T_{p,q}$ be such as in Figure 2.12. First, we label the edges of the ovals $C_p^{(j)}, 1 \leq j \leq q$ as the same in case (1).

Secondly we label the edges of the circles $C_q^{(j)}, 1 \leq j \leq p$ begin with the edges of the inner most circle $C_q^{(1)}$ as follows: $g(u_1u_2) = 2pq+4, g(u_2u_3) = 2pq+2, g(u_3u_4) = 2pq+6, g(u_4u_5) = 2pq+8, g(u_5u_6) = 2pq+12, g(u_6u_7) = 2pq+10, g(u_7u_8) = 2pq+14, g(u_8u_9) = 2pq+16, g(u_9u_{10}) = 2pq+20, g(u_{10}u_{11}) = 2pq+18, g(u_{11}u_{12}) = 2pq+22, \dots, g(u_{q-9}u_{q-8}) = 2q(p+1)-18, g(u_{q-8}u_{q-7}) = 2q(p+1)-16, g(u_{q-7}u_{q-6}) = 2q(p+1)-12, g(u_{q-6}u_{q-5}) = 2q(p+1)-14, g(u_{q-5}u_{q-4}) = 2q(p+1)-10, g(u_{q-4}u_{q-3}) = 2q(p+1)-8, g(u_{q-3}u_{q-2}) = 2q(p+1)-4, g(u_{q-2}u_{q-1}) = 2q(p+1)-6, g(u_{q-1}u_q) = 2q(p+1)$ and $g(v_qv_1) = 2q(p+1)-2$. Next the edges of outer circle $C_q^{(p)}$ are labeled as: $g(u_{(p-1)q+1}u_{(p-1)q+i+1}) = 2q(p+1)+2i, 1 \leq i \leq q-2, g(u_{q-1}u_{pq}) = 2q(p+2), g(u_{pq}u_{(p-1)q+1}) = 2q(p+2)-2$.

Then, we label the edges the circle $C_q^{(2)}$ as follows: $g(u_{q+1}u_{q+2}) = 2q(p+2)+4, g(u_{q+2}u_{q+3}) = 2q(p+2)+2, g(u_{q+3}u_{q+4}) = 2q(p+2)+6, g(u_{q+4}u_{q+5}) = 2q(p+2)+8, g(u_{q+5}u_{q+6}) = 2q(p+2)+12, g(u_{q+6}u_{q+7}) = 2q(p+2)+10, g(u_{q+7}u_{q+8}) = 2q(p+2)+16, g(u_{q+8}u_{q+9}) = 2q(p+2)+14, g(u_{q+9}u_{q+10}) = 2q(p+2)+20, g(u_{q+10}u_{q+11}) = 2q(p+2)+18, \dots, g(u_{2q-9}u_{2q-8}) = 2q(p+3)-16, g(u_{2q-8}u_{2q-7}) = 2q(p+3)-18, g(u_{2q-7}u_{2q-6}) = 2q(p+3)-14, g(u_{2q-6}u_{2q-5}) = 2q(p+3)-12, g(u_{2q-5}u_{2q-4}) = 2q(p+3)-8, g(u_{2q-4}u_{2q-3}) = 2q(p+3)-10, g(u_{2q-3}u_{2q-2}) = 2q(p+3)-4, g(u_{2q-2}u_{2q-1}) = 2q(p+3)-6, g(u_{2q-1}u_{2q}) = 2q(p+3)-2$ and $g(u_{2q}u_{q+1}) = 2q(p+3)$.

Then we label the edges of circles $C_q^{(3)}, C_q^{(5)}, \dots, C_q^{(p-2)}$ and $C_q^{(4)}, C_q^{(6)}, \dots, C_q^{(p-1)} \bmod 4pq$ as the same in case (1).

Thus, the labels of the vertices of the inner most circle $C_q^{(1)} \bmod 4pq$ are given as: $g^*(u_1) \equiv 2pq+2q+6, g^*(u_2) \equiv 2pq+2q+10, g^*(u_3) \equiv 2pq+2q+12, g^*(u_4) \equiv 2pq+2q+18, g^*(u_5) \equiv 2pq+2q+24, g^*(u_6) \equiv 2pq+2q+26, g^*(u_7) \equiv 2pq+2q+28, g^*(u_8) \equiv 2pq+2q+34, g^*(u_9) \equiv 2pq+2q+40, g^*(u_{10}) \equiv 2pq+2q+42, g^*(u_{11}) \equiv 2pq+2q+44, g^*(u_{12}) \equiv 2pq+$

$2q + 50, \dots, g^*(u_{q-8}) \equiv 2pq + 6q - 30, g^*(u_{q-7}) \equiv 2pq + 6q - 28, g^*(u_{q-6}) \equiv 2pq + 6q - 22, g^*(u_{q-5}) \equiv 2pq + 6q - 16, g^*(u_{q-4}) \equiv 2pq + 6q - 14, g^*(u_{q-3}) \equiv 2pq + 6q - 12, g^*(u_{q-2}) \equiv 2pq + 6q - 6, g^*(u_{q-1}) \equiv 2pq + 6q$ and $g^*(u_q) \equiv 2pq + 6q + 2$.

The labels of the vertices of the outer circle $C_q^{(p)} \bmod 4pq$ are the same in case (1).

The labels of the vertices of the circle $C_q^{(2)} \bmod 4pq$ are:

$g^*(u_{q+1}) \equiv 2pq + 6q + 8, g^*(u_{q+2}) \equiv 2pq + 6q + 10, g^*(u_{q+3}) \equiv 2pq + 6q + 12, g^*(u_{q+4}) \equiv 2pq + 6q + 18, g^*(u_{q+5}) \equiv 2pq + 6q + 24, g^*(u_{q+6}) \equiv 2pq + 6q + 26, g^*(u_{q+7}) \equiv 2pq + 6q + 28, g^*(u_{q+8}) \equiv 2pq + 6q + 34, g^*(u_{q+9}) \equiv 2pq + 6q + 40, g^*(u_{q+10}) \equiv 2pq + 6q + 42, \dots, g^*(u_{2q-8}) \equiv 2pq + 10q - 30, g^*(u_{2q-7}) \equiv 2pq + 10q - 28, g^*(u_{2q-6}) \equiv 2pq + 10q - 22, g^*(u_{2q-5}) \equiv 2pq + 10q - 16, g^*(u_{2q-4}) \equiv 2pq + 10q - 14, g^*(u_{2q-3}) \equiv 2pq + 10q - 10, g^*(u_{2q-2}) \equiv 2pq + 10q - 6, g^*(u_{2q-1}) \equiv 2pq + 10q - 4, g^*(u_{2q}) \equiv 2pq + 10q + 2$.

The labels of the vertices of circles $C_q^{(3)}, C_q^{(5)}, \dots, C_q^{(p-2)}$ and $C_q^{(4)}, C_q^{(6)}, \dots, C_q^{(p-1)} \bmod 4pq$ as the same in case (1).

□

Illustration: The e. e. g. l. of $T_{8,9}$, $T_{10,9}$, $T_{12,9}$ and $T_{14,9}$ are presented in Figure 2.13, respectively.

(a) $T_{8,9}$

(b) $T_{10,9}$

(c)T_{12,9}

(d) $T_{14,9}$ Figure 2.13: The e. e. g. l. of $T_{8,9}$, $T_{10,9}$, $T_{12,9}$ and $T_{14,9}$

Theorem 2.0.7. Assume that $p > 1$ is an odd number that belongs to $\mathbf{Z}^+$ and $q > 1$ is an odd number that belongs to $\mathbf{Z}^+$. Then the torus grid graph $T_{p,q}$ is an e. e. g. graph.

Proof. By taking $n = |V(T_{p,q})| = pq$ and $m = |E(T_{p,q})| = 2pq$ and $r = \max\{m, n\} = 2pq$ and let $g : E(T_{p,q}) \longrightarrow \{2, 4, \dots, 4pq - 2\}$, then there are four cases:

Case 1 $q \equiv 0 \pmod{8}$. Assume that $T_{p,q}$, is given as in Figure 2.14. Here, we label the edges of the ovals $C_p^{(j)}$, $1 \leq j \leq q$ starting with the edges of the oval $C_p^{(1)}$ as follows :

Turn clockwise to label the edges

$$u_{(p-1)q+1}u_{(p-2)q+1}, u_{(p-1)q+2}u_{(p-2)q+2}, u_{(p-1)q+3}u_{(p-2)q+3}, \\ \dots, u_{pq-1}u_{(p-1)q-1}, u_{pq}u_{(p-1)q}$$

by $2, 4, 6, \dots, 2q - 2, 2q$, then turn counter-clockwise to label the edges

$$u_{(p-2)q+1}u_{(p-4)q+1}, u_{(p-1)q}u_{(p-3)q}, u_{(p-1)q-1}u_{(p-3)q-1}, \\ \dots, u_{(p-2)q+3}u_{(p-4)q+3}, u_{(p-2)q+2}u_{(p-4)q+2}$$

by $2q + 2, 2q + 4, 2q + 6, \dots, 4q - 2, 4q$ and so on. If $p \equiv 1 \pmod{4}$, we turn clockwise to label the edges

$$u_{q+1}u_1, u_{q+2}u_2, u_{q+3}u_3, \dots, u_{2q-1}u_{q-1}, u_{2q}u_q$$

by $(p-1)q + 2, (p-1)q + 4, (p-1)q + 6, \dots, (p+1)q - 2, (p+1)q$, next we turn-counterclockwise to label the edges

$$u_1u_{2q+1}, u_qu_{3q}, u_{q-1}u_{3q-1}, \dots, u_3u_{2q+3}, u_2u_{2q+2}$$

by $(p+1)q + 2, (p+1)q + 4, (p+1)q + 6, \dots, (p+3)q - 2, (p+3)q$, then we turn clockwise to label the edges

$$u_{2q+1}u_{4q+1}, u_{2q+2}u_{4q+2}, u_{2q+3}u_{4q+3}, \dots, u_{3q-1}u_{5q-1}, u_{3q}u_{5q}$$

by $(p+3)q + 2, (p+3)q + 4, (p+3)q + 6, \dots, (p+5)q - 2, (p+5)q$ and so on.

Figure 2.14: $T_{p,q,p}$ is even and $q \equiv 0 \pmod{8}$

If $p \equiv 3 \pmod{4}$, we turn counter-clockwise to label the edges

$$u_{q+1}u_1, u_{2q}u_q, u_{2q-1}u_{q-1}, \dots, u_{q+3}u_3, u_{q+2}u_2$$

by $(p-1)q+2, (p-1)q+4, (p-1)q+6, \dots, (p+1)q-2, (p+1)q$, next we turn clockwise to label the edges

$$u_1u_{2q+1}, u_2u_{2q+2}, u_3u_{2q+3}, \dots, u_{q-1}u_{3q-1}, u_qu_{3q}$$

by $(p+1)q+2, (p+1)q+4, (p+1)q+6, \dots, (p+3)q-2, (p+3)q$, then turn counter-clockwise anticlockwise to label the edges

$$u_{2q+1}u_{4q+1}, u_{3q}u_{5q}, u_{3q-1}u_{5q-1}, \dots, u_{2q+3}u_{4q+3}, u_{2q+2}u_{4q+2}$$

by $(p+3)q+2, (p+3)q+4, (p+3)q+6, \dots, (p+5)q-2, (p+5)q, (p+5)q-2, (p+5)q$ and so on.

Finally, we turn clockwise to label the edges

$$u_{(p-3)q+1}u_{(p-1)q+1}, u_{(p-3)q+2}u_{(p-1)q+2}, u_{(p-3)q+3}u_{(p-1)q+3}, \\ \dots, u_{(p-2)q-1}u_{pq-1}, u_{(p-2)q}u_{pq}$$

by $2q(p-1)+2, 2q(p-1)+4, 2q(p-1)+6, \dots, 2pq-2, 2pq$. Second, we label the edges of the circles $C_q^{(j)}, 1 \leq j \leq p$ starting with the edges of the inner most circle $C_q^{(1)}$ as follows:

$g(u_1u_2) = 2pq+2, g(u_2u_3) = 2pq+4, g(u_3u_4) = 2pq+8, g(u_4u_5) = 2pq+6, g(u_5u_6) = 2pq+10, g(u_6u_7) = 2pq+12, g(u_7u_8) = 2pq+16, g(u_8u_9) = 2pq+14, g(u_9u_{10}) = 2pq+18, g(u_{10}u_{11}) = 2pq+20, g(u_{11}u_{12}) = 2pq+24, \dots, g(u_{q-9}u_{q-8}) = 2q(p+1)-20, g(u_{q-8}u_{q-7}) = 2q(p+1)-16, g(u_{q-7}u_{q-6}) = 2q(p+1)-14, g(u_{q-6}u_{q-5}) = 2q(p+1)-10, g(u_{q-5}u_{q-4}) = 2q(p+1)-12, g(u_{q-4}u_{q-3}) = 2q(p+1)-8, g(u_{q-3}u_{q-2}) = 2q(p+1)-6, g(u_{q-2}u_{q-1}) = 2q(p+1)-2, g(u_{q-1}u_q) = 2q(p+1)-4$ and $g(u_qu_1) = 2q(p+1)$, next we label the edges of outer circle $C_q^{(p)}$ as follows: $g(u_{(p-1)q+i}u_{(p-1)q+i-1}) = 2q(p+1)+2i, 1 \leq i \leq q-1, g(u_{pq}u_{q(p-1)+1}) = 2q(p+2)$, then we label the edges the circle $C_q^{(2)}$ as : $g(u_{q+1}u_{q+2}) = 2q(p+2)+2, g(u_{q+2}u_{q+3}) = 2q(p+2)+6, g(u_{q+3}u_{q+4}) = 2q(p+2)+4, g(u_{q+4}u_{q+5}) = 2q(p+2)+8, g(u_{q+5}u_{q+6}) = 2q(p+2)+10, g(u_{q+6}u_{q+7}) = 2q(p+2)+14, g(u_{q+7}u_{q+8}) = 2q(p+2)+12, g(u_{q+8}u_{q+9}) = 2q(p+2)+16, g(u_{q+9}u_{q+10}) = 2q(p+2)+18, g(u_{q+10}u_{q+11}) = 2q(p+2)+22, \dots, g(u_{2q+9}u_{2q+8}) = 2q(p+3)-16, g(u_{2q+8}u_{2q+7}) = 2q(p+3)-12, g(u_{2q+7}u_{2q+6}) = 2q(p+3)-14, g(u_{2q+6}u_{2q+5}) = 2q(p+3)-$

$10, g(u_{2q+5}u_{2q+4}) = 2q(p+3)-8, g(u_{2q+4}u_{2q+3}) = 2q(p+3)-4, g(u_{2q+3}u_{2q+2}) = 2q(p+3)-6, g(u_{2q+2}u_{2q+1}) = 2q(p+3)-2, g(u_{2q-1}u_{2q}) = 2q(p+3)-2$ and $g(u_{2q}u_{q+1}) = 2q(p+3)$. Then we label the edges of circles

$C_q^{(3)}, C_q^{(5)}, \dots, C_q^{(p-2)}$ as follows:

$g(u_{(j-1)q+i}u_{(j-1)q+i+1}) = 4q\left(p - \frac{j-1}{2}\right) + 2i$ and $g(u_{jq}u_{(j-1)q+1}) = 4q(p - (2j-4)) - 2, 3 \leq j \leq p-2$ where j is odd and $1 \leq i \leq q-1$. Finally, we label the edges of the circles $C_q^{(4)}, C_q^{(6)}, \dots, C_q^{(p-1)}$ as follows:

$g(u_{(j-1)q+i}u_{(j-1)q+i+1}) = 2q(p+j)+2i$ and $g(u_{jq}u_{(j-1)q+1}) = 2q(p+j+1), 4 \leq j \leq q-1$ where j is even and $1 \leq i \leq q-1$. Thus we have the labels of the vertices of the inner most circle $C_q^{(1)} \bmod 4pq$ are $g^*(u_1) \equiv 2pq + 2q + 6, g^*(u_2) \equiv 2pq + 2q + 10, g^*(u_3) \equiv 2pq + 2q + 16, g^*(u_4) \equiv 2pq + 2q + 18, g^*(u_5) \equiv 2pq + 2q + 20, g^*(u_6) \equiv 2pq + 2q + 26, g^*(u_7) \equiv 2pq + 2q + 32, g^*(u_8) \equiv 2pq + 2q + 34, g^*(u_9) \equiv 2pq + 2q + 36, g^*(u_{10}) \equiv 2pq + 2q + 42, g^*(u_{11}) \equiv 2pq + 2q + 48, g^*(u_{12}) \equiv 2pq + 2q + 50, \dots, g^*(u_{q-8}) \equiv 2pq + 6q - 32, g^*(u_{q-7}) \equiv 2pq + 6q - 26, g^*(u_{q-6}) \equiv 2pq + 6q - 20, g^*(u_{q-5}) \equiv 2pq + 6q - 18, g^*(u_{q-4}) \equiv 2pq + 6q - 16, g^*(u_{q-3}) \equiv 2pq + 6q - 10, g^*(u_{q-2}) \equiv 2pq + 6q - 4, g^*(u_{q-1}) \equiv 2pq + 6q - 2$ and $g^*(u_q) \equiv 2pq + 6q$. The labels of the vertices of the outer circle $C_q^{(p)} \bmod 4pq$ are:

$g^*(u_{(p-1)q+1}) \equiv 2pq + 4q + 6, g^*(u_{(p-1)q+i}) \equiv 2pq + 2q + 8i - 2, 2 \leq i \leq q$

The labels of the vertices of the outer circle $C_q^{(2)} \bmod 4pq$ are: $g^*(u_{q+1}) \equiv 2pq + 6q + 6, g^*(u_{q+2}) \equiv 2pq + 6q + 12, g^*(u_{q+3}) \equiv 2pq + 6q + 14, g^*(u_{q+4}) \equiv 2pq + 6q + 18, g^*(u_{q+5}) \equiv 2pq + 6q + 22, g^*(u_{q+6}) \equiv 2pq + 6q + 28, g^*(u_{q+7}) \equiv 2pq + 6q + 30, g^*(u_{q+8}) \equiv 2pq + 6q + 32, g^*(u_{q+9}) \equiv 2pq + 6q + 38, g^*(u_{q+10}) \equiv 2pq + 6q + 44, \dots, g^*(u_{2q-8}) \equiv 2pq + 10q - 30, g^*(u_{2q-7}) \equiv 2pq + 10q - 24, g^*(u_{2q-6}) \equiv 2pq + 10q - 22, g^*(u_{2q-5}) \equiv 2pq + 10q - 20, g^*(u_{2q-4}) \equiv 2pq + 10q - 14, g^*(u_{2q-3}) \equiv 2pq + 10q - 8, g^*(u_{2q-2}) \equiv 2pq + 10q - 6, g^*(u_{2q-1}) \equiv 2pq + 10q - 4$ and $g^*(u_{2q}) \equiv 2pq + 10q + 2$. The labels of the vertices of circles $C_q^{(3)}, C_q^{(5)}, \dots, C_q^{(p-2)} \bmod 4pq$ are: $g^*(u_{jq+i}) \equiv 2pq - 2(j-1)q + 4i + 2, 2 \leq j \leq p-3$ where j is even and $1 \leq i \leq p$.

Case 2 $p \equiv 3 \pmod{8}$. Let $T_{p,q}$ be as in Figure 2.14. First, we label the edges of the ovals $C_p^{(j)}$, $1 \leq j \leq n$ as the same in case (1). Secondly, we label the edges of the circles $C_q^{(j)}$, $1 \leq j \leq p$ begin with the edges of the inner most circle $C_q^{(1)}$ as follows: $g(u_1u_2) = 2pq + 2, g(u_2u_3) = 2pq + 4, g(u_3u_4) = 2pq + 8, g(u_4u_5) = 2pq + 6, g(u_5u_6) = 2pq + 10, g(u_6u_7) = 2pq + 12, g(u_7u_8) = 2pq + 16, g(u_8u_9) = 2pq + 14, g(u_9u_{10}) = 2pq + 18, g(u_{10}u_{11}) = 2pq + 20, g(u_{11}u_{12}) = 2pq + 24, \dots, g(u_{q-9}u_{q-8}) = 2q(p+1) - 18, g(u_{q-8}u_{q-7}) = 2q(p+1) - 14, g(u_{q-7}u_{q-6}) = 2q(p+1) - 16, g(u_{q-6}u_{q-5}) = 2q(p+1) - 12, g(u_{q-5}u_{q-4}) = 2q(p+1) - 10, g(u_{q-4}u_{q-3}) = 2q(p+1) - 6, g(u_{q-3}u_{q-2}) = 2q(p+1) - 8, g(u_{q-2}u_{q-1}) = 2q(p+1), g(u_{q-1}u_q) = 2q(p+1) - 7$ and $g(u_qu_1) = 2q(p+1) - 2$. Then we label the edges of circle $C_q^{(2)}$ as follows: $g(u_{q+1}u_{q+2}) = 2q(p+2) + 2, g(u_{q+2}u_{q+3}) = 2q(p+2) + 6, g(u_{q+3}u_{q+4}) = 2q(p+2) + 4, g(u_{q+4}u_{q+5}) = 2q(p+2) + 8, g(u_{q+5}u_{q+6}) = 2q(p+2) + 10, g(u_{q+6}u_{q+7}) = 2q(p+2) + 14, g(u_{q+7}u_{q+8}) = 2q(p+2) + 12, g(u_{q+8}u_{q+9}) = 2q(p+2) + 16, g(u_{q+9}u_{q+10}) = 2q(p+2) + 18, g(u_{q+10}u_{q+11}) = 2q(p+2) + 22, \dots, g(u_{2q-9}u_{2q-8}) = 2q(p+3) - 16, g(u_{2q-8}u_{2q-7}) = 2q(p+3) - 18, g(u_{2q-7}u_{2q-6}) = 2q(p+3) - 14, g(u_{2q-6}u_{2q-5}) = 2q(p+3) - 12, g(u_{2q-5}u_{2q-4}) = 2q(p+3) - 8, g(u_{2q-4}u_{2q-3}) = 2q(p+3) - 10, g(u_{2q-3}u_{2q-2}) = 2q(p+3) - 6, g(u_{2q-2}u_{2q-1}) = 2q(p+3) - 4, g(u_{2q-1}u_{2q}) = 2q(p+3)$ and $g(u_{2q}u_{q+1}) = 2q(p+3) - 2$. Then we label the edges of circles $C_q^{(3)}, C_q^{(4)}, \dots, C_q^{(p)}$ as the same as case (1). Thus, we have the labels of the vertices of the inner most circle $C_q^{(1)} \pmod{4pq}$ are: $g^*(u_1) \equiv 2pq + 2q + 4, g^*(u_2) \equiv 2pq + 2q + 10, g^*(u_3) \equiv 2pq + 2q + 16, g^*(u_4) \equiv 2pq + 2q + 18, g^*(u_5) \equiv 2pq + 2q + 20, g^*(u_6) \equiv 2pq + 2q + 26, g^*(u_7) \equiv 2pq + 2q + 32, g^*(u_8) \equiv 2pq + 2q + 34, g^*(u_9) \equiv 2pq + 2q + 36, g^*(u_{10}) \equiv 2pq + 2q + 42, g^*(u_{11}) \equiv 2pq + 2q + 48, g^*(u_{12}) \equiv 2pq + 2q + 50, \dots, g^*(u_{q-8}) \equiv 2pq + 6q - 28, g^*(u_{q-7}) \equiv 2pq + 6q - 26, g^*(u_{q-6}) \equiv 2pq + 6q - 24, g^*(u_{q-5}) \equiv 2pq + 6q - 18, g^*(u_{q-4}) \equiv 2pq + 6q - 12, g^*(u_{q-3}) \equiv 2pq + 6q - 10, g^*(u_{q-2}) \equiv 2pq + 6q - 4, g^*(u_{q-1}) \equiv 2pq + 6q$ and $g^*(u_q) \equiv 2pq + 6q - 2$. The labels of the vertices of the circle $C_q^{(2)} \pmod{4pq}$ are as follows: $g^*(u_{q+1}) \equiv 2pq + 6q + 4, g^*(u_{q+2}) \equiv 2pq + 6q + 12, g^*(u_{q+3}) \equiv 2pq + 6q + 14, g^*(u_{q+4}) \equiv 2pq + 6q + 16, g^*(u_{q+5}) \equiv 2pq + 6q + 22, g^*(u_{q+6}) \equiv 2pq + 6q + 28, g^*(u_{q+7}) \equiv 2pq + 6q + 30, g^*(u_{q+8}) \equiv 2pq + 6q + 32, g^*(u_{q+9}) \equiv 2pq + 6q + 38, g^*(u_{q+10}) \equiv 2pq + 6q + 44, \dots, g^*(u_{2q-8}) \equiv 2pq + 10q - 30, g^*(u_{2q-7}) \equiv 2pq + 10q - 28, g^*(u_{2q-6}) \equiv 2pq + 10q - 22, g^*(u_{2q-5}) \equiv 2pq + 10q - 16, g^*(u_{2q-4}) \equiv 2pq + 10q - 14, g^*(u_{2q-3}) \equiv 2pq + 10q - 12, g^*(u_{2q-2}) \equiv 2pq + 10q - 6, g^*(u_{2q-1}) \equiv 2pq + 10q$ and $g^*(u_{2p}) \equiv 2pq + 10q + 2$. The labels of the vertices of circles $C_q^{(3)}, C_q^{(4)}, \dots, C_q^{(p)} \pmod{4pq}$ are the same as in case (1).

Case 3 $p \equiv 5 \pmod{8}$. Let $T_{p,q}$ be as in Figure 14. First, label the edges of the ovals $C_p^{(j)}$, $1 \leq j \leq q$ as the same in case (1). Secondly, we label the edges of the circles $C_q^{(j)}$, $1 \leq j \leq p$ begin with the edges of the inner most circle $C_q^{(1)}$ as follows: $g(u_1u_2) = 2pq + 4$, $g(u_2u_3) = 2pq + 2$, $g(u_3u_4) = 2pq + 6$, $g(u_4u_5) = 2pq + 8$, $g(u_5u_6) = 2pq + 12$, $g(u_6u_7) = 2pq + 10$, $g(u_7u_8) = 2pq + 14$, $g(u_8u_9) = 2pq + 16$, $g(u_9u_{10}) = 2pq + 20$, $g(u_{10}u_{11}) = 2pq + 18$, $g(u_{11}u_{12}) = 2pq + 22$, $\dots$, $g(u_{q-9}u_{q-8}) = 2q(p+1) - 18$, $g(u_{q-8}u_{q-7}) = 2q(p+1) - 14$, $g(u_{q-7}u_{q-6}) = 2q(p+1) - 16$, $g(u_{q-6}u_{q-5}) = 2q(p+1) - 12$, $g(u_{q-5}u_{q-4}) = 2q(p+1) - 10$, $g(u_{q-4}u_{q-3}) = 2q(p+1) - 6$, $g(u_{q-3}u_{q-2}) = 2q(p+1) - 8$, $g(u_{q-2}u_{q-1}) = 2q(p+1)$, $g(u_{q-1}u_q) = 2q(p+1) - 4aqdg(u_qu_1) = 2q(p+1)g(v_{(m-1)n+i}v_{(m-1)n+i+1}) = 2n(m+1) + 2i$, $3 \leq i \leq n-2$ and $g(v_{mn}v_{n(m-1)+1}) = 2n(m+2)$. Next we label the edges the circle $C_q^{(2)}$ as follows: $g(u_{q+1}u_{q+2}) = 2q(p+2) + 2$, $g(u_{q+2}u_{q+3}) = 2q(p+2) + 6$, $g(u_{q+3}u_{q+4}) = 2q(p+2) + 4$, $g(u_{q+4}u_{q+5}) = 2q(p+2) + 8$, $g(u_{q+5}u_{q+6}) = 2q(p+2) + 10$, $g(u_{q+6}u_{q+7}) = 2q(p+2) + 14$, $g(u_{q+7}u_{q+8}) = 2q(p+2) + 12$, $g(u_{q+8}u_{q+9}) = 2q(p+2) + 16$, $g(u_{q+9}u_{q+10}) = 2q(p+2) + 18$, $g(u_{q+10}u_{q+11}) = 2q(p+2) + 22$, $\dots$, $g(u_{2q-9}u_{2q-8}) = 2q(p+3) - 16$, $g(u_{2q-8}u_{2q-7}) = 2q(p+3) - 12$, $g(u_{2q-7}u_{2q-6}) = 2q(p+3) - 14$, $g(u_{2q-6}u_{2q-5}) = 2q(p+3) - 10$, $g(u_{2q-5}u_{2q-4}) = 2q(p+3) - 8$, $g(u_{2q-4}u_{2q-3}) = 2q(p+3) - 4$, $g(u_{2q-3}u_{2q-2}) = 2q(p+3) - 4$, $g(u_{2q-2}u_{2q-1}) = 2q(p+3) - 6$, $g(u_{2q-1}u_{2q}) = 2q(p+3) - 2$ and $g(u_{2q}u_{q+1}) = 2q(p+3)$.

Then we label the edges of circles $C_q^{(3)}, C_q^{(4)}, \dots, C_q^{(p)}$ and as the same in case (1).

Thus, the labels of the vertices of the inner most circle $C_q^{(1)} \pmod{4pq}$ are: $g^*(u_1) \equiv 2pq + 2q + 6$, $g^*(u_2) \equiv 2pq + 2q + 10$, $g^*(u_3) \equiv 2pq + 2q + 12$, $g^*(u_4) \equiv 2pq + 2q + 18$, $g^*(u_5) \equiv 2pq + 2q + 24$, $g^*(u_6) \equiv 2pq + 2q + 26$, $g^*(u_7) \equiv 2pq + 2q + 28$, $g^*(u_8) \equiv 2pq + 2q + 34$, $g^*(u_9) \equiv 2pq + 2q + 40$, $g^*(u_{10}) \equiv 2pq + 2q + 42$, $g^*(u_{11}) \equiv 2pq + 2q + 44$, $g^*(u_{12}) \equiv 2pq + 2q + 50$, $\dots$, $g^*(u_{q-8}) \equiv 2pq + 6q - 28$, $g^*(u_{q-7}) \equiv 2pq + 6q - 26$, $g^*(u_{q-6}) \equiv 2pq + 6q - 24$, $g^*(u_{q-5}) \equiv 2pq + 6q - 18$, $g^*(u_{q-4}) \equiv 2pq + 6q - 12$, $g^*(u_{q-3}) \equiv 2pq + 6q - 10$, $g^*(u_{q-2}) \equiv 2pq + 6q - 4$, $g^*(u_{q-1}) \equiv 2pq + 6q$ and $g^*(u_q) \equiv 2pq + 6q - 2$. The labels of the vertices of the outer circle $C_q^{(2)} \pmod{4pq}$ are as follows:

$g^*(u_{q+1}) \equiv 2pq + 6q + 6$, $g^*(u_{q+2}) \equiv 2pq + 6q + 12$, $g^*(u_{q+3}) \equiv 2pq + 6q + 14$, $g^*(u_{q+4}) \equiv 2pq + 6q + 16$, $g^*(u_{q+5}) \equiv 2pq + 6q + 22$, $g^*(u_{q+6}) \equiv 2pq + 6q + 28$, $g^*(u_{q+7}) \equiv 2pq + 6q + 30$, $g^*(u_{q+8}) \equiv 2pq + 6q + 32$, $g^*(u_{q+9}) \equiv 2pq + 6q + 38$, $g^*(u_{q+10}) \equiv 2pq + 6q + 44$, $\dots$, $g^*(u_{2q-8}) \equiv 2pq + 10q - 30$, $g^*(u_{2q-7}) \equiv 2pq + 10q - 24$, $g^*(u_{2q-6}) \equiv 2pq + 10q - 22$, $g^*(u_{2q-5}) \equiv 2pq + 10q - 20$, $g^*(u_{2q-4}) \equiv 2pq + 10q - 14$, $g^*(u_{2q-3}) \equiv 2pq + 10q - 8$, $g^*(u_{2q-2}) \equiv$

$2pq + 10q - 6, g^*(u_{2q-1}) \equiv 2pq + 10q - 4, g^*(u_{2q}) \equiv 2pq + 10q + 2$. Then we label the edges of circles $C_q^{(3)}, C_q^{(4)}, \dots, C_q^{(p)}$ and $\text{mod } 4pq$ as the same in case (1).

Case 4 $p \equiv 7 \pmod{8}$. Let $T_{p,q}$ be as in Figure 2.14. First, we label the edges of the ovals $C_p^{(j)}, 1 \leq j \leq q$ as the same in case (1). Secondly, we label the edges of the circles $C_q^{(j)}, 1 \leq j \leq p$ begin with the edges of the inner most circle $C_q^{(1)}$ as follows: $g(u_1u_2) = 2pq + 4, g(u_2u_3) = 2pq + 2, g(u_3u_4) = 2pq + 6, g(u_4u_5) = 2pq + 8, g(u_5u_6) = 2pq + 12, g(u_6u_7) = 2pq + 10, g(u_7u_8) = 2pq + 14, g(u_8u_9) = 2pq + 16, g(u_9u_{10}) = 2pq + 20, g(u_{10}u_{11}) = 2pq + 18, g(u_{11}u_{12}) = 2pq + 22, \dots, g(u_{q-9}u_{q-8}) = 2q(p+1) - 20, g(u_{q-8}u_{q-7}) = 2q(p+1) - 16, g(u_{q-7}u_{q-6}) = 2q(p+1) - 14, g(u_{q-6}u_{q-5}) = 2q(p+1) - 10, g(u_{q-5}u_{q-4}) = 2q(p+1) - 12, g(u_{q-4}u_{q-3}) = 2q(p+1) - 8, g(u_{q-3}u_{q-2}) = 2q(p+1) - 6, g(u_{q-2}u_{q-1}) = 2q(p+1) - 2, g(u_{q-1}u_q) = 2q(p+1) - 4$ and $g(u_qu_1) = 2q(p+1)$. Next, we label the edges the circle $C_q^{(2)}$ as follows: $g(u_{q+1}u_{q+2}) = 2q(p+2) + 2, g(u_{q+2}u_{q+3}) = 2q(p+2) + 6, g(u_{q+3}u_{q+4}) = 2q(p+2) + 4, g(u_{q+4}u_{q+5}) = 2q(p+2) + 8, g(u_{q+5}u_{q+6}) = 2q(p+2) + 10, g(u_{q+6}u_{q+7}) = 2q(p+2) + 14, g(u_{q+7}u_{q+8}) = 2q(p+2) + 12, g(u_{q+8}u_{q+9}) = 2q(p+2) + 16, g(u_{q+9}u_{q+10}) = 2q(p+2) + 18, g(u_{q+10}u_{q+11}) = 2q(p+2) + 22, \dots, g(u_{2q-9}u_{2q-8}) = 2q(p+3) - 16, g(u_{2q-8}u_{2q-7}) = 2q(p+3) - 18, g(u_{2q-7}u_{2q-6}) = 2q(p+3) - 14, g(u_{2q-6}u_{2q-5}) = 2q(p+3) - 12, g(u_{2q-5}u_{2q-4}) = 2q(p+3) - 8, g(u_{2q-4}u_{2q-3}) = 2q(p+3) - 10, g(u_{2q-3}u_{2q-2}) = 2q(p+3) - 6, g(u_{2q-2}u_{2q-1}) = 2q(p+3) - 4, g(u_{2q-1}u_{2q}) = 2q(p+3)$ and $g(u_{2q}u_{q+1}) = 2q(p+3) - 2$. Hence, we label the edges of circles $C_q^{(3)}, C_q^{(4)}, \dots, C_q^{(p)}$ as the same in case (1).

Thus, the labels of the vertices of the inner most circle $C_q^{(1)} \text{ mod } 4pq$ are: $g^*(u_1) \equiv 2pq + 2q + 8, g^*(u_2) \equiv 2pq + 2q + 10, g^*(u_3) \equiv 2pq + 2q + 12, g^*(u_4) \equiv 2pq + 2q + 18, g^*(u_5) \equiv 2pq + 2q + 24, g^*(u_6) \equiv 2pq + 2q + 26, g^*(u_7) \equiv 2pq + 2q + 28, g^*(u_8) \equiv 2pq + 2q + 34, g^*(u_9) \equiv 2pq + 2q + 40, g^*(u_{10}) \equiv 2pq + 2q + 42, g^*(u_{11}) \equiv 2pq + 2q + 44, g^*(u_{12}) \equiv 2pq + 2q + 50, \dots, g^*(u_{q-8}) \equiv 2pq + 6q - 32, g^*(u_{q-7}) \equiv 2pq + 6q - 26, g^*(u_{q-6}) \equiv 2pq + 6q - 20, g^*(u_{q-5}) \equiv 2pq + 6q - 18, g^*(u_{q-4}) \equiv 2pq + 6q - 16, g^*(u_{q-3}) \equiv 2pq + 6q - 10, g^*(u_{q-2}) \equiv 2pq + 6q - 4, g^*(u_{q-1}) \equiv 2pq + 6q - 2$ and $g^*(u_q) \equiv 2pq + 6q$.

The labels of the vertices of the circle $C_q^{(2)} \text{ mod } 4pq$ are:

$$g^*(u_{q+1}) \equiv 2pq + 6q + 4, g^*(u_{q+2}) \equiv 2pq + 6q + 12, g^*(u_{q+3}) \equiv 2pq + 6q + 14, g^*(u_{q+4}) \equiv 2pq + 6q + 16, g^*(u_{q+5}) \equiv 2pq + 6q + 22, g^*(u_{q+6}) \equiv 2pq + 6q + 28, g^*(u_{q+7}) \equiv 2pq + 6q + 30, g^*(u_{q+8}) \equiv 2pq + 6q + 32, g^*(u_{q+9}) \equiv$$

circle $C_q^{(2)} \bmod 4pq$ are:

$$\begin{aligned} g^*(u_{q+1}) &\equiv 2pq+6q+4, g^*(u_{q+2}) \equiv 2pq+6q+12, g^*(u_{q+3}) \equiv 2pq+6q+14, \\ g^*(u_{q+4}) &\equiv 2pq+6q+16, g^*(u_{q+5}) \equiv 2pq+6q+22, g^*(u_{q+6}) \equiv 2pq+6q+28, \\ g^*(u_{q+7}) &\equiv 2pq+6q+30, g^*(u_{q+8}) \equiv 2pq+6q+32, g^*(u_{q+9}) \equiv 2pq+6q+38, \\ g^*(u_{q+10}) &\equiv 2pq+6q+44, \dots, g^*(u_{2q-8}) \equiv 2pq+10q-30, \\ g^*(u_{2q-7}) &\equiv 2pq+10q-28, g^*(u_{2q-6}) \equiv 2pq+10q-22, g^*(u_{2q-5}) \equiv 2pq+10q-16, \\ g^*(u_{2q-4}) &\equiv 2pq+10q-14, g^*(u_{2q-3}) \equiv 2pq+10q-12, g^*(u_{2q-2}) \equiv 2pq+10q-6, \\ g^*(u_{2q-1}) &\equiv 2pq+10q, g^*(u_{2q}) \equiv 2pq+10q+2. \end{aligned}$$

The labels of the vertices of circles $C_q^{(3)}, C_q^{(4)}, \dots, C_q^{(p)} \bmod 4pq$ as the same in case (1). $\square$

Illustration: The e. e. g. l. of $T_{9,9}, T_{11,9}, T_{13,9}$ and $T_{15,9}$ are presented in Figure 15, respectively.

(a) $T_{9,9}$

(b) $T_{11,9}$

(c) $T_{13,9}$

(d) $T_{15,9}$ Figure 2.15: The e. e. g. l. of $T_{9,9}$, $T_{11,9}$, $T_{13,9}$ and $T_{15,9}$

Conclusion

In this paper, we have used the connection of labeling of graphs with modular arithmetic and the theory of numbers in general. Also, we have given a detailed study for e.e.g.l. of all cases of members of the torus grid graphs. Moreover, we have given the necessary and sufficient conditions for e.e.g.l. of other important families which is including rectangular $P_m \times P_n$ grid graphs that should be taken into consideration in future studies of e.e.g.l.

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